Representative Variable Annuity Policy Selection using Latin Hypercube Sampling

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Abstract

Valuation and risk management of large portfolios of variable annuity policies are a big challenge to insurance companies because pricing a large portfolio of variable annuity policies is a time consuming task. Recently, a method based on clustering and kriging has been proposed to address the computational problem arising from this area. In this method, the value of the portfolio is estimated by the kriging method from the values of the representative variable annuity policies, which are selected by a clustering method. However, thousands of representative policies are required in order to obtain relatively accurate estimations. This paper proposes a Latin hypercube sampling method for selecting representative variable annuity policies. Our test results show that the Latin hypercube sampling method is superior to the clustering method in that only a few hundred representative policies selected by the Latin hypercube sampling method are enough to produce accurate estimations.

Keywords: Variable annuity, Monte Carlo simulation, Portfolio valuation, Portfolio pricing, Data clustering, Latin hypercube sampling, Kriging

1. Introduction

A variable annuity (VA) refers to an attractive life insurance product that provides upside participation and downside protection in both bull and bear markets. Once an investor enters into a variable annuity contract with an insurance company, the investor agrees to make one lump-sum or a series

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of purchase payments to the insurance company and the insurance company agrees to make benefit payments to the investor beginning immediately or at some future date. In a variable annuity contract, the investor's money is invested in a basket of mutual funds, which include bond funds and equity funds. When a variable annuity matures, the benefit of the contract is equal to the market value of the accumulated purchase payments. Variable annuity has other names such as segregated fund, guaranteed investment fund, unit-linked life insurance, equity-linked life insurance, or participating life insurance (Armstrong, 2001)

A main feature of variable annuities is that they contain guarantees. For example, every VA contract contains the guaranteed minimum death benefit (GMDB) (Milevsky and Posner, 2001; Nteukam T. et al., 2011; Chi and Lin, 2012; Gerber et al., 2012; Gao and Ulm, 2012; Gerber et al., 2013) because the variable annuity is a type of life insurance. VA contracts also include the guaranteed minimum withdrawal benefit (GMWB) (Milevsky and Salisbury, 2006; Chen and Forsyth, 2008; Chen et al., 2008; Xu and Wang, 2009; Ko et al., 2010; Peng et al., 2010; Kolkiewicz and Liu, 2012; Yang and Dai, 2013), the guaranteed minimum maturity benefit (GMAB) (Zhuo and Park, 2006; Jiang and Chang, 2010), and the guaranteed minimum income benefit (GMIB) (Marshall et al., 2010; Bacinello et al., 2011). These guarantees are optional in that a policyholder can purchase these guarantees for additional fees. All the guarantees are financial guarantees that cannot be adequately addressed by traditional actuarial approaches (Boyle and Hardy, 1997; Hardy, 2000). Dynamic hedging (Boyle and Hardy, 1997; Hardy, 2003) is a popular risk management approach for variable annuities and is adopted by many insurance companies.

Since VA contracts embedding guarantees are relatively complex, the calculation of their fair market values cannot be done in closed form except for special cases (Gerber and Shiu, 2003; Feng and Volkmer, 2012). In practice, insurance companies rely on the Monte Carlo simulation method to determine the fair market values of VA contracts. However, using the Monte Carlo simulation method to value a large portfolio of VA contract is time consuming because every VA contract needs to be projected over many risk-neutral scenarios for a long time horizon.

To make dynamic hedging work for a large portfolio of VA policies, an insurance company needs to calculate the Greeks (e.g., dollar Delta and dollar Rho) of the big portfolio on a daily basis in order to incorporate the changes in the portfolio and the market. In particular, the insurance company needs to complete the calculation of the Greeks over night between today's market close and tomorrow's market open. In order to complete the computationally intensive calculation, insurance companies employ many computers to do the calculation. For example, GPUs (Graphics Processing Unit) have been used to value VA contracts (Phillips, 2012; NVIDIA, 2012).

