

Frees – Insurance Portfolio Risk Retention

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Insurance Portfolio Risk Retention

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Abstract. In this paper, I introduce a tool for managing a portfolio of insurance risks. This tool is based on changes in the risk profile when changes in a risk parameter, such as a deductible, coinsurance, or upper policy limit, are made. I refer to the new statistic as a risk measure relative marginal, or RM^2 , for short, change. The theory underpinning the development of this tool draws from the management science literature on quantile sensitivity and from the financial risk management literature on granularity theory.

By examining data from the Wisconsin Local Government Property Fund, I show how RM^2 changes can be used by a policyholder to select an effective risk mitigation strategy. I also show how it can be used by an insurer to identify the best and worst risks in terms of opportunities for risk management. The RM^2 changes reflect the underlying dependence structure of risks; I use an elliptical copula framework to demonstrate the sensitivity of risk mitigation strategy to the dependence structure.

1 Introduction and Background

1.1 Basic Set-Up

Individuals, corporations, and government entities regularly manage financial risks Y . To motivate the discussion, think of Y as an insurance risk, specifically, the amount of a loss due to an insured event such as an automobile accident or damage to a home. In designing and establishing contracts, it is common for insurers to take on only a portion of the risk through the use of deductibles, coinsurance, and upper policy limits. For example, if a person purchases an insurance contract with a deductible d , then that person, the insured, is responsible for loss in the amount of $\min(Y, d)$ whereas the insurer's responsibility is $\max(0, Y - d)$. In general, I use the notation $g_\theta(Y)$ to denote the retained claims of the insurer, where g is a known function and θ is the risk retention parameter (such as a deductible), both set in the insurance contract.

The insurer is assumed to have a portfolio, or collection, of risks Y_1, \dots, Y_p and is typically interested in the distribution of the sum of risks. It is interesting to understand how this

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distribution changes as the insurer changes contractual parameter θ . To be specific, suppose that the parameter θ affects only the i th risk. In this context, the retained claims for the risk is $g_\theta(Y_i)$, the other retained losses are denoted as $S_{(i)} = \sum_{j \neq i} Y_j$, and the portfolio sum of retained claims is $S_g = g_\theta(Y_i) + S_{(i)}$.

To assess the uncertainty of a risk, this article employs the concept of a “risk-measure,” described further in Section 2.3.3. As an important special case that provides the foundation for the study of the more general class, I focus on the quantile. A quantile is also known as the “value at risk,” denoted as VaR , and is commonly used in the banking and insurance industries. In general, one can interpret the risk measure as the amount of assets needed by the holder of risk as a protection against the risk’s uncertainty.

1.2 Quantile Sensitivities

To understand how a risk measure, in particular a quantile, changes when a parameter changes, we first review the literature on quantile sensitivities, as in [Hong \(2009\)](#). Consider a generic random vector X and let $h(X; \theta)$ be a function that depends on X and parameter θ . I assume that θ is univariate. If θ is multivariate, then we may treat each dimension as a univariate parameter, while fixing other dimensions as constant. Let $F_h(x; \theta) = \Pr(h(X; \theta) \leq x)$ be the distribution function of the random variable $h(X; \theta)$ and, for a fixed α , define the quantile function $q_\alpha(\theta) = F_h^{-1}(\alpha)$, recognizing the dependence of the quantile on the parameter θ . Assume smoothness of F_h in both arguments x and θ . With this, we have $\alpha = F_h(q_\alpha(\theta))$. Differentiating with respect to θ yields

$$0 = \partial_\theta F_h(q_\alpha(\theta)) + f_h(q_\alpha(\theta)) \partial_\theta q_\alpha(\theta).$$

This uses the short-hand notation $\partial_\theta = \frac{\partial}{\partial \theta}$. Thus,

$$q'_\alpha(\theta) = \partial_\theta q_\alpha(\theta) = -\frac{\partial_\theta F_h(q_\alpha(\theta))}{f_h(q_\alpha(\theta))} = -\frac{\partial_\theta F_h(x; \theta)}{\partial_x F_h(x; \theta)} \Big|_{x=q_\alpha(\theta)}. \quad (1)$$

The derivative of the quantile function in equation (1) is known as a “quantile sensitivity.”

To interpret the quantile sensitivity, [Hong \(2009\)](#) showed how to write it as a conditional expectation. Using notation from [Jiang and Fu \(2015\)](#), who provide less stringent conditions for the relationship to hold, we have

$$\mathbb{E}(\partial_\theta h(X; \theta) | h(X; \theta) = x) = -\frac{\partial_\theta F_h(x; \theta)}{\partial_x F_h(x; \theta)}. \quad (2)$$

In management science, these results have been utilized as part of the general simulation literature on evaluating derivatives, cf., [Fu \(2008\)](#). For quantile sensitivities, most of the work in this active stream has focused on estimation using identically and independently distributed data as in, e.g. [Fu et al. \(2009\)](#).

For a special case, one can think of $X = (Y_i, S_{(i)})$, the i th risk and the sum of other risks in a portfolio, and the function $h(X; \theta) = \theta Y_i + S_{(i)} = S$. In this context, equation (2) was derived by [Gourieroux et al. \(2000\)](#) in the financial risk management literature. Specifically, they showed

$$q'_\alpha(\theta) = \mathbb{E}(Y_i | S = q_\alpha(\theta)).$$

A related result was given by [Tasche \(1999\)](#) for the “conditional tail expectation” risk measure.

In financial risk management, the study of portfolio loss contributions can be applied to (a) market, (b) credit, or (c) operational losses. According to [Rosen and Saunders \(2010\)](#), decomposing portfolio risk into different sources is a fundamental problem in financial risk management. Here, the focus is on the $S = \theta Y_i + S_{(i)}$ that allows the analyst to analyze the change in a risk measure due to the change in the i th risk/investment. The risk measures RM are based on a positively homogeneous functions of degree one, $RM(tx) = tRM(x)$.

In financial risk management, there is an interest in decomposing risk measures (including VaR) into their systemic risk and idiosyncratic contributions. The systemic risk factors are sources of uncertainty that are common to risks in the portfolio (e.g., market risk factors such as interest rates, exchange rates, and so forth). The systematic portfolio distribution is the $E[S_g|Z]$ where Z is the set of systemic risk factors. The technique is to examine the portfolio risk measure as the sum of the systematic portfolio risk measure plus a so-called “granularity adjustment” that is essentially a second order Taylor series expansion of the quantile (around the “infinitely granular” portfolio $E[S_g|Z]$).

This is an active literature with recent contributions by [Gagliardini and Gouriéroux \(2014\)](#), [Gagliardini and Gouriéroux \(2013\)](#), and [Fermanian \(2014\)](#). The literature emphasizes:

- Non-tradeable as well as tradeable securities (e.g., loans, mortgages, life insurance contracts, or credit default swaps).
- Dynamic aspects, such as portfolios hedged on a daily basis.
- Multi-dimensional risk factors, e.g., market risk factors such as interest rates, exchange rates, and so forth.

1.3 Insurance Context

In the scope of the paper, it will take very little additional effort to generalize the work on quantile sensitivities to risk measure sensitivities. A risk measure is a functional of the insurance loss distribution function F . To be specific, in equation (13) that follows, this article emphasizes risk measures of the form $R(F) = \int_0^1 q_\alpha(\theta) dK(\alpha)$, where $K(\cdot)$ is some appropriately chosen weight function.

This paper considers quantile sensitivities for the portfolio random variable $S_g = g(Y_i; \theta) + S_{(i)}$, where $g(\cdot; \theta)$ is a known function up to parameter θ . The choice of g includes the linear case $g(y; \theta) = \theta y$ as well as $g(y; \theta) = \max(0, y - \theta)$ for a deductible. To give the reader additional flavor for the insurance applications, in Section 2 I summarize the single policy case ($S_{(i)} \equiv 0$). Here, we will see that not only may the distribution of Y_i contain mass points but, even in the case that Y_i has continuous distribution, the distribution $g(Y_i; \theta)$ often contains mass points. This feature naturally distinguishes this work from the earlier literature.

Also in Section 2, I rescale the quantile sensitivity by the derivative of the expectation. This rescaling does not cause additional technical complications but will help with the interpretation of results. Specifically, assume that there are opportunities for changing risks through the risk-sharing parameters. For example, an insurer can reduce both the price and

the risk measure by increasing the deductible. We examine small changes in the risk measure per unit change in price of the form

$$\frac{\partial_{\theta} R(g(Y))}{\partial_{\theta} P(g(Y))} \quad (3)$$

where θ is a risk retention parameter. I call this a *Risk Measure Relative Marginal* change, or RM^2 change, for short. As we will see, quantifying RM^2 changes can provide risk managers with valuable perspective on managing risks.

The main contribution of the paper is the introduction of RM^2 changes to a portfolio of risks in Section 3. Section 3.1 focuses on the case of a portfolio of independent risks and Section 3.2 provides extensions to a portfolio of dependent risks. I initially consider general multivariate distributions but then specialize consideration to multivariate elliptical copulas in Section 3.2.2.

Many readers will find the probabilistic results presented in Section 3 to be relatively straightforward and easy to describe; their impact on a substantive discipline, insurance, is developed in the empirical Section 4. This section considers experience from a government insurer, the Wisconsin Local Government Property Insurance Fund (LGPIF). This fund insures buildings and contents of local government entities, such as villages and schools, that need to manage tight fiscal budgets prudently. This section shows how the RM^2 measures can be used to give advice to policyholder risk managers as well as the insurer's risk manager.

This section helps to emphasize the contribution of this paper. The novelty is due to our consider of heterogeneous populations (that we model using regression techniques), portfolio participation that allows for discontinuities, and general dependence structures.

I close in Section 5 with a summary and a few concluding remarks.

2 Single Policy Risk Retention

2.1 Foundations

Consider a loss Y with distribution function F . For a fixed $0 < \alpha < 1$, let $\xi_{\alpha} = F^{-1}(\alpha)$ be the corresponding quantile.

For a risk Y with distribution function F , use the notation $g(Y)$ to denote the insured loss, the amount that they insurer will pay, and $Y - g(Y)$ to denote those losses retained by the insured. For an important special case that covers many situations of interest, consider the risk-sharing function

$$g(y; d, c, u) = \begin{cases} 0 & y < d \\ c(y - d) & d \leq y < u \\ c(u - d) & y \geq u \end{cases} . \quad (4)$$

Here, d is the deductible, c is the coinsurance amount, and u is the upper limit of coverage. For example, one might have a prescription drug coverage with a deductible $d = 100$, coinsurance $c = 0.80$, and upper limit $u = 2,100$. With these parameters, a loss of 50 yields a $g(50) = 0$ claim. A loss of 150 yields a $g(150) = 40$ claim. A loss of 3,200 yields a $g(3200) = 1600$ claim. More general risk-sharing plans are described in Section 2.3.2.

Define the risk-sharing function defined in equation (4). To collect notation, recall that c is the coinsurance amount, d is the deductible, and u is for upper limit. The risk parameters may be collected into a vector $\theta = (d, c, u)$. I also use the parameter $b = c(u - d)$ which represents the maximum payout. Returning to equation (4), the distribution function of $g(Y)$ is

$$F_{g(Y;\theta)}(z) = \begin{cases} F(d) & z = 0 \\ F\left(\frac{z}{c} + d\right) & z < b \\ 1 & z \geq b \end{cases} . \quad (5)$$

When it exists, define the probability density function $F' = f$. There is additional discreteness due to the parameters d and u (or b). See Figure 1 to visualize the distribution function of the insured loss.

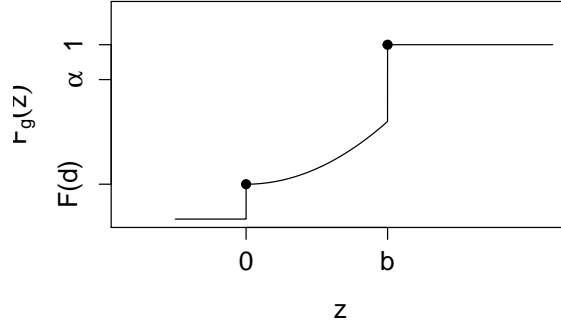


Figure 1: Single Policy Insured Loss Distribution Function.

Now use Figure 1 to help determine the quantile of the insured loss distribution, say, $\xi_{g,\alpha}$. From the figure, if $F(d) < \alpha < F(b-)$, then one finds $\xi_{g,\alpha}$ as the solution z of the equation $\alpha = F\left(\frac{z}{c} + d\right)$. Straight-forward algebra shows this to be $\xi_{g,\alpha} = c(\xi_\alpha - d)$. From the figure, if $\alpha \geq F(b-)$, then $\xi_{g,\alpha} = b = c(u - d)$. Summarizing, we have

$$\xi_{g,\alpha} = \begin{cases} 0 & \alpha < F(d) \\ c(\xi_\alpha - d) & F(d) \leq \alpha < F(b-) \\ c(u - d) & \alpha \geq F(b-) \end{cases} . \quad (6)$$

We will also need the mean insured loss. Use integration by parts to get

$$\begin{aligned} E g(Y; \theta) &= \int_d^u c(y - d)dF(y) + c(u - d)(1 - F(u)) \\ &= c \int_d^u (1 - F(y))dy. \end{aligned} \quad (7)$$

assuming that d is finite.

2.2 Risk Measure Relative Marginal Changes

In this paper, I wish to focus on how much a risk measure, such as $\xi_{g,\alpha}$, and a premium, such as the expected insured loss $E g(Y; \theta)$, change when a risk parameter changes. Table 1 summarizes these marginal changes.

It is interesting to think about how much a risk measure changes when a parameter changes, per unit change of a premium. I refer to this as the *risk measure relative marginal* (RM^2) change for insured losses. To illustrate for the coinsurance parameter c , I have

$$\frac{\partial_c \xi_{g,\alpha}}{\partial_c E g(Y; \theta)} = \begin{cases} 0 & \alpha < F(d) \\ \frac{\xi_{\alpha-d}}{\int_d^u (1-F(y)) dy} & F(d) \leq \alpha < F(b-) \\ \frac{u-d}{\int_d^u (1-F(y)) dy} & \alpha \geq F(b-) \end{cases}, \quad (8)$$

Results for other parameters appear in Table 1.

Table 1: Marginal Changes of Expected Insured Losses and Quantiles

Summary Measure	Parameter (θ_r)		
	d	c	u
$\partial_\theta E g(Y; \theta)$	$-c(1 - F(d))$	$\int_d^u (1 - F(y)) dy$	$c(1 - F(u))$
$\partial_\theta \xi_{g,\alpha}$	$\begin{cases} 0 & \alpha < F(d) \\ -c & \alpha \geq F(d) \end{cases}$	$\begin{cases} 0 & \alpha < F(d) \\ \xi_{\alpha-d} & F(d) \leq \alpha < F(b-) \\ u-d & \alpha \geq F(b-) \end{cases}$	$\begin{cases} 0 & \alpha < F(b-) \\ c & \alpha \geq F(b-) \end{cases}$
RM^2	$\begin{cases} 0 & \alpha < F(d) \\ \frac{1}{1-F(d)} & \alpha \geq F(d) \end{cases}$	$\begin{cases} 0 & \alpha < F(d) \\ \frac{\xi_{\alpha-d}}{\int_d^u (1-F(y)) dy} & F(d) \leq \alpha < F(b-) \\ \frac{u-d}{\int_d^u (1-F(y)) dy} & \alpha \geq F(b-) \end{cases}$	$\begin{cases} 0 & \alpha < F(b-) \\ \frac{1}{1-F(u)} & \alpha \geq F(b-) \end{cases}$

Note that the risk parameter c enters into the RM^2 changes in Table 1 only implicitly through the parameter b . As a potentially interesting benchmark, I also remark that if d equals the lower limit of support of F , then

$$\lim_{u \rightarrow \infty} \frac{\partial_c \xi_{g,\alpha}}{\partial_c E g(Y; \theta)} = \frac{\xi_{\alpha}}{E Y}.$$

Given knowledge of the distribution, one can use RM^2 values in Table 1 to select the preferred marginal risk management strategy. To illustrate, first note that all marginal changes are non-negative, so 0 is the best one can do. If the current set of risk parameters is such that $F(d) < \alpha < F(b-)$, then changing the upper limit u is the best choice, as Table 1 indicates that marginal changes in u mean no change in the risk parameter per unit of the insured loss. The risk manager retains more premiums (expected claims) but there is no change in the risk measure.

Suppose instead that $\alpha \geq F(b-)$. In this situation, one prefers to change the deductible compared to the upper limit. This is because $F(d) \leq F(u)$ and so

$$\frac{1}{1 - F(d)} \leq \frac{1}{1 - F(u)}.$$

That is, for a change in the deductible that gives that same change in retained premiums, one gets a smaller increase in the risk measure, when compared to upper limit. For an analytic comparison involving the coinsurance parameter, consider the following

Example. Pareto Distribution. Let Y follow a Pareto distribution with distribution function $F(y) = 1 - \left(\frac{\eta}{y}\right)^\gamma$, where $y \geq \eta$. Then, straightforward calculations show that $\xi_\alpha = \eta(1 - \alpha)^{-1/\gamma}$ and

$$\int_d^u (1 - F(y))dy = \int_d^u \left(\frac{\eta}{y}\right)^\gamma dy = \frac{\eta^\gamma}{\gamma - 1} (u^{-\gamma+1} - d^{-\gamma+1}).$$

To compare changes in deductibles to changes in coinsurance, we use Table 1 to get

$$\frac{\partial_d \xi_{g,\alpha}}{\partial_d \mathbb{E} g(Y; \theta)} = \left(\frac{d}{\eta}\right)^\gamma \geq? \frac{\eta(1 - \alpha)^{-1/\gamma}}{\frac{\eta^\gamma}{\gamma-1} (u^{-\gamma+1} - d^{-\gamma+1})} = \frac{\partial_c \xi_{g,\alpha}}{\partial_c \mathbb{E} g(Y; \theta)}$$

This is equivalent to

$$1 \geq? \frac{(\gamma - 1)\eta(1 - \alpha)^{-1/\gamma}}{d \left\{ \left(\frac{d}{u}\right)^{\gamma-1} - 1 \right\}}$$

which holds for $\gamma > 1$ (the right-hand term is negative). Thus, for this distribution, risk measure, and premium, coinsurance represents the preferred risk mitigation strategy. □

2.3 Extensions of the Basic Single Policy Framework

2.3.1 Retained Claims

To see some other choices of the $g(\cdot)$ function, one can consider the policyholder's viewpoint of retained claims, $Y - g(Y; \theta)$. To this end, use equation (4) to define complementary risk-sharing function

$$g_{RC}(y; d, c, u) = y - g(y; d, c, u) = \begin{cases} y & y < d \\ y - c(y - d) & d \leq y < u \\ y - c(u - d) & y > u \end{cases} . \quad (9)$$

Analogous to equation (5), the distribution function of retained claims is

$$F_{g_{RC}(Y; d, c, u)}(z) = \begin{cases} F(z) & z < d \\ F\left(\frac{z - cd}{1 - c}\right) & d \leq z < (1 - c)u + cd \\ F(z + c(u - d)) & z \geq (1 - c)u + cd \end{cases} . \quad (10)$$

Unlike the insured loss transform, the retained claims transform does not induce discontinuity at the deductible d and upper limit u .

From this, it is straightforward to calculate quantiles of the retained claims distribution. These are

$$\begin{aligned}\xi_{g_{RC},\alpha} &= \begin{cases} \xi_\alpha & \alpha < F(d) \\ \xi_\alpha - c(\xi_\alpha - d) & F(d) \leq \alpha < F(u) \\ \xi_\alpha - c(u - d) & \alpha \geq F(u) \end{cases} \\ &= \xi_\alpha - \xi_{g,\alpha}.\end{aligned}\tag{11}$$

Further, the expected retained claims are simply expected losses minus expected insured losses, that is, $E g_{RC}(Y; \theta) = E Y - E g(Y; \theta)$; recall that expected insured losses are given in equation (7). Thus, the marginal change in expected retained claims is simply negative one times the marginal change in expected insured losses given in Table 1, that is, $\partial_\theta E g_{RC}(Y; \theta) = -\partial_\theta E g(Y; \theta)$, for each parameter θ . Putting this together, the risk measure relative marginal change for retained claims exactly equals risk measure relative marginal change for insured losses.

2.3.2 Indemnification Functions

More generally, it is helpful to consider a generic ‘‘indemnification’’ function $g(\cdot)$. We now only require that $g(y) \leq y$ and that it is monotonically increasing; compare this to the special case in equation (4). Our interest is when $g(\cdot) = g_\theta(\cdot)$ may be a function of one or more parameters θ .

Because of the monotonicity requirement, we have that $\xi_{g,\alpha} = g(\xi_\alpha)$. We can think about the insurer as responsible for insured losses $g(y)$ and the insured as retaining uninsured claims $y - g(y)$. Assume that this function is also monotonic increasing and denote the quantile as $\xi_{g_{RC},\alpha}$. Based on these two observations, we have that

$$\xi_\alpha = g(\xi_{g,\alpha}) + (I - g)(\xi_{g,\alpha}) = \xi_{g,\alpha} + \xi_{g_{RC},\alpha},$$

so equation (11) holds more generally. Further, with the monotonicity, it is straight-forward to compute a marginal change in the value at risk

$$\partial_\theta \xi_{g,\alpha} = \partial_\theta g_\theta(z)|_{z=\xi_\alpha} = g'_\theta(\xi_\alpha),\tag{12}$$

cf., Assa (2015). This gives us another way to interpret the calculations for $\partial_\theta \xi_{g,\alpha}$ in Table 1. Note that the direct calculation of equation (12) is available in instances where the general expression in equation (1) is not. To illustrate, consider $g(\cdot)$ in equation (4) with $d = 0$, $c = 1$ and α such that $F(u) < \alpha < 1$. Then, equation (12) provides an expression for $\partial_\theta \xi_{g,\alpha}$ whereas the conditions are not satisfied for (1) to hold.

2.3.3 Risk Measures

As will be emphasized in the empirical Section 4, the choice of the risk measure matters. So, we seek a theory that encompasses a broad choice of risk measures. Using notation from Serfling (1980) (page 265), we consider general risk measures of the form

$$R(F) = \int_0^1 F^{-1}(t) dK(t) = \int_0^1 J(t)F^{-1}(t)dt + \sum_{j=1}^m a_j F^{-1}(\alpha_j).\tag{13}$$

This is known as a “distortion” risk measure and encompasses many important measures as special cases, cf., [Assa \(2015\)](#).

This general form encompasses many important special cases.

- We have already considered the *VaR* (quantile) risk measure where one chooses $J(t) \equiv 0$, $m = 1$, and $a_1 = 1$.
- The expected value is also a special case of $R(F)$ by using $J(t) \equiv 1$ and $m = 0$.
- The conditional tail expectation is another special case, where one chooses $m = 0$ and $J(t) = 0$ for $0 \leq t < \alpha$ and $J(t) = 1$ for $\alpha \leq t \leq 1$. Here, α ($0 \leq \alpha < 1$) is the threshold level.
- The risk measure from the proportional hazards transform arises when $m = 0$ and $J(t) = \alpha(1 - t)^{\alpha-1}$. Here, $0 < \alpha \leq 1$.

An immediate implication of equations (11) and (13) is that

$$R(F) = R(F_g) + R(F_{gRC}) \tag{14}$$

so that if we choose risk parameters using one measure, implications are clear for the other (assuming that the choice of risk parameters does not affect the loss distribution, a type of moral hazard). Moreover, if the insured and insurer share a single policy and use the same risk measure, their interests conflict. This is because making the risk measure small for one party means making it large for the other.