Although using many computers or GPUs can speed up the calculation, this approach is not scalable. In other words, if the number of VA policies in a portfolio doubles, then the insurance company needs to double the number of computers or GPUs in order to complete the calculation within the same time interval. In addition, buying or renting many computers or GPUs is expensive and can cost the insurance company a lot of money annually.

Gan (2013) proposed a method based on data clustering and machine learning to address the computational problem arising from the VA area from the perspective of software. In this method, a set of representative VA policies is first selected from the big portfolio via a clustering method. Then the set of representative policies is valued via the Monte Carlo simulation method. Finally, the value of the whole portfolio is estimated from the representative policies' values through a machine learning method called kriging. Since the set of representative policies is much smaller than the whole portfolio, this process is able to calculate the value of the portfolio quickly.

In order for the aforementioned method to produce relatively accurate estimation, however, thousands of representative policies are required. One major reason is that the set of representative VA policies selected by the clustering method does not include boundary VA policies, which are important to increase the accuracy of the Kriging method (Kleijnen and van Beers, 2004). In this paper, we propose a Latin hypercube sampling method (McKay et al., 1979; Pistone and Vicario, 2010; Petelet et al., 2010; Viana, 2013) to select representative policies. We shall compare the two methods for selecting representative VA policies in terms of accuracy.

The remaining of the paper is structured as follows. Section 2 gives a brief review of the method based on clustering and kriging that was proposed by Gan (2013). Section 3 introduces a Latin hypercube sampling method used to select representative VA contracts. Section 4 presents some numerical results of applying the Latin hypercube sampling method to select representative VA contracts. Section 5 concludes the paper and gives a survey of future work.

2. The Method based on Clustering and Kriging

In this section, we give a brief description of the method proposed by Gan (2013). The method consists of two major steps: the clustering step and the kriging step.

2.1. The Clustering Method

In the clustering step, the k-prototypes algorithm (Huang, 1998; Gan et al., 2007) was used to select representative VA policies from a large portfolio of VA policies. The k-prototypes algorithm is a clustering algorithm that was developed to cluster mixed-type data.

Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ denote the portfolio of VA contracts, where n is the number of VA contracts and $\mathbf{x}_i$ represents the ith VA contract. Without loss of generality, we assume that a VA contract is characterized by d attributes (e.g., gender, age, account value, etc.) and that the first d_1 attributes are numeric and the last $d_2 = d - d_1$ attributes are categorical. Then the distance between two records $\mathbf{x}$ and $\mathbf{y}$ in X can be defined as (Huang, 1998):

$$D(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{h=1}^{d_1} (x_h - y_h)^2 + \sum_{h=d_1+1}^{d} \delta(x_h, y_h)}, \tag{1}$$

where x_h and y_h are the hth component of $\mathbf{x}$ and $\mathbf{y}$, respectively, and $\delta(\cdot, \cdot)$ is the simple matching distance defined as

$$\delta(x_h, y_h) = \begin{cases} 0, & \text{if } x_h = y_h, \\ 1, & \text{if } x_h \neq y_h. \end{cases} \tag{2}$$

The numerical values in the above distance definition are normalized so that for each $h = 1, 2, \ldots, d_1$, the standard deviation of x_{1h}, x_{2h}, $\ldots$, x_{nh} is one, where x_{ih} is the hth component of the ith VA contract $\mathbf{x}_i$. Normalizing the numerical attributes prevents an individual numerical attribute dominates the distance calculation.

The objective function that the k-prototypes algorithm tries to minimize is defined as

$$P = \sum_{j=1}^{k} \sum_{\mathbf{x} \in C_j} D^2(\mathbf{x}, \boldsymbol{\mu}_j), \tag{3}$$

where $D(\cdot,\cdot)$ is defined in Equation (1), k is the number of clusters, C_j is the jth cluster, and $\boldsymbol{\mu}_j$ is the center or prototype of cluster C_j.