Return now to the general problem introduced in Section 1.2 where $F_h(x; \theta)$ is the distribution function of the random variable $h(X; \theta)$ with quantile $q_\alpha(\theta)$. We have already thought about discrete components, so consider only the continuous portion of equation (13) and assume that we can interchange differentiation and integration to get

$$\partial_\theta R(F_h) = \int_0^1 J(t) \partial_\theta F^{-1}(t) dt = \int_0^1 J(t) q'_t(\theta) dt.$$

Use equation (1) and a change of variables ($u = q_t(\theta)$) to express this as

$$\partial_\theta R(F_h) = \int_0^1 J(t) \frac{\partial_\theta F_h(q_t(\theta))}{f_h(q_t(\theta))} dt = \int \partial_\theta F_h(u) J(F_h(u)) du.$$

This expression reinforces the idea that determining marginal changes in the distribution function ($\partial_\theta F_h(\cdot)$) represents the main technical difficulty and so is the focus of this paper.

Additional discussion and examples of risk measures can be found in Appendix Section 6.7.

3 Risk Retention in a Portfolio

In contrast to making decisions about each risk in isolation of the others, this section focuses on a collection, or portfolio, of risks. For example, an insurer’s risk manager may be responsible for contracts with several policyholders. Or, an individual may have several policies, such as auto, homeowners, umbrella insurance, and so forth, to protect him or her against insurable risks.

As introduced in Section 1.3, think of a sum S_g of retained claims. Let $g(Y_i; \theta)$ be the retained claims from the i th risk and let $S_{(i)}$ be the sum of retained claims from all other risks. Although all risks may have deductibles, upper limits, and coinsurance, we focus on only changes in the i th risk that has parameter vector θ .

Our goal is to develop RM^2 changes for the portfolio S_g ; that is, we are interested in the changes in the *portfolio* based on changes in risk retention parameters for the i th risk. Because risks are additive, $S_g = g(Y_i; \theta) + S_{(i)}$, and most premium functionals are linear, we will easily be able to analyze changes in premiums by looking at individual contract changes, as in Section 2.2. The more difficult problem is to understand changes in the risk measures. We do this by analyzing the convolution of distribution functions of $g(Y_i; \theta)$ and $S_{(i)}$; Sections 3.1 and 3.2 handle the independent and dependent cases, respectively.

3.1 Portfolio Distribution Based on Independent Risks

For convenience, I begin with a list of assumptions.

1. The i portfolio distribution is continuous with density $f_{S_{(i)}}(s)$.
2. The distribution for the i th risk may be mixed. Specifically, let f_{ki} denote the k th mass point of the Y_i distribution, $k = 1, 2, \dots$ and, when it exists, let $f_i(s)$ be the density function.
3. For the indemnification function $g(y; \theta)$, assume
 - It is potentially bounded, so that $b_1(\theta) = b_1 \leq g(y) \leq b_2 = b_2(\theta)$ for all y . Allow $b_1 = -\infty$ and $b_2 = \infty$ for no bounds. Also use $b'_1 = \partial_\theta b_1(\theta)$ and $b'_2 = \partial_\theta b_2(\theta)$.
 - For all $b_1 < z < b_2$, assume that $g^{-1}(z) = G(\theta, z)$ is uniquely defined.
 - Assume that partial derivatives exist a.e. and use the notation $G_1(\theta, z) = \partial_\theta G(\theta, z)$ and $G_2(\theta, z) = \partial_z G(\theta, z)$.
4. Assume that $F_i(G(\theta, \cdot))$ is differentiable a.e. in θ
5. Assume that $S_{(i)}$ and Y_i are independent random variables.

With these assumptions, we may let A_i denote the set of mass points of the $g(Y_i)$ distribution. This may include b_1 and b_2 , if they are finite. Further, it includes mass points of the Y_i distribution, if they are in the interval (b_1, b_2) . Specifically, define

$$A_{i,g} = \{g_k : g_k = g(f_{ki}), b_1 < g_k < b_2, k = 1, \dots\}$$

and

$$A_i = A_{i,g} \cup \{b_j : -\infty < b_j < \infty, F_{g,i}(b_j) > F_{g,i}(b_j-), j = 1, 2\}.$$

I summarize the main result in the following

Proposition 1. Consider the portfolio random variable S_g with distribution function $F_{S,g,\theta}(t) = \Pr(S_g \leq t)$ and α quantile $F_{S,g,\theta}^{-1}(\alpha) = \xi_{S,g,\alpha}$. Use the above assumptions. Then, the portfolio density, $\partial_t F_{S,g,\theta}(t)$, is

$$f_{S,g}(t) = \sum_{g_k \in A_i} f_{S_{(i)}}(t - g_k) \Pr(g(Y_i) = g_k) + \int_{t-b_2}^{t-b_1} f_i(G(\theta, t-s)) G_2(\theta, t-s) dF_{S_{(i)}}(s). \quad (15)$$

The marginal change in the distribution function, $\partial_\theta F_{S,g,\theta}(t)$, is

$$A_1(t) = \int_{t-b_2}^{t-b_1} f_i(G(t-s)) G_1(\theta, t-s) dF_{S_{(i)}}(s) - F_i(G(b_1)) f_{S_{(i)}}(t-b_1) b'_1 - [1 - F_i(G(b_2))] f_{S_{(i)}}(t-b_2) b'_2. \quad (16)$$

Thus, the portfolio quantile sensitivity is

$$\partial_\theta F_{S,g,\theta}^{-1}(\alpha) = \frac{-A_1(\xi_{S,g,\alpha})}{f_{S,g}(\xi_{S,g,\alpha})}. \quad (17)$$

3.1.1 Portfolio Independent Risks Special Cases

To see how to use Proposition 1, let us consider a number of important special cases.

Special Case 1. Linear Retention. A simple case of interest is $g(y) = cy$, where c is a positive constant. With this choice, $b_1 = -\infty$ and $b_2 = \infty$. Further, $g^{-1}(z) = G(z) = z/c$, $G_1(\theta, z) = -z/c^2$ and $G_2(\theta, z) = 1/c$. Thus, Proposition 1 reduces to

$$\partial_\theta F_{S,g,\theta}^{-1}(\alpha) = \frac{1}{f_{S,g}(\xi_{S,g,\alpha})} \int_{-\infty}^{\infty} f_i\left(\frac{\xi_{S,g,\alpha} - s}{c}\right) \frac{\xi_{S,g,\alpha} - s}{c^2} dF_{S_{(i)}}(s).$$

Use a change of variable $z = (\xi_{S,g,\alpha} - s)/c$ to get

$$\begin{aligned} \partial_\theta F_{S,g,\theta}^{-1}(\alpha) &= \frac{1}{f_{S,g}(\xi_{S,g,\alpha})} \int_{-\infty}^{\infty} f_i(z) \frac{z}{c} f_{S_{(i)}}(\xi_{S,g,\alpha} - cz) (-c) dz \\ &= \mathbf{E}(Y | S_g = \xi_{S,g,\alpha}), \end{aligned}$$

assuming the expectation exists. As described in Section 1, this intuitively appealing result is due to [Gourieroux et al. \(2000\)](#) and illustrates a more general relationship in equation (2). □

Special Case 2. Insurance Retention. The specification of g is given in equation (4), so that $b_1 = 0$ and $b_2 = b = c(u-d)$. Easy calculations show that $g^{-1}(z) = G(z) = \frac{z}{c} + d$ on $b_1 = 0 < z < b = b_2$. Further, $G_2(\theta, z) = \frac{1}{c}$ and

$$G_1(\theta, z) = \partial_\theta G(z) = \partial_\theta \left(\frac{z}{c} + d \right) = \begin{cases} 1 & \theta = d \\ -\frac{z}{c^2} & \theta = c \\ 0 & \theta = u \end{cases}$$

Thus, equations (15) and (16) can be written as

$$f_{S,g}(t) = \frac{1}{c} \int_{t-b}^t f_i(G(t-s)) dF_{S_{(i)}}(s) + F_i(d) f_{S_{(i)}}(t) + [1 - F_i(u)] f_{S_{(i)}}(t-b)$$

and

$$A_1(t) = \int_{t-b}^t f_i(G(t-s)) G_1(\theta, t-s) dF_{S(i)}(s) - [1 - F_i(u)] f_{S(i)}(t-b) b'_2(\theta).$$

We now examine each of the three parameter choices. To begin, for $\theta = d$, we have $b'_2 = \partial_d b_2(\theta) = -c$. After a bit of algebra,

$$\partial_d F_{S,g,\theta}^{-1}(\alpha) = \frac{-A_1(\xi_{S,g,\alpha})}{f_{S,g}(\xi_{S,g,\alpha})} = -c \left(1 - F_i(d) \frac{f_{S(i)}(\xi_{S,g,\alpha})}{f_{S,g}(\xi_{S,g,\alpha})} \right). \quad (18)$$

In the same way, for $\theta = c$, we have

$$\begin{aligned} \partial_c F_{S,g,\theta}^{-1}(\alpha) &= \frac{1}{f_{S,g}(\xi_{S,g,\alpha})} \left\{ (u-d)(1-F_i(u)) f_{S(i)}(\xi_{S,g,\alpha}-b) \right. \\ &\quad \left. + \frac{1}{c^2} \int_{\xi_{S,g,\alpha}-b}^{\xi_{S,g,\alpha}} (\xi_{S,g,\alpha}-s) f_i \left(\frac{\xi_{S,g,\alpha}-s}{c} + d \right) dF_{S(i)}(s) \right\} \end{aligned} \quad (19)$$

and, for $\theta = u$,

$$\partial_u F_{S,g,\theta}^{-1}(\alpha) = c(1 - F_i(u)) \frac{f_{S(i)}(\xi_{S,g,\alpha}-b)}{f_{S,g}(\xi_{S,g,\alpha})}.$$

The marginal changes in expectations are given Table 1, so these results immediately yield our RM^2 coefficients. For the deductible and upper limit, these are

$$\frac{\partial_d F_{S,g,\theta}^{-1}(\alpha)}{\partial_d \mathbb{E} g(Y_i; \theta)} = \frac{1 - F_i(d) \frac{f_{S(i)}(\xi_{S,g,\alpha})}{f_{S,g}(\xi_{S,g,\alpha})}}{1 - F_i(d)} \quad (20)$$

and

$$\frac{\partial_u F_{S,g,\theta}^{-1}(\alpha)}{\partial_u \mathbb{E} g(Y_i; \theta)} = \frac{f_{S(i)}(\xi_{S,g,\alpha}-b)}{f_{S,g}(\xi_{S,g,\alpha})}. \quad (21)$$

The form of RM^2 changes in equations (20) and (21) suggests that, if the i th risk is a small portion of the portfolio, then each ratio will be close to one. This will be seen in our empirical Section 4.

The marginal change for the coinsurance is a bit more complex. With equation (19), we can write

$$\frac{\partial_c F_{S,g,\theta}^{-1}(\alpha)}{\partial_c \mathbb{E} g(Y_i; \theta)} = \frac{\partial_c F_{S,g,\theta}^{-1}(\alpha)}{\int_d^u (1 - F_i(y)) dy}. \quad (22)$$

□

Special Case 3. Call Option. We can also think about y as the return on an investment and examine a type of financial derivative, a call option, with pay-off $g(y) = \max(y - K, 0)$, where K is known as the “strike” price. This can be formulated as a particular instance of Special Case 1, with $K = d$, $c = 1$, and $u = \infty$. With equation (18), this yields,

$$\partial_K F_{S,g,\theta}^{-1}(\alpha) = F_i(K) \frac{f_{S(i)}(\xi_{S,g,\alpha})}{f_{S,g}(\xi_{S,g,\alpha})} - 1,$$

□

Special Case 4. Put Option. For a put option, consider pay-off $g(y) = \max(K - y, 0)$, where K is the strike price. With this, we have $b_1 = 0$ and $b_2 = \infty$. Easy calculations show that $g^{-1}(z) = G(z) = K - z$, so that $G_1(\theta, z) = 1$ and $G_2(\theta, z) = -1$.

Equations (15) and (16) can be written as

$$f_{S,g}(\xi_{S,g,\alpha}) = - \int_{-\infty}^{\xi_{S,g,\alpha}} f_i(G(\xi_{S,g,\alpha} - s)) dF_{S_{(i)}}(s) + F_i(K) f_{S_{(i)}}(\xi_{S,g,\alpha})$$

and

$$\begin{aligned} A_1(t) &= \int_{-\infty}^{\xi_{S,g,\alpha}} f_i(G(\xi_{S,g,\alpha} - s)) dF_{S_{(i)}}(s) \\ &= F_i(K) f_{S_{(i)}}(\xi_{S,g,\alpha}) - f_{S,g}(\xi_{S,g,\alpha}). \end{aligned}$$

Thus,

$$\partial_K F_{S,g,\theta}^{-1}(\alpha) = 1 - F_i(K) \frac{f_{S_{(i)}}(\xi_{S,g,\alpha})}{f_{S,g}(\xi_{S,g,\alpha})},$$

which, not surprisingly, equals a minus one times the call option result.

□

3.2 Portfolio Distribution Based on Dependent Risks

In this section, assume a portfolio of p risks, Y_1, \dots, Y_p , with an as yet unspecified dependence structure. Further define $\mathbf{Y}_{(i)}$ to be the collection of random variables Y_1, \dots, Y_p excluding the i th one and, as above, let $S_{(i)}$ denote their sum.

3.2.1 Portfolio Distribution Function

I summarize the result in the following

Proposition 2. Use the assumptions and notation of Proposition 1, omitting the fifth assumption of independence. Then, the marginal change in the distribution function, $\partial_\theta F_{S,g,\theta}(t)$, is

$$\begin{aligned} A_2(t) &= E_{\mathbf{Y}_{(i)}} \left\{ f_{i|(i)}(G(t - S_{(i)})) G_1(\theta, t - S_{(i)}) I(t - b_2 \leq S_{(i)} < t - b_1) \right\} \\ &\quad - F_i(G(b_1)) f_{S_{(i)}}(t - b_1) b_1'(\theta) - [1 - F_i(G(b_2))] f_{S_{(i)}}(t - b_2) b_2'(\theta). \end{aligned} \quad (23)$$

The marginal change in the quantile has the form

$$\partial_\theta F_{S,g,\theta}^{-1}(\alpha) = \frac{-A_2(\xi_{S,g,\alpha})}{f_{S,g}(\xi_{S,g,\alpha})}. \quad (24)$$

The assumption of independence among risks allows us to write, in the appendix, the distribution function of the portfolio as convolution of two distributions, and then to apply calculus technique to determine changes in the quantiles. The case of dependence is more complex. The proof of the proposition is in Appendix Section 6.2. For a sketch of the approach, Appendix Section 6.2 first demonstrates that the distribution function of the portfolio sum S_g is

$$\Pr(S_g \leq t) = \mathbf{E}_{\mathbf{Y}_{(i)}} \{F_{i|(i)}(G(t - S_{(i)})) I(t - b_2 \leq S_{(i)} \leq t - b_1)\} + F_{S_{(i)}}(t - b_2). \quad (25)$$

Here, $\mathbf{E}_{\mathbf{Y}_{(i)}}$ is the conditional expectation given $\mathbf{Y}_{(i)}$ and $F_{i|(i)}$ is the corresponding distribution function of Y_i given $\mathbf{Y}_{(i)}$. Next, I take derivatives of $\Pr(S_g \leq t)$ with respect to the parameters. For the details, see Appendix Section 6.2 where I demonstrate equation (23) that extends equation (16).

The RM^2 measures in equations (20), (21), and (22) follow in the same way as the case of independent risks.

Special Case: Normal Distribution. To provide additional intuition, suppose that $Y_i \sim N(\mu_i, \sigma_i^2)$, that is, the i th risk is normally distributed with mean μ_i and variance σ_i^2 , $S_{(i)} \sim N(\mu_{(i)}, \sigma_{(i)}^2)$ and that $Cov(Y_i, S_{(i)}) = \rho_i \sigma_i \sigma_{(i)}$. As $\sigma_{(i)}^2 \rightarrow \infty$, it is straight-forward to show (omitted here) that the upper limit and deductible RM^2 changes as in equations (21) and (20) each approach 1.

This is not the case for the coinsurance measure. Although one could use equations (23) and (24), a more direct approach is available if we consider the special case of no deductibles ($d = 0$) and no upper limits ($u = \infty$) so that $g(Y_i) = cY_i$ and $S_g = cY_i + S_{(i)}$. With these assumptions, $S_g \sim N(c\mu_i + \mu_{(i)}, c^2\sigma_i^2 + \sigma_{(i)}^2 + 2c\rho_i\sigma_i\sigma_{(i)})$ and

$$F_{S,g,\theta}^{-1}(\alpha) = c\mu_i + \mu_{(i)} + z_\alpha \sqrt{c^2\sigma_i^2 + \sigma_{(i)}^2 + 2c\rho_i\sigma_i\sigma_{(i)}},$$

where $z_\alpha = \Phi^{-1}(\alpha)$, the α^{th} quantile from the standard normal distribution. From this, $\partial_c \mathbf{E} S_g = \mu_i$ and so

$$\begin{aligned} RM^2 &= \frac{\partial_c F_{S,g,\theta}^{-1}(\alpha)}{\partial_c \mathbf{E} S_g} = 1 + \frac{z_\alpha}{\mu_i} \frac{1}{2} (c^2\sigma_i^2 + \sigma_{(i)}^2 + 2c\rho_i\sigma_i\sigma_{(i)})^{-1/2} (2c\sigma_i^2 + 2\rho_i\sigma_i\sigma_{(i)}) \\ &= 1 + z_\alpha \frac{\sigma_i}{\mu_i} \frac{c\sigma_i + \rho_i\sigma_{(i)}}{\sqrt{c^2\sigma_i^2 + \sigma_{(i)}^2 + 2c\rho_i\sigma_i\sigma_{(i)}}}. \end{aligned} \quad (26)$$

Note that if $\sigma_{(i)}^2 = 0$, then we are in the single policy case and $RM^2 = 1 + z_\alpha \sigma_i / \mu_i$ which is intuitively appealing. As $\sigma_{(i)}^2 \rightarrow \infty$, then $RM^2 \rightarrow 1 + z_\alpha \rho_i \sigma_i / \mu_i$ which can be greater or less than 1 depending on the sign of ρ_i .

Further, it is straight-forward to show (omitted here) that RM^2 in equation (26) is increasing in ρ_i . Intuitively, this means that, other things being equal, more correlated risks require a greater amount of capital for protection against unanticipated uncertainties.

□

3.2.2 Elliptical Copulas

The main difficulty with the marginal change expression in equation (23) is in computing the conditional density $f_{i|(i)}$. To this end, assume a portfolio of p risks, Y_1, \dots, Y_p , whose dependence structure can be quantified using an elliptical copula. Naturally, alternative dependence models can be employed but I find elliptical copulas relatively straightforward to implement and to interpret.

Assume an association matrix Σ and let Σ_{11i} be the matrix Σ after removing the i th row and column. Define Σ_{12i} to be the i th column of Σ , after removing the i th row. Also use $\Sigma_{2\cdot 1i} = 1 - \Sigma'_{12i} \Sigma_{11i}^{-1} \Sigma_{12i}$.

For easy reference, the Appendix Section 6.3 collects details about multivariate elliptical copulas. For example, the conditional density is

$$f_{i|(i)}(y_i) = f_{(Y_i|\mathbf{Y}_{(i)})}(y_i) = f_i(y_i) \frac{g_1(\frac{1}{2}(\nu_i - \mu_{2\cdot \nu i})^2 / \Sigma_{2\cdot 1i})}{g_1(\nu_i^2 / 2)} \frac{1}{\sqrt{\Sigma_{2\cdot 1i}}}, \quad (27)$$

where $\nu_j = H^{-1}(F_j(y_j)) = H^{-1}(F_j)$, $j = 1, \dots, n$, $\boldsymbol{\nu}_{(i)} = (\nu_1, \dots, \nu_{i-1}, \nu_{i+1}, \dots, \nu_p)'$ and

$$\mu_{2\cdot \nu i} = -\boldsymbol{\nu}'_{(i)} \Sigma_{11i}^{-1} \Sigma_{12i}.$$

The functions g_1 and H vary by choice of the copula. For example, for the Gaussian copula, $g_1(z) = e^{-z}$ and H is the standard normal distribution function Φ . More details are in Appendix Section 6.3.

Compared to other copulas, elliptical copulas have the advantage that the conditional distribution is relatively easy to compute for a wide range of associations and that simulation is straightforward.

4 Local Government Property Insurance Fund (LGPIF)

To see the RM^2 measures work in action, consider the building and contents coverage of the Local Government Property Insurance Fund (LGPIF). Here, claims experience from a sample of over 1,000 policyholders was considered over years 2006-2010. Based on available rating factors including coverage and entity type, the Tweedie distribution was fit. From the fitted model, we have available estimated means for each policyholder, as well as shape and scale parameters. For additional background on this fund, see Frees and Lee (2015).

For the purposes of illustration, I assume a deductible equal to the smaller of 5,000 and 20 percent of the mean is in place for each policyholder. I also assume an upper limit at the 95th percentile of the distribution for each policyholder. As noted above, the current coinsurance parameter is not critical for demonstration purposes and so I use $c = 1$, for full participation. Section 4.1 summarizes calculations for a single policyholder. Section 4.2 extends considerations to a portfolio of 75 randomly selected schools that belong to the LGPIF assuming independence and Section 4.3 covers dependent risks.

4.1 Single Policy Risk Retention

To be concrete, I start with one policyholder, the Madison Metropolitan School District, that has an estimated mean $\hat{\mu} = 154,644.70$. With the estimated Tweedie shape parameter

1.670612 and dispersion parameter 164.6253, easy calculations give an estimated distribution function at the deductible $\hat{F}(d) = 0.4881266$ and at the upper coverage limit $\hat{F}(u) = 0.95$. With this distribution and these parameters, the estimated 95th percentile is $\hat{\xi}_{0.95} = 727,320$ and the area under the curve of the survival function $(1 - F(y))$ between the deductible and upper bound is 134,413.9. From this, one can calculate the coinsurance relative marginal risk measure RM^2 to be 5.3738. Table 2 summarizes the calculation for several other RM^2 changes.

Table 2: RM^2 Changes for the Madison Metropolitan School District

Deductible	Coinsurance Parameter – Percentile (α)					Upper Limit
	80	85	90	95	99	
1.7994	1.9346	2.6218	3.6208	5.3738	9.5321	20.000

In the same way, the relative marginal changes in the risk measures for all policyholders were calculated. Figure 2 compares the deductible to the coinsurance marginal change for our portfolio sample. The line $y = x$ is superimposed; a point above this line indicates that the deductible relative marginal change is smaller than the corresponding coinsurance marginal change and so is preferred.

As indicated in Table 1, the coinsurance relative marginal change depends on the risk preference parameter α although the deductible change does not. For interpretation, note that the larger is the risk preference parameter α , the greater is the risk measure ξ_α . This indicates a greater need for capital to protect against unanticipated claims and so is a measure of risk aversion. It is interesting in Figure 2 that, for several policyholders, the most conservative risk managers (with high values of α) will prefer the deductible strategy whereas a corresponding more aggressive risk manager (with lower values of α) will prefer the coinsurance strategy.

□

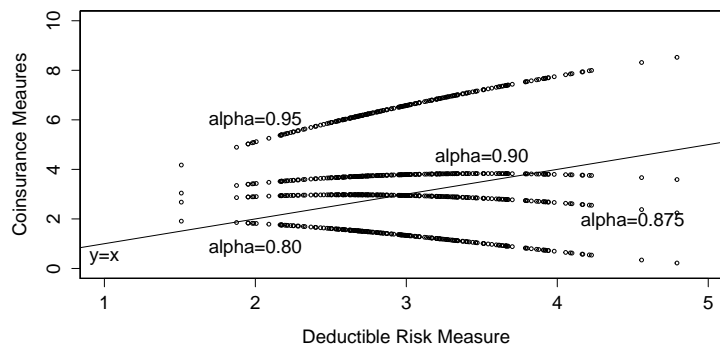


Figure 2: Comparison of Relative Marginal Changes in Risk Measures.