The k-prototypes algorithm proceeds iteratively in order to find a solution that minimizes the objective function defined in Equation (3). In other words, the k-prototypes algorithm repeats updating the cluster memberships given the cluster centers and updating the cluster centers given the cluster memberships until some stop condition is satisfied. For a detail description of the iterative process, readers are referred to (Huang, 1998; Gan, 2013).

2.2. The Kriging method

In the approach proposed by Gan (2013), the ordinary kriging method (Isaaks and Srivastava, 1990) was used to estimate the fair market value and the Greeks of the whole portfolio from the representative VA policies.

Let $\mathbf{z}_1$, $\mathbf{z}_2$, ..., $\mathbf{z}_k$ be the representative VA contracts obtained from the clustering algorithm. For every $j = 1, 2, \ldots, k$, let y_j be the fair value of $\mathbf{z}_j$ that is calculated by the Monte Carlo method. Then we use the Kriging method to estimate the fair value of the VA contract $\mathbf{x}_i$ as

$$\hat{y}_i = \sum_{j=1}^{k} w_{ij} \cdot y_j, \tag{4}$$

where $w_{i1}, w_{i2}, \ldots, w_{ik}$ are the Kriging weights.

The Kriging weights $w_{i1}, w_{i2}, \ldots, w_{ik}$ are obtained by solving the following linear equation system

$$\begin{pmatrix} V_{11} & \cdots & V_{1k} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{k1} & \cdots & V_{kk} & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_{i1} \\ \vdots \\ w_{ik} \\ \theta_i \end{pmatrix} = \begin{pmatrix} D_{i1} \\ \vdots \\ D_{ik} \\ 1 \end{pmatrix}, \tag{5}$$

where θ_i is a control variable used to make sure the sum of the Kriging weights is equal to one,

$$V_{rs} = \alpha + \exp\left(-\frac{3}{\beta} D(\mathbf{z}_r, \mathbf{z}_s)\right), \quad r, s = 1, 2, \ldots, k,$$

and

$$D_{ij} = \alpha + \exp\left(-\frac{3}{\beta} D(\mathbf{x}_i, \mathbf{z}_j)\right), \quad j = 1, 2, \ldots, k.$$

Here the distance function $D(\cdot, \cdot)$ is defined in Equation (1), and $\alpha \geq 0$ and $\beta > 0$ are two parameters. Since $D(\mathbf{z}_r, \mathbf{z}_s) > 0$ for all $1 \leq r < s \leq k$, the above linear equation system has a unique solution (Isaaks and Srivastava, 1990).

The fair value of the portfolio X is equal to the sum of the fair values of all VA contracts in X, i.e.,

$$\hat{Y} = \sum_{i=1}^{n} \hat{y}_i = \sum_{i=1}^{n} \sum_{j=1}^{k} w_{ij} \cdot y_j = \sum_{j=1}^{k} w_j \cdot y_j, \tag{6}$$

where

$$w_j = \sum_{i=1}^{n} w_{ij}.$$

The fair value $\hat{Y}$ of the portfolio can be calculated efficiently by solving w_1, w_2, ..., w_k from the following linear equation system

$$\begin{pmatrix} V_{11} & \cdots & V_{1k} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{k1} & \cdots & V_{kk} & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_k \\ \theta \end{pmatrix} = \begin{pmatrix} D_1 \\ \vdots \\ D_k \\ n \end{pmatrix}, \tag{7}$$

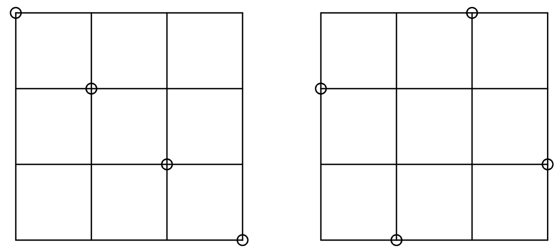
where

$$D_j = \sum_{i=1}^{n} D_{ij}, \quad j = 1, 2, \ldots, k.$$

In fact, Equation (7) is obtained by summing both sides of Equation (5) from $i = 1$ to n. For a detail discussion of the parameters used in the ordinary kriging method, readers are referred to (Isaaks and Srivastava, 1990) and (Gan, 2013).