4.2 Portfolio Risk Retention – Independent Risks

Most policies in the LGPIF contain several coverages, such as buildings and contents, motor vehicle coverage (on the fleet of cars owned by the entity), and equipment (such as tractors and park benches). Thus, a policy risk manager may be concerned with a portfolio of $p = 3$ coverages and wish to develop the Section 3 measures.

Instead, I continue to focus on building and contents coverage and take the perspective of an insurer’s risk manager. For a portfolio of 75 randomly selected schools in the LGPIF, I calculated the RM^2 measures using equations (20), (21) and (22). To begin, Table 3 shows these calculations for one risk in the portfolio, the Madison Metropolitan School District. This table shows, for each risk preference level α , that the deductible RM^2 is less than the corresponding coinsurance and upper limit relative risk measures. This table emphasizes that, unlike the single policy measures in Table 2, deductible and upper limit measures vary with the risk preference level.

A comparison of single policy measures in Table 2 and portfolio measures in Table 3 also serves to reinforce the differences between the two approaches. Under the single policy approach, the relative marginal risk measures are the same for both the insured and insurer. In contrast, for the portfolio approach, the insurer typically considers several risks simultaneously and policy changes for each individual risk need to be considered in the context of the rest of the portfolio. Results in Table 3 are not of interest to the risk manager for the Madison Metropolitan School District but they are of interest to the insurer that has the Madison school district as an insured in a larger portfolio.

Table 3: Portfolio RM^2 Changes for the Madison Metropolitan School District

Risk Parameter	Coinsurance Parameter – Percentile (α)				
	80	85	90	95	99
Deductible	1.1872	1.2278	1.2852	1.3607	1.4749
Coinsurance	1.5404	1.7264	1.8901	2.1849	2.8885
Upper Limit	1.8488	2.2536	2.7079	3.4406	5.2799

Given the upper limits and deductibles selected for illustration purposes, it turned out that the relative risk marginal for the upper limit (equation (21)) was greater than that for the deductible (equation (20)). Thus, changing the upper limit is always a worse choice than changing the deductible and so are not be considered further here. Figure 3 compares the coinsurance and deductible measures. For comparison purposes, only a deductible corresponding to $\alpha = 90\%$ is plotted along with several coinsurance measures (the deductible measure exhibited smaller variation in our sample than the coinsurance measures). The line $y = x$ helps assess relative size of the measures. Most observations are above this line suggesting that deductibles represent the preferred risk management strategy.

□

It is interesting to compare the relative marginal risk measures based on a single policy versus the portfolio perspective. Figure 3 provides such a comparison, plotting each RM^2 versus the estimated mean of the risk, $\hat{\mu}_i$. Shown in the figure is coinsurance RM^2 at an $\alpha = 90\%$ level. The left-hand panel exhibits an inverse relationship between RM^2 and

size, as proxied through the estimated mean, whereas the right-hand panel exhibits a strong linear relationship. This figure suggests that decisions based on a single policy measure can be quite different than those based on a portfolio perspective.

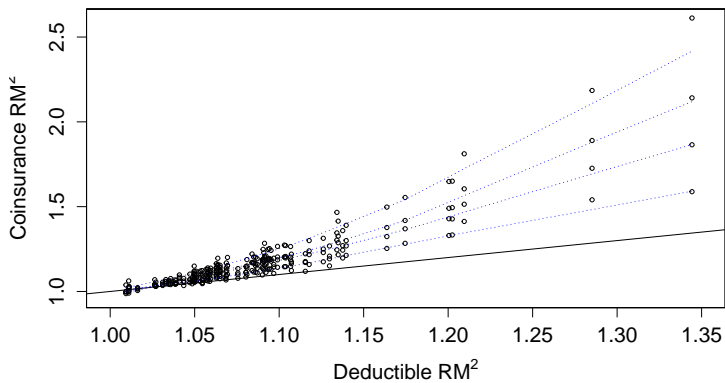


Figure 3: Comparison of Portfolio Relative Marginal Changes in Risk Measures.

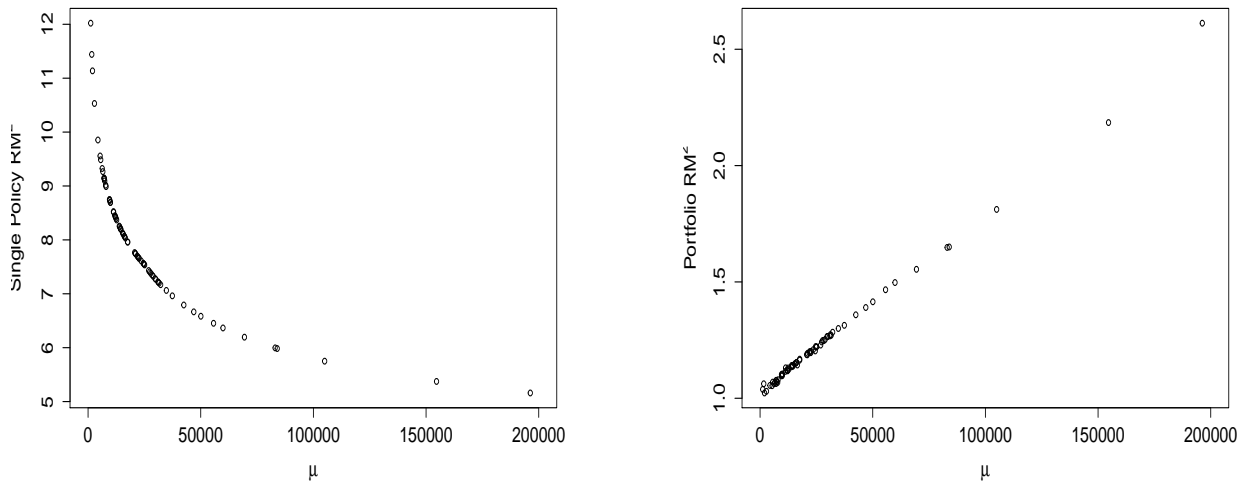


Figure 4: Comparison of Single Policy to Portfolio RM^2 Changes in Risk Measures.

Special Case: Normal Distribution. To help explain the different behaviors of single policy and portfolio RM^2 measures in Figure 4, consider the special case of normally distributed risks, returning to the calculations summarized in equation (26).

Using moments from the Tweedie distribution as a benchmark, we have $\sigma_i^2 = \phi \mu_i^p$ where ϕ is a dispersion parameter. With this, we have $RM^2 = 1 + z_\alpha \sqrt{\phi} \mu_i^{p/2-1}$. For the LGPIF

data, $p \approx 1.67$ resulting in $RM^2 \approx_{LGPIF} 1 + z_\alpha \sqrt{\phi} \mu_i^{-0.165}$. This is consistent with the left-hand panel in Figure 4 in the sense that risks with a larger mean have a smaller value of single policy RM^2 .

Now suppose that $\sigma_{(i)}^2 > 0$ and $\rho_i = 0$, the case of a risk that is independent of the rest of the portfolio. Then,

$$RM^2 = 1 + z_\alpha \frac{\sigma_i}{\mu_i} \frac{c\sigma_i}{\sqrt{Var S_g}} \approx_{LGPIF} 1 + z_\alpha \phi \frac{c\mu_i^{0.67}}{\sqrt{Var S_g}}.$$

This is consistent with the right-hand panel of Figure 4, larger policy policies have larger values of portfolio RM^2 .

□

For this portfolio, the risk manager would focus on the deductible relative marginal risk measure. Figure 5 shows the distribution of RM^2 measures. It is apparent that, for the deductible selected in this illustration, that small size risks are the best for the insurer in the sense that a change in the deductible represents a small change in the required capital, per unit change of premium.

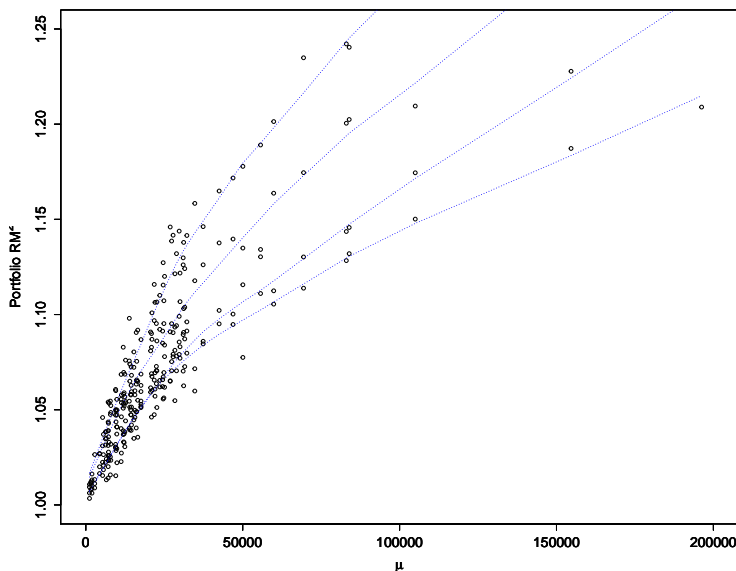


Figure 5: Portfolio Deductible RM^2 Changes by Size $\hat{\mu}$

4.3 Portfolio Risk Retention – Dependent Risks

To assess the effects of dependence in the portfolio of LGPIF insured school districts, in this paper I assume an exchangeable (constant) dependence structure. Specifically, I assume

$\Sigma = (1 - \rho)\mathbf{I} + \rho\mathbf{J}$, where Σ is the association matrix defined in Section 3.2.2, \mathbf{I} is a $p \times p$ identity matrix ($p = 75$ for the LGPIF data), \mathbf{J} is a $p \times p$ matrix of ones, and ρ is the association parameter. This association structure is common in longitudinal data, cf., Frees (2004), where it may be due to a latent variable that is common to all risks, such as statewide funding availability to the schools.

To interpret the effects of dependence, I begin with a typical risk in the portfolio, again focusing on the Madison Metropolitan School District. The results in Table 4 show the dramatic effects of dependence. It is interesting that all three risk measures decrease as the dependence increases. Intuitively, this is because the portfolio risk becomes more concentrated as the dependence increases. Thus, the increase in the amount of assets needed as protection against the risk’s uncertainty, per unit change in premium (expected value), decreases with increasing dependence.

Table 4 also shows the preferred risk management tools switch when dependence is accounted for. In the independence case, the risk manager would seek changes in the deductible, as this results in the smallest increase in the risk measure per unit change in premium. However, even at a moderate level of dependence corresponding to $\rho = 0.2$, the coinsurance, or pro-rata, strategy becomes preferred.

Table 4: Portfolio Dependent Risks RM^2 Changes for the Madison Metropolitan School District

Risk Parameter	Association Parameter – Percentile (ρ)				
	0	0.2	0.4	0.6	0.8
Deductible	1.306	1.220	1.149	1.107	1.046
Coinsurance	2.235	0.727	0.399	0.306	0.294
Upper Limit	3.485	1.655	1.228	1.103	1.095

This phenomena of coinsurance becoming a preferred tool is not restricted to the Madison Metropolitan School District. Figure 6 compares the deductible to coinsurance RM^2 changes for all 75 risks in our portfolio over several hypothesized values of the dependence parameter ρ . The top cluster corresponds to $\rho = 0$; almost all 75 risks are above the line $y = x$, indicating that the deductible RM^2 is less than the coinsurance RM^2 and so is generally preferred.

Other clusters are below the line $y = x$, indicating that the coinsurance is preferred when compared to the deductible strategy. For example, the cluster corresponding to $\rho = 0.2$ is closest to the line and the cluster corresponding to $\rho = 0.4$ is just below that.