3. A Latin Hypercube Sampling Method

Latin hypercube sampling (LHS) is a statistical method for generating plausible design points from multiple dimensional spaces that are used to conduct computer experiments. Figure 1 gives two examples of Latin hypercube designs with 4 points on a 2-dimensional area. From the figure we see that there is only one sample point in each row and each column. For



(a) (b)

Figure 1: Two examples of Latin hypercube designs with 4 divisions and 2 variables.

more information about LHS, readers are referred to (McKay et al., 1979), (Pistone and Vicario, 2010), (Petelet et al., 2010), and (Viana, 2013).

When the number of divisions and the number of variables increase, the number of Latin hypercubes increases exponentially (McKay and Wanless, 2008). For example, there are

$$64 \times 4! \times (3!)^3 = 331,776$$

Latin hypercubes with 4 divisions and 3 variables. As a result, one way to find a good Latin hypercube design is to generate Latin hypercube samples randomly and select the best one from the samples.

In this section, we introduce a LHS method for selecting representative VA policies. The LHS method introduced here is very similar to the MATLAB function `lhsdesign` in that both methods produce maximin Latin hypercube designs. However, the `lhsdesign` function was developed for numerical variables only. The LHS method introduced in this section is able to handle both numerical and categorical variables.

To describe the LHS method, we assume that a VA contract is characterized by d attributes (e.g., gender, age, account value, etc.) and that the first d_1 attributes are numerical and the remaining $d_2 = d - d_1$ attributes are categorical. For $j = 1, 2, \ldots, d_1$, let L_j and H_j denote the minimum and maximum values that the jth numerical variable can take. That is,

$$L_j = \min\{x_j : \mathbf{x} \in X\}, \quad H_j = \max\{x_j : \mathbf{x} \in X\}, \tag{8}$$

where x_j denotes the jth component of $\mathbf{x}$ and $X = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ denote the portfolio of VA contracts. For $j = d_1 + 1$, $d_1 + 2$, $\ldots$, d, let N_j denote the number of distinct values that the jth categorical variable can take, i.e.,

$$N_j = \left|\{x_j : \mathbf{x} \in X\}\right|, \tag{9}$$

where $|\cdot|$ denote the number of elements in a set.

Suppose that we want to generate a Latin hypercube design with k design points, where $k \geq 2$. To do that, we first divide the range of each of the d_1 numerical variable into k divisions. For each $l = 1, 2, \ldots, k$, the lth division of the jth dimension is given by

$$I_l = \left(L_j + \left(l - \frac{3}{2}\right) \frac{H_j - L_j}{k-1}, L_j + \left(l - \frac{1}{2}\right) \frac{H_j - L_j}{k-1}\right].$$

Since

$$\bigcup_{l=1}^{k} I_l = \left(L_j - \frac{H_j - L_j}{2(k-1)}, H_j + \frac{H_j - L_j}{2(k-1)}\right] \subset [L_j, H_j],$$

the union of the k divisions covers the whole range of the jth variable. For each of the remaining categorical variables, we just treat each category as a division.

Let $\mathcal{H}$ be a set of d-dimensional points defined to be

$$\mathcal{H} = \{(a_1, a_2, \ldots, a_d)\} \tag{10}$$

such that for $j = 1, 2, \ldots, d_1$,

$$a_j \in \left\{L_j + (l-1)\frac{H_j - L_j}{k-1}, l = 1, 2, \ldots, k\right\},$$

and for $j = d_1 + 1, d_1 + 2, \ldots, d$,

$$a_j \in \{A_{jl}, l = 1, 2, \ldots, N_j\},$$

where A_{j1}, A_{j2}, …, A_{jN_j} are the distinct categories of the jth variable and L_j, H_j, and N_j are defined in Equations (8) and (9). There are many points in the set $\mathcal{H}$. In fact, we have