The form and strength of dependence among risks with the LGPIF is an empirical question. It is not unreasonable to posit dependence not only according to entity type (e.g., school versus court house) but also over spatial/geographic differences and over time. The point of this demonstration is to show that dependence can not only have an important quantitative impact on the risk measures but qualitative as well, we have seen instances where the preferred risk strategy switches from deductible to coinsurance with a changing dependence.

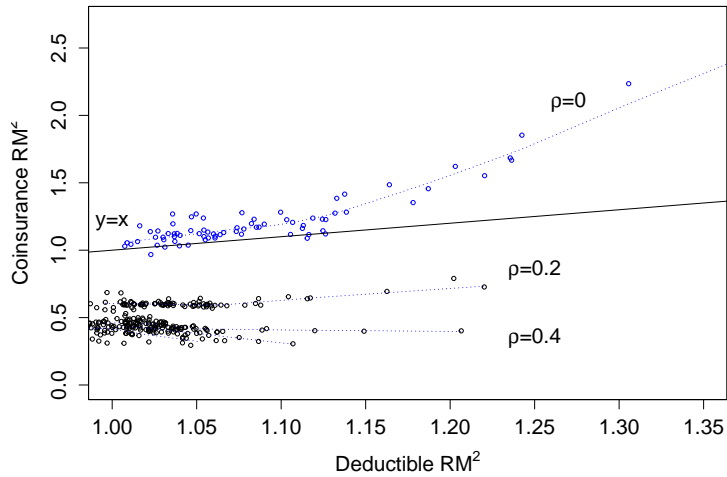


Figure 6: Portfolio RM^2 Changes for Dependent Risks. Coinsurance Changes Compared to Deductible Changes by Several Association Parameters ρ .

5 Summary and Concluding Remarks

For the manager of a single risk, this paper provides a new tool for deciding the most cost-effective way of insuring a risk. I have focused on a choice among a deductible, coinsurance, and upper policy limits because these represent widely used parameters of risk-sharing agreements.

The approach assumes knowledge of the risk's distribution. Although this may not be available to an individual policyholder, it can be easily estimated by an insurer based on a collection of comparable risks. Restricting attention to single risks, the RM^2 measure is the same between the insured and insurer so that it is easy for an insurer to calibrate RM^2 measures for policyholders.

The RM^2 approach focuses on local changes in premiums and in risk measures. Using local changes, this paper shows how to accommodate “background risks” that I interpret to be other risks in the portfolio. These background risks may include other coverages (different risks), even in the case of individual policyholder, or other policies, especially for the insurer that may have a collection of policies. Naturally, an insurer may have a combination, representing a collection of policies with multiple coverages.

In the both the single policy and the portfolio context, the RM^2 measure is a tool that a risk manager can use to decide on the most risk effective strategy for managing a risk (e.g., the choice of deductible, coinsurance, or upper limit). Further, the portfolio manager may identify which risk is the best (and worst) in sense of requiring the least amount of necessary capital per unit increase in premium. Risks are naturally allowed to be heterogenous (for example, in a regression context). Further, risks may depend on one another. This paper provides explicit tools for modeling risks via elliptical copulas although the framework certainly applies to other dependency structures.

The paper has focused on the quantile, or Value at Risk, for two reasons. First, this risk measure is being used extensively in the banking and insurance industries. Second, as seen in section 2.3.3, it is a foundation for a broad class of risk measures. The work in this paper can be readily extended to this broader class.

This paper has focused on insurance applications because portfolio selection methods in finance rely heavily on concentrating our systematic risks. There may also be applications in finance where the RM^2 approach is useful but the insurance context seems like a natural arena for introducing the approach of this paper.

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6 Appendices

6.1 Proof of Proposition 1

In the following, I give a proof of the proposition based on straight-forward, yet complicated, calculus. Let

$$F_{g,i}(t) = \Pr(g(Y_i) \leq t) = F_i(g^{-1}(t)) = F_i(G(\theta, t))$$

be the distribution function of $g(Y_i)$. When it exists, the density function is

$$f_{g,i}(t) = \partial_t F_i(G(\theta, t)) = f_i(G(\theta, t))G_2(\theta, t).$$

The distribution function of the portfolio is

$$\begin{aligned} \Pr(S_g \leq t) &= \Pr(S_{(i)} + g(Y_i) \leq t) = \int F_{S_{(i)}}(t - z) dF_{g,i}(z) \\ &= \sum_{g_k \in A_i} F_{S_{(i)}}(t - g_k) \Pr(g(Y_i) = g_k) + \int_{b_1}^{b_2} F_{S_{(i)}}(t - z) f_{g,i}(z) dz. \end{aligned}$$

With this, the density function of the portfolio is

$$\begin{aligned}
f_{S,g}(t) &= \partial_t \Pr(S_g \leq t) \\
&= \sum_{g_k \in A_i} f_{S_{(i)}}(t - g_k) \Pr(g(Y_i) = g_k) + \int_{b_1}^{b_2} f_{S_{(i)}}(t - z) f_{g,i}(z) dz \\
&= \sum_{g_k \in A_i} f_{S_{(i)}}(t - g_k) \Pr(g(Y_i) = g_k) + \int_{t-b_2}^{t-b_1} f_{g,i}(t - s) f_{S_{(i)}}(s) ds \\
&= \sum_{g_k \in A_i} f_{S_{(i)}}(t - g_k) \Pr(g(Y_i) = g_k) + \int_{t-b_2}^{t-b_1} f_i(G(\theta, t - s)) G_2(\theta, t - s) dF_{S_{(i)}}(s).
\end{aligned}$$

This establishes equation (15).

Another way to write the distribution function of the portfolio is

$$\begin{aligned}
\Pr(S_g \leq t) &= \Pr(g(Y_i) + S_{(i)} \leq t) \\
&= \int_{-\infty}^{t-b_2} F_{i,g(Y_i;\theta)}(t - s) dF_{S_{(i)}}(s) + \int_{t-b_2}^{t-b_1} F_{i,g(Y_i;\theta)}(t - s) dF_{S_{(i)}}(s) \\
&\quad + \int_{t-b_1}^{\infty} F_{i,g(Y_i;\theta)}(t - s) dF_{S_{(i)}}(s) \\
&= \Pr(S_{(i)} \leq t - b_2) + \int_{t-b_2}^{t-b_1} F_{i,g(Y_i;\theta)}(t - s) dF_{S_{(i)}}(s) + 0 \\
&= \Pr(S_{(i)} \leq t - b_2) + \int_{t-b_2}^{t-b_1} F_i(G(t - s)) dF_{S_{(i)}}(s). \tag{28}
\end{aligned}$$

Using the Leibniz rule, take derivatives with respect to θ to get

$$\begin{aligned}
A_1(t) &= \partial_\theta \Pr(S_g \leq t) \\
&= \partial_\theta F_{S_{(i)}}(t - b_2) + \partial_\theta \left(\int_{t-b_2}^{t-b_1} F_i(G(t - s)) dF_{S_{(i)}}(s) \right) \\
&= f_{S_{(i)}}(t - b_2) \partial_\theta(t - b_2) + \int_{t-b_2}^{t-b_1} f_i(G(t - s)) \partial_\theta G(\theta, t - s) dF_{S_{(i)}}(s) \\
&\quad + F_i(G(b_1)) f_{S_{(i)}}(t - b_1) \partial_\theta(t - b_1) - F_i(G(b_2)) f_{S_{(i)}}(t - b_2) \partial_\theta(t - b_2) \\
&= -f_{S_{(i)}}(t - b_2) b_2' + \int_{t-b_2}^{t-b_1} f_i(G(t - s)) G_1(\theta, t - s) dF_{S_{(i)}}(s) \\
&\quad - F_i(G(b_1)) f_{S_{(i)}}(t - b_1) b_1' + F_i(G(b_2)) f_{S_{(i)}}(t - b_2) b_2'.
\end{aligned}$$

This establishes equation (16).

□

6.2 Proof of Proposition 2

To prove Proposition 2, we first need the distribution function of the portfolio sum S_g . As in equation (28),

$$\begin{aligned}
\Pr(S_g \leq t) &= \Pr(g(Y_i) + S_{(i)} \leq t) = \mathbf{E}_{\mathbf{Y}_{(i)}} \Pr(g(Y_i) \leq t - S_{(i)} | \mathbf{Y}_{(i)}) \\
&= \mathbf{E}_{\mathbf{Y}_{(i)}} \left\{ \Pr(g(Y_i) \leq t - S_{(i)}, t - b_2 \leq S_{(i)} \leq t - b_1 | \mathbf{Y}_{(i)}) \right. \\
&\quad \left. + \Pr(g(Y_i) \leq t - S_{(i)}, S_{(i)} \leq t - b_2 | \mathbf{Y}_{(i)}) \right\} \\
&= \mathbf{E}_{\mathbf{Y}_{(i)}} \left\{ F_{i|(i)}(G(t - S_{(i)})) I(t - b_2 \leq S_{(i)} \leq t - b_1) \right\} + F_{S_{(i)}}(t - b_2).
\end{aligned}$$

thus establishing equation (25).

I now take derivatives of $\Pr(S_g \leq t)$ with respect to θ . For the second term on the right-hand side, we have

$$\partial_\theta F_{S_{(i)}}(t - b_2) = -f_{S_{(i)}}(t - b_2)b'_2.$$

Assume sufficient conditions hold to interchange the derivative and the expectation. I will evaluate

$$\begin{aligned}
&\partial_\theta \mathbf{E}_{\mathbf{Y}_{(i)}} \left\{ F_{i|(i)}(G(t - S_{(i)})) I(t - b_2 \leq S_{(i)} \leq t - b_1) \right\} \\
&= \partial_\theta \mathbf{E}_{\mathbf{Y}_{(i)}} \left\{ F_{i|(i)}(G(t - S_{(i)})) I(S_{(i)} \leq t - b_1) \right\} \\
&\quad - \partial_\theta \mathbf{E}_{\mathbf{Y}_{(i)}} \left\{ F_{i|(i)}(G(t - S_{(i)})) I(S_{(i)} \leq t - b_2) \right\}.
\end{aligned}$$

Then, using the chain rule and the Leibniz rule,

$$\begin{aligned}
&\partial_\theta \mathbf{E}_{\mathbf{Y}_{(i)}} \left\{ F_{i|(i)}(G(t - S_{(i)})) I(S_{(i)} \leq t - b_1) \right\} \\
&= \partial_\theta \int \cdots \int \left\{ F_{i|(i)}(G(\theta, t - S_{(i)})) I(S_{(i)} \leq t - b_1) \right\} \prod_{j \neq i} dF_{Y_j}(Y_j) \\
&= \int \cdots \int \left\{ \partial_\theta F_{i|(i)}(G(\theta, t - S_{(i)})) \right\} I(S_{(i)} \leq t - b_1) \prod_{j \neq i} dF_{Y_j}(Y_j) \\
&\quad + F_{i|(i)}(G(\theta, t - (t - b_1))) \left\{ \partial_\theta \int \cdots \int I(S_{(i)} \leq t - b_1) \prod_{j \neq i} dF_{Y_j}(Y_j) \right\} \\
&= \int \cdots \int \left\{ f_{i|(i)}(G(\theta, t - S_{(i)})) G_1(\theta, t - S_{(i)}) \right\} I(S_{(i)} \leq t - b_1) \prod_{j \neq i} dF_{Y_j}(Y_j) \\
&\quad + F_{i|(i)}(d) \left\{ \partial_\theta F_{S_{(i)}}(t - b_1) \right\} \\
&= \int \cdots \int f_{i|(i)}(G(\theta, t - S_{(i)})) G_1(\theta, t - S_{(i)}) I(S_{(i)} \leq t - b_1) \prod_{j \neq i} dF_{Y_j}(Y_j) \\
&\quad - F_{i|(i)}(d) f_{S_{(i)}}(t - b_1) b'_1(\theta).
\end{aligned}$$

Thus,

$$\begin{aligned} & \partial_{\theta} \mathbf{E}_{\mathbf{Y}_{(i)}} \left\{ F_{i|(i)}(G(t - S_{(i)})) I(t - b_2 \leq S_{(i)} \leq t - b_1) \right\} \\ = & \int \cdots \int f_{i|(i)}(G(t - S_{(i)})) G_1(\theta, t - S_{(i)}) I(t - b_2 \leq S_{(i)} \leq t - b_1) \prod_{j \neq i} dF_{Y_j}(Y_j) \\ & + F_{i|(i)}(u) f_{S_{(i)}}(t - b_2) b_2'(\theta) - F_{i|(i)}(d) f_{S_{(i)}}(t - b_1) b_1'(\theta). \end{aligned}$$

Note that $\mathbf{E}_{\mathbf{Y}_{(i)}} F_{i|(i)}(z) = \Pr(Y_i \leq z) = F_i(z)$ for $z = d, u$. This is sufficient to demonstrate equation (23). □

6.3 Properties of Multivariate Elliptical Copulas

This appendix collects properties about elliptical distributions useful for regression modeling. Foundational references include [Cambanis et al. \(1981\)](#) and [Fang et al. \(1990\)](#). A similar collection appears in [Frees and Wang \(2006\)](#).