$$|\mathcal{H}| = k^{d_1} \prod_{j=d_1+1}^{d} N_j.$$

The first step of the LHS method is to select k points from the set $\mathcal{H}$ with the best score, which is to be defined. Let H be a subset of $\mathcal{H}$ with k elements. The score of the set H is defined to be the minimum distance between any pairs of distinct points in H. That is,

$$S(H) = \min\{M(\mathbf{a}, \mathbf{b}) : \mathbf{a} \in H, \mathbf{b} \in H, \mathbf{a} \neq \mathbf{b}\}, \tag{11}$$

where $M(\mathbf{a}, \mathbf{b})$ is the distance between $\mathbf{a}$ and $\mathbf{b}$ given by

$$M(\mathbf{a}, \mathbf{b}) = \sum_{j=1}^{d_1} \frac{(k-1)|a_j - b_j|}{H_j - L_j} + \sum_{j=d_1+1}^{d} \delta(a_j, b_j), \tag{12}$$

where a_j and b_j are the jth components of $\mathbf{a}$ and $\mathbf{b}$, respectively, and $\delta(\cdot, \cdot)$ is defined in Equation (2). The larger the score, the better the Latin hypercube design. An optimal Latin hypercube design with k points is defined as

$$H^* = \underset{H \subset \mathcal{H}, |H|=k}{\text{argmax}} \ S(H). \tag{13}$$

Since the set $\mathcal{H}$ contains huge number of points, finding an optimal Latin hypercube design with k points from $\mathcal{H}$ is not easy. To find such an optimal Latin hypercube design, we randomly generate many (e.g., 500) Latin hypercube designs and select the one with the largest score. To generate a random Latin hypercube design $H = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ with k points, we proceed as follows:

1. For each $j = 1, 2, \ldots, d_1$, we randomly generate k uniform real numbers from the interval $[0, 1]$. Suppose that these random numbers are r_{j1}, r_{j2}, …, r_{jk}. Since these numbers are random real numbers, they are mutually distinct in general. We sort the k real numbers in an ascending order such that

   $$r_{ji_1} < r_{jr_2} < \cdots < r_{jr_k},$$

   where $(i_1, i_2, \ldots, i_k)$ is a permutation of $(1, 2, \ldots, k)$. Then we define the first d_1 coordinates of the k design points as

   $$a_{jl} = L_j + (i_l - 1)\,\frac{H_j - L_j}{k-1}, \quad j = 1, \ldots, d_1,\ l = 1, \ldots, k.$$

   For each $j = 1, 2, \ldots, d_1$, the coordinates of the k design points at the jth dimension are mutually distinct.
2. For each $j = d_1 + 1, d_1 + 2, \ldots, d$, we randomly generate k uniform integers from $\{1, 2, \ldots, N_j\}$. For portfolios of variable annuity policies, we usually have $k > N_j$, that is, the number of design points is larger than the number of values that a categorical variable can take. Suppose that these random integers are $i_1, i_2, \ldots, i_k$. Then we define the remaining d_2 coordinates of the k design points as

   $$a_{jl} = A_{ji_l}, \quad j = d_1 + 1, \ldots, d,\ l = 1, \ldots, k,$$

where A_{j1}, A_{j2}, ..., A_{jN_j} are the distinct categories of the jth variable.

Once we find a Latin hypercube design $H^* = \{\mathbf{a}_1^*, \mathbf{a}_2^*, \ldots, \mathbf{a}_k^*\}$ using the above procedure. The second step of the LHS method is to find k representative VA policies that are close to the k design points in H^*. In particular, the VA policy that is close to $\mathbf{a}_i^*$ is determined by

$$\mathbf{z}_i = \underset{\mathbf{x}\in X}{\operatorname{argmin}}\, M(\mathbf{a}_i^*, \mathbf{x}), \quad i = 1, 2, \ldots, k,$$

where $M(\cdot, \cdot)$ is defined in Equation (12).