Multivariate Elliptical Distributions

Assume that a random d -dimensional vector \mathbf{Z}_c has a multivariate elliptical distribution, written as $\mathbf{Z}_c \sim E_d(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$, with density function

$$h_{\mathbf{Z}_c}(\mathbf{z}) = \frac{c_d}{\sqrt{\det(\boldsymbol{\Sigma})}} g_d \left(\frac{1}{2} (\mathbf{z} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{z} - \boldsymbol{\mu}) \right) = h(\mathbf{z}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, d). \quad (29)$$

The function $g_d(\cdot)$ should satisfy $\int_0^\infty x^{d/2-1} g_d(x) dx < \infty$ and is known as the density generator function. From [Landsman and Valdez \(2003\)](#), special cases of interest in insurance work include:

Multivariate Distribution	Density $g_d(x)$
Normal distribution	e^{-x}
t -distribution with r degrees of freedom	$(1 + 2x/r)^{-(d+r)/2}$
Cauchy	$(1 + 2x)^{-(d+1)/2}$
Logistic	$e^{-x}/(1 + e^{-x})^2$
Exponential power	$\exp(-rx^s)$

Now suppose $p = p_1 + p_2$ and that we partition \mathbf{Z}_c , $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ such that

$$\mathbf{Z}_c = \begin{pmatrix} \mathbf{Z}_{c1} \\ \mathbf{Z}_{c2} \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

An important property of elliptical distributions is that the marginal distributions are from the same family as the parent joint distribution (see, for example, [Fang et al. \(1990\)](#), or [Hult and Lindskog \(2002\)](#)). Specifically, we have that $\mathbf{Z}_{c1} \sim E_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}, \psi)$ is a $p_1 \times 1$ vector,

and similarly for \mathbf{Z}_{c2} . Further, under mild regularity conditions, from Cambanis, Huang and Simons (1981, Corollary 5), we have that $\mathbf{Z}_{c2}|\mathbf{Z}_{c1} = \mathbf{z}_1 \sim E_{p2}(\boldsymbol{\mu}_{2.1}, \boldsymbol{\Sigma}_{2.1}, \psi)$, where

$$\boldsymbol{\mu}_{2.1} = \boldsymbol{\mu}_2 - (\mathbf{z}_1 - \boldsymbol{\mu}_1)' \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12} \quad \text{and} \quad \boldsymbol{\Sigma}_{2.1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12}.$$

Thus, from equation (29), we may write the conditional density as

$$\begin{aligned} h_{\mathbf{Z}_{c2}|\mathbf{Z}_{c1}=\mathbf{z}_1}(\mathbf{z}) &= \frac{c_{p2}}{\sqrt{\det(\boldsymbol{\Sigma}_{2.1})}} g_{p2} \left(\frac{1}{2} (\mathbf{z} - \boldsymbol{\mu}_{2.1})' \boldsymbol{\Sigma}_{2.1}^{-1} (\mathbf{z} - \boldsymbol{\mu}_{2.1}) \right) \\ &= h(\mathbf{z}; \boldsymbol{\mu}_{2.1}, \boldsymbol{\Sigma}_{2.1}, p2). \end{aligned} \quad (30)$$

Elliptical Copulas

We are now ready to define the elliptical copula, a function defined for all $(u_1, u_2, \dots, u_p)'$, by

$$C(u_1, \dots, u_p) = H_{\mathbf{Z}_c}(H^{-1}(u_1), \dots, H^{-1}(u_p)).$$

Because copulas are concerned primarily with relationships, we may restrict our considerations to the case where $\boldsymbol{\mu} = \mathbf{0}$ and diagonal elements of $\boldsymbol{\Sigma}$ are 1. Thus, the marginal distributions are identical so that $H_j = H$ with density $h(z) = c_1 g_1(z^2/2)$.

The corresponding probability density function

$$c(u_1, \dots, u_p) = h(H^{-1}(u_1), \dots, H^{-1}(u_p); \boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\Sigma}, p) \prod_{j=1}^p \frac{1}{h(H^{-1}(u_j))} \quad (31)$$

can be evaluated using equation (29).

The conditional density function

$$\begin{aligned} c(u_{p_1+1}, \dots, u_p | u_1, \dots, u_{p_1}) &= \frac{c(u_1, \dots, u_p)}{c(u_1, \dots, u_{p_1})} \\ &= \frac{h(H^{-1}(u_1), \dots, H^{-1}(u_p); \boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\Sigma}, p)}{h(H^{-1}(u_1), \dots, H^{-1}(u_{p_1}); \boldsymbol{\mu}_1 = \mathbf{0}, \boldsymbol{\Sigma}_1, p_1)} \prod_{j=p_1+1}^p \frac{1}{h(H^{-1}(u_j))} \\ &= h(H^{-1}(u_{p_1+1}), \dots, H^{-1}(u_p)); \boldsymbol{\mu}_{2.u}, \boldsymbol{\Sigma}_{2.1}, p2) \prod_{j=p_1+1}^p \frac{1}{h(H^{-1}(u_j))} \end{aligned} \quad (32)$$

can be evaluated as in equation (30) with

$$\boldsymbol{\mu}_{2.u} = -(H^{-1}(u_1), \dots, H^{-1}(u_{p_1})) \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12}.$$

Conditional Distributions

Consider the set of random variables $\mathbf{Z} = (Z_1, \dots, Z_p)'$ with marginal distribution functions $F_j(z_j) = \Pr(Z_j \leq z_j) = F_j$. Here, we use $\mathbf{z} = (z_1, \dots, z_p)'$ for the corresponding realizations of \mathbf{Z} . The joint distribution function of \mathbf{Z} can be expressed as a function of the marginal

distributions through the copula function $F(z_1, \dots, z_p) = C(F_1, \dots, F_p)$, where C is an elliptical copula.

Assuming continuity, the joint density function is given by

$$f(z_1, \dots, z_p) = c(F_1, \dots, F_p) \prod_{j=1}^p f_j,$$

where $f_j(z_j) = f_j$ is the corresponding marginal density function.

Now partition the $p \times 1$ random vector

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix}$$

where \mathbf{Z}_1 and \mathbf{Z}_2 have dimensions $p_1 \times 1$ and $p_2 \times 1$, respectively (recall, $p_1 + p_2 = p$).

Thus, using equations (32) and (33), the predictive distribution is

$$\begin{aligned} f(z_{p_1+1}, \dots, z_p | z_1, \dots, z_{p_1}) &= \frac{f(z_1, \dots, z_p)}{f(z_1, \dots, z_{p_1})} = \frac{c(F_1, \dots, F_p)}{c(F_1, \dots, F_{p_1})} \prod_{j=p_1+1}^p f_j \\ &= \frac{h(H^{-1}(F_1), \dots, H^{-1}(F_p); \boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\Sigma}, p)}{h(H^{-1}(F_1), \dots, H^{-1}(F_{p_1}); \boldsymbol{\mu}_1 = \mathbf{0}, \boldsymbol{\Sigma}_1, p_1)} \prod_{j=p_1+1}^p \frac{f_j}{h(H^{-1}(F_j))} \\ &= \frac{h(\nu_1, \dots, \nu_p; \boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\Sigma}, p)}{h(\nu_1, \dots, \nu_{p_1}; \boldsymbol{\mu}_1 = \mathbf{0}, \boldsymbol{\Sigma}_1, p_1)} \prod_{j=p_1+1}^p \frac{f_j}{h(\nu_j)} \\ &= h(\nu_{p_1+1}, \dots, \nu_p; \boldsymbol{\mu}_{2 \cdot \nu}, \boldsymbol{\Sigma}_{2 \cdot 1}, p_2) \prod_{j=p_1+1}^p \frac{f_j}{h(\nu_j)}. \end{aligned}$$

where $\nu_j = H^{-1}(F_j(z_j)) = H^{-1}(F_j)$, $j = 1, \dots, p$ and

$$\boldsymbol{\mu}_{2 \cdot \nu} = -(\nu_1, \dots, \nu_{p_1}) \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12}.$$

In particular, with $p_1 = p - 1$ and equation (29), we have

$$\begin{aligned} f(z_p | z_1, \dots, z_{p-1}) &= f_p(z_p) \frac{h(\nu_p; \boldsymbol{\mu}_{2 \cdot \nu}, \boldsymbol{\Sigma}_{2 \cdot 1}, 1)}{h(\nu_p)} \\ &= f_p(z_p) \frac{g_1(\frac{1}{2}(\nu_p - \boldsymbol{\mu}_{2 \cdot \nu})^2 / \boldsymbol{\Sigma}_{2 \cdot 1})}{g_1(\nu_p^2 / 2)} \frac{1}{\sqrt{\boldsymbol{\Sigma}_{2 \cdot 1}}} \end{aligned} \quad (33)$$

6.4 Insurance Economics Framework

There is a long history of determining risk retention parameters in the insurance economics literature and so I first review this framework. Using this approach, an optimal choice of risk retention parameters can be determined through expected utility. To illustrate, a classic problem in insurance economics is to find the value of c to maximize expected utility,

$$\text{E } u(w_o - (1 - c)y - (1 + \lambda)\text{E } cy), \quad (34)$$

where $u(\cdot)$ denotes an insured’s utility function and w_0 represents initial wealth. Under mild conditions, the result of Mossin in 1968 indicates that full coverage corresponding to $c = 1$ is optimal when insurance is fair, $\lambda = 0$. When a loading is in place, $\lambda > 0$, then a partial insurance coverage corresponding to a value of $c < 1$ is optimal, cf., [Schlesinger \(2013\)](#). Schlesinger also notes that this insurance result has a portfolio interpretation. One can define $A = w_0 - (1 + \lambda)E y$ to be a non-risky asset and the weighted average, $Y(c) = (1 - c)A + c(A + y)$, to be a combination of a non-risky and risky asset y . In this sense, the insurance choice problem is equivalent to the portfolio problem in financial economics. As another example, the optimality of deductible policies ($d > 0$) was first established by Arrow in 1974. See [Gollier \(2013\)](#), for a description of this and related results.

A set of related results involve the notion of a “background risk,” where the initial wealth in equation (34) is no longer treated as a known constant but is itself a random variable. In the portfolio context, this has a natural interpretation where $Y(c) = (1 - c)A + c(A + y)$ represents a combination of a two risky assets. Typically, results on optimal coverage assume independence between two risky assets. See the review article of [Schlesinger \(2013\)](#) for more background and additional references.

The insurance economics approach to selection of risk retention parameters is useful but, by design, does rely on individual-specific preferences. As noted by [Schlesinger \(2013\)](#) (page 167), “... whereas most financial assets are readily tradeable and have a risk that relates to the marketplace, insurance is a contract contingent on an individual’s personal wealth. This personal nature is what distinguishes it from other financial assets.” In contrast, the Section 4 empirical portion of this paper considers instances of insurance traded from one company to another, where wealth and risk preferences play a smaller role. Moreover, we seek to understand measures that summarize uncertainty in the tails of the distribution that the supply and demand orientation of expected utility is not well suited to handle, cf., [Powers and Zanjani \(2013\)](#) for additional discussion of insurance economics approaches in comparison to risk measures, the next topic.