4. Numerical Experiments

In this section, we present some numerical results of applying the k-prototypes method (cf. Section 2) and the LHS method (cf. Section 3) to select representative VA policies. To do the test, we generate a portfolio of 200,000 synthetic VA contracts. The attributes and their ranges of values are shown in Table 1. For each synthetic VA contract, the value of an attribute is generated from a uniform distribution with the corresponding range given in Table 1. The total account value of this portfolio is 51,065,089,363.74 dollars.

Attribute	Values
Guarantee type	{GMDB only, GMDB + GMWB}
Gender	{Male, Female}
Age	{20, 21, 22, ..., 60}
Account value	$[10000, 500000]$
GMWB withdrawal rate	{0.04, 0.05, 0.06, 0.07, 0.08}
Maturity	{10, 11, 12, ..., 25}

Table 1: Variable annuity attributes and their ranges of values.

To test the effectiveness of the two methods for selecting representative VA policies, we use the ordinary kriging method (cf. Section 2) to estimate the fair market value, dollar Delta, and dollar Rho of the whole portfolio. In addition, we test the two methods with different numbers of representative VA policies. In particular, we conducted tests with 50, 100, 200, 300, 400, 500, 1000, and 2000 representative VA policies. In all the test cases, we used 500 iterations in the LHS method. That is, 500 Latin hypercube designs are randomly generated and the one with the largest score is selected.

In our first test, we used the k-prototypes algorithm and the LHS method to select two sets of 50 representative VA policies. Then we used the two sets of representative VA policies with the ordinary kriging method to estimate the fair market value, dollar Delta, and dollar Rho of the whole portfolio. Table 2 shows the results and the differences. The first row (MC) shows the fair market value, dollar Delta, and dollar Rho of the portfolio calculated by the Monte Carlo method with 1000 risk-neutral scenarios. The second row (KP) and the third row (LHS) show the numbers estimated from the representative VA policies selected by the k-prototypes method and the LHS method, respectively. The last four rows show the dollar difference and the percentage difference.

	Fair Market Value	Dollar Delta	Dollar Rho
MC	3,003,947,180	(8,150,275,955)	(9,736,358)
KP	3,425,223,434	(9,253,343,920)	(11,322,564)
LHS	3,019,066,798	(7,946,383,820)	(9,700,733)
KP-MC	421,276,254	(1,103,067,964)	(1,586,206)
LHS-MC	15,119,619	203,892,136	35,625
(KP-MC)/MC	14.02%	13.53%	16.29%
(LHS-MC)/MC	0.50%	-2.50%	-0.37%

Table 2: The fair market values, dollar Deltas, and dollar Rhos estimated from 50 representative VA policies.

From Table 2 we see that a set of 50 representative VA policies selected by the LHS method gives very good estimations of the fair market value, dollar Delta, and dollar Rho of the whole portfolio. The percentage differences of fair market value and dollar Rho are less than 1%. However, the differences from the k-prototypes algorithm are between 13% to 16%. The difference of the dollar Delta from the k-prototypes algorithm is more than 1 billion.

In our second test, we used the two methods to select 100 representative VA policies from the whole portfolio. The results of this test case are shown in Table 3. The results from the two methods improve from the previous test. However, the differences from the k-prototypes algorithm are still around 10%. The differences from the LHS method are around 0.5%. The test

	Fair Market Value	Dollar Delta	Dollar Rho
MC	3,003,947,180	(8,150,275,955)	(9,736,358)
KP	3,304,156,326	(8,924,741,716)	(10,843,115)
LHS	3,016,679,402	(8,180,679,337)	(9,789,810)
KP-MC	300,209,146	(774,465,760)	(1,106,757)
LHS-MC	12,732,222	(30,403,381)	(53,451)
(KP-MC)/MC	9.99%	9.50%	11.37%
(LHS-MC)/MC	0.42%	0.37%	0.55%

Table 3: The fair market values, dollar Deltas, and dollar Rhos estimated from 100 representative VA policies.

results show that the 100 representative VA policies selected by the LHS method are very good in terms of estimation accuracy.