6.5 Risk Measures Framework

Following the lead from the banking and insurance industries, I focus on quantifying risk retention using risk measures. A risk measure is essentially a functional of the insurance loss distribution function F . To be specific, in equation (13) that follows, this article emphasizes risk measures of the form $R(F) = \int_0^1 F^{-1}(t) dK(t)$, where $K(\cdot)$ is some appropriately chosen weight function. An important special case is the value at risk, denoted as VaR , which is simply the α quantile for a specific value of α in $(0, 1)$.

There are a variety of properties and axioms that a well-designed risk measure can satisfy, depending on the purpose of the measure. Risk measures are primarily used for two purposes, premium determination and capital budgeting. Classical applications of risk measures in insurance are for the former, cf., [Young \(2004\)](#). The banking industry focuses on the use of risk measures for capital budgeting and allocation purposes, cf., [Embrechts et al. \(2005\)](#) and [Embrechts and Hofert \(2013\)](#).

As with utility theoretic approaches, there is a long, although less well-known, literature on the choice of optimal risk retention parameters beginning with the work of Borch in 1960; see, for example, [Assa \(2015\)](#) for a recent overview. As described in Assa, in insurance

one worries about optimal decisions from both the insured and the insurer’s viewpoint. However, suggesting the use of risk transfers using risk measures such as VaR is not without controversy. See, for example, [Guerra and Centeno \(2012\)](#) for a discussion of potential drawbacks of this approach.

Much of the work done in insurance is for sharing risk between an insurer and a re-insurer, an “insurer of insurers.” As such, there is a focus on optimal arrangements of a single policy, the subject of our Section 2.

Under restrictive assumptions, both the utility theoretic and the risk measure approaches reduce to thinking about a premium as an expected value and measuring the uncertainty through the variance. This is the classic mean-variance trade-off of the Markowitz model of portfolios in finance. This is the approach currently taken by actuarial textbooks; for example, in this context, [Gray and Pitts \(2012\)](#) (Chapter 5) compares risk-sharing arrangements involving deductibles, coinsurance, and upper limits. This paper provides a tool to make such comparisons using presumably more appropriate measures of risk.

6.6 Notes on Monte-Carlo Evaluation

Evaluation of the risk measure relative marginals in equations (20), (21), and (22), can be readily accomplished via Monte-Carlo. The strategy is to first generate a replication of the portfolio of losses Y_1, \dots, Y_p and from this, compute the sum, S_g and the sum, $S_{(i)}$, that excludes the i th loss Y_i . Then, from many replications of S_g , one computes an approximation of the quantile $\xi_{S_g, \alpha}$ and the densities evaluated at this quantile, $f_{S_g}(\xi_{S_g, \alpha})$ and $f_{S_{(i)}}(\xi_{S_g, \alpha})$, as well as the density $f_{S_{(i)}}(\xi_{S_g, \alpha} - b)$. From replications of Y_i , it is straight-forward to evaluate the distribution function at the deductible and upper limit, $F_i(d)$ and $F_i(u)$. Slightly more complicated is the numerical integration required for the integral in denominator of equation (22) but, as this regards a single policy, it is readily achievable.

The most complicated term is the integral in equation (19). For this term, one strategy is to simulate many replications of the sum $S_{(i)}$ and then use this distribution for a Monte-Carlo evaluation of the integral. This is the approach taken in this paper.

6.7 Notes on Risk Measures

To see how the Section 2.3.3 introduction works in the simpler case of retained claims, use equations (11) and (13) to write

$$\begin{aligned}
 R(F_{g_{RC}}) &= \int_0^1 J(t) F_{g_{RC}}^{-1}(t) dt + \sum_{j=1}^m a_j F_{g_{RC}}^{-1}(\alpha_j) \\
 &= R(F) - c \int_{F(d)}^{F(u)} J(t) (\xi_t - d) dt - c(u - d) \int_{F(u)}^1 J(t) dt \\
 &\quad - c \sum_{j=1}^m a_j \{ (\xi_{\alpha_j} - d) I(F(d) \leq \alpha_j < F(u)) + (u - d) I(\alpha_j \geq F(u)) \}
 \end{aligned}$$

For the conditional tail expectation, we have

$$\begin{aligned} R(F_{g_{RC}}) &= R(F) - c \int_{F(d)}^{F(u)} I(t \geq \alpha) (\xi_t - d) dt - c(u - d) \int_{F(u)}^1 I(t \geq \alpha) dt \\ &= \begin{cases} R(F) - c(u - d)(1 - F(u)) - c \int_{\alpha}^{F(u)} (\xi_t - d) dt & \alpha < F(u) \\ R(F) - c(u - d)(1 - \alpha) & \alpha \geq F(u) \end{cases}. \end{aligned}$$

Under mild regularity conditions (allowing the interchange of the integral and derivative), use equations (12) and (13) to get the marginal change in the risk measure

$$\partial_{\theta} R(F) = \int_0^1 g'_{\theta}(\xi_t) dK(t), \quad (35)$$

cf., [Assa \(2015\)](#).

Special Case: Conditional Tail Expectation. For an explicit example, consider the conditional tail expectation so that the risk measure is $R(F) = \int_{\alpha}^1 \xi_t dt$. Using equation (4), for the deductible we have $\partial_d g(\xi_t) = -cI(t \geq F(d))$. This yields

$$\partial_d R(F) = -c(1 - \max(\alpha, F(d))).$$

In the same way, for the upper limit, we have

$$\partial_u R(F) = c(1 - \max(\alpha, F(u))).$$

The case of coinsurance is a bit more complex. Using equation (4), we have

$$\partial_c g(\xi_t) = \begin{cases} 0 & t < F(d) \\ \xi_t - d & F(d) \leq t < F(u) \\ u - d & t \geq F(u) \end{cases}$$

Thus,

$$\partial_c R(F) = \begin{cases} \int_{F(d)}^{F(u)} (\xi_t - d) dt + (1 - F(u))(u - d) & \alpha < F(d) \\ \int_{\max(\alpha, F(d))}^{F(u)} (\xi_t - d) dt + (1 - F(u))(u - d) & F(d) \leq \alpha < F(u) \\ (1 - \alpha)(u - d) & \alpha \geq F(u) \end{cases}$$

□

Special Case: Conditional Tail Expectation and the Pareto Distribution. With the Pareto distribution, recall that $F(y) = 1 - \left(\frac{y}{a}\right)^{\gamma}$ and $\xi_t = \eta(1 - t)^{-1/\gamma}$. With this, it is easy to compute $\partial_d R(F)$ and $\partial_u R(F)$. Further, with

$$\int_a^b \xi_t dt = \eta \int_a^b (1 - t)^{-1/\gamma} dt = \frac{\eta\gamma}{\gamma - 1} \left\{ (1 - a)^{(\gamma-1)/\gamma} - (1 - b)^{(\gamma-1)/\gamma} \right\},$$

we have for $F(d) \leq \alpha < F(u)$

$$\begin{aligned} \partial_c R(F) &= \int_{\max(\alpha, F(d))}^{F(u)} (\xi_t - d) dt + (1 - F(u))(u - d) \\ &= \frac{\eta\gamma}{\gamma - 1} \left\{ (1 - \max[\alpha, F(d)])^{(\gamma-1)/\gamma} - (1 - F(u))^{(\gamma-1)/\gamma} \right\} \\ &\quad - d(F(u) - \max[\alpha, F(d)]) + (1 - F(u))(u - d) \end{aligned}$$

which can be readily computed. Other cases follow in the same way. □

6.8 Special Case: Mixture of Exponentials

It is difficult to get analytic results because there are few simple distributions that can represent sums of non-identical random variables, even in the case of independence. So, one could look to identical distributions or investigate the “usual suspects,” including the normal and Poisson distributions. Here, I investigate mixtures of exponentials, a flexible form that allows for closed-form expressions. However, as will be seen, there are serious limitations to this assumption. Most applications will use numerical integration, as will be described in Section 4.

Assume that the density of the i th risk can be described by a mixture of exponentials of the form

$$f_i(t) = \sum_{j=1}^{m_i} a_{ij} \lambda_{ij} \exp(-\lambda_{ij} t)$$

where there are m_i exponential distributions, each with parameter λ_{ij} and $\{a_{ij}\}$ are non-negative parameters that describe the proportion of mixing such that $a_{i1} + \dots + a_{iM_i} = 1$.

In the same way, the rest of the portfolio, $S_{(i)}$, has density

$$f_{S_{(i)}}(t) = \sum_{j=1}^{m_{(i)}} a_{(i)j} \lambda_{(i)j} \exp(-\lambda_{(i)j} t). \tag{36}$$

For our RM^2 formula, we need the density of the portfolio distribution

$$\begin{aligned}
f_{S,g}(t) &= \int_0^t f_{g(Y_i)}(t-s) f_{S_{(i)}}(s) ds = \int_{t-b}^t f_i \left(\frac{t-s}{c} + d \right) f_{S_{(i)}}(s) ds \\
&= \sum_{j=1}^{m_i} \sum_{k=1}^{m^{(i)}} a_{ij} \lambda_{ij} a_{(i)j} \lambda_{(i)j} \int_{t-b}^t \left(\exp(-\lambda_{ij}(\frac{t-s}{c} + d)) \exp(-\lambda_{(i)j}s) \right) ds \\
&= \sum_{j=1}^{m_i} \sum_{k=1}^{m^{(i)}} a_{ij} \lambda_{ij} a_{(i)j} \lambda_{(i)j} \exp \left(-\lambda_{ij}(\frac{t}{c} + d) \right) \int_{t-b}^t \exp[-s(\lambda_{(i)j} - \frac{\lambda_{ij}}{c})] ds \\
&= \sum_{j=1}^{m_i} \sum_{k=1}^{m^{(i)}} a_{ij} \lambda_{ij} a_{(i)j} \lambda_{(i)j} \exp \left(-\lambda_{ij}(\frac{t}{c} + d) \right) \frac{\exp[-(\lambda_{(i)j} - \lambda_{ij}/c)(t-b)] - \exp[-(\lambda_{(i)j} - \lambda_{ij}/c)t]}{\lambda_{(i)j} - \lambda_{ij}/c} \\
&= \sum_{j=1}^{m_i} \sum_{k=1}^{m^{(i)}} a_{ij} \lambda_{ij} a_{(i)j} \lambda_{(i)j} \exp(-\lambda_{ij}d) \frac{\exp[-\lambda_{(i)j}(t-b) - \lambda_{ij}b/c] - \exp[-\lambda_{(i)j}t]}{\lambda_{(i)j} - \lambda_{ij}/c},
\end{aligned}$$

which is complex but easy to compute. With a value for the quantile $\xi_{S,g,\alpha}$, this provides a mechanism to compute RM^2 changes in equations (21) and (20).

To evaluate marginal change for coinsurance, one can evaluate the integral inside equation (19),

$$\frac{1}{c^2} \int_{t-b}^t (t-s) f_i \left(\frac{\xi_{S,g,\alpha} - s}{c} + d \right) f_{S_{(i)}}(s) ds,$$

using $t = \xi_{S,g,\alpha}$, in the same fashion.

Thus, using mixtures of exponentials is a desirable approach, allowing for closed form expressions of densities and distribution functions of convolutions (although not quantiles). With actuarial science applications, it has the flexibility to represent many distributions of interest, cf., Klugman and Rioux (2006) and Lee and Lin (2010). The limitation of this approach is that one needs a form for the portfolio density such as in equation (36) that depends on the risk being omitted. Thus, one can think of applications where a key risk is omitted and mixture of exponentials has the flexibility to provide close approximations to a risk of interest. However, for other applications, one would like to perform analyses omitting each risk, one at time, and consider the entire portfolio. In this case, without further restrictions on parameters in equation (36), there is no guarantee of consistency in the analysis. In contrast, in Section 4, we pursue a simulation approach that does guarantee this consistency when evaluating all the risks in a portfolio.

□