	Fair Market Value	Dollar Delta	Dollar Rho
MC	3,003,947,180	(8,150,275,955)	(9,736,358)
KP	3,151,068,016	(8,609,921,702)	(10,328,238)
LHS	3,052,868,370	(8,268,011,986)	(9,911,503)
KP-MC	147,120,836	(459,645,747)	(591,880)
LHS-MC	48,921,190	(117,736,031)	(175,145)
(KP-MC)/MC	4.90%	5.64%	6.08%
(LHS-MC)/MC	1.63%	1.44%	1.80%

Table 4: The fair market values, dollar Deltas, and dollar Rhos estimated from 200 representative VA policies.

In our third test, we used the two methods to select 200 representative VA policies. Table 4 shows the results of this test case. Comparing Table 4 with Table 3, we see that the results from the k-prototypes algorithm improve. However, the differences from the LHS method increase about 1%. This is counter-intuitive because a larger number of representative VA policies should lead to a better estimation. It is possible that the best Latin hypercube with 200 design points was not found by the LHS method.

In our fourth test, we used the two methods to select 300 representative VA policies from the whole portfolio. The results of this test case are shown in Table 5. The results from the clustering method are worse than those from

	Fair Market Value	Dollar Delta	Dollar Rho
MC	3,003,947,180	(8,150,275,955)	(9,736,358)
KP	3,229,513,256	(8,757,006,759)	(10,632,954)
LHS	3,019,466,305	(8,215,952,368)	(9,787,228)
KP-MC	225,566,076	(606,730,803)	(896,596)
LHS-MC	15,519,125	(65,676,413)	(50,870)
(KP-MC)/MC	7.51%	7.44%	9.21%
(LHS-MC)/MC	0.52%	0.81%	0.52%

Table 5: The fair market values, dollar Deltas, and dollar Rhos estimated from 300 representative VA policies.

the third test. However, the results from the LHS method are better than those from the third test. Comparing Table 5 and Table 3, the results from the LHS method are similar.

	Fair Market Value	Dollar Delta	Dollar Rho
MC	3,003,947,180	(8,150,275,955)	(9,736,358)
KP	3,174,485,382	(8,609,900,099)	(10,347,897)
LHS	3,013,144,924	(8,171,077,894)	(9,736,581)
KP-MC	170,538,202	(459,624,143)	(611,539)
LHS-MC	9,197,745	(20,801,938)	(222)
(KP-MC)/MC	5.68%	5.64%	6.28%
(LHS-MC)/MC	0.31%	0.26%	0.00%

Table 6: The fair market values, dollar Deltas, and dollar Rhos estimated from 400 representative VA policies.

In our fifth test, we used the two methods to select 400 representative VA policies. The results of this test case are shown in Table 6, from which we see that the results from the LHS method are really good in terms of accuracy. The differences from the LHS method are less that 0.5%. In particular, the dollar difference of the dollar Rho is around 200 dollars. However, the differences from the clustering method are still more than 5%.

In our sixth test, we used the two methods to select 500 representative VA policies from the whole portfolio. Table 7 shows the results of this test case. From the table we see that the differences from the k-prototypes algorithm are around 5%. The differences from the LHS method are within 0.5%.

	Fair Market Value	Dollar Delta	Dollar Rho
MC	3,003,947,180	(8,150,275,955)	(9,736,358)
KP	3,153,864,868	(8,584,797,784)	(10,304,245)
LHS	3,008,948,423	(8,181,891,156)	(9,751,202)
KP-MC	149,917,688	(434,521,829)	(567,887)
LHS-MC	5,001,243	(31,615,200)	(14,844)
(KP-MC)/MC	4.99%	5.33%	5.83%
(LHS-MC)/MC	0.17%	0.39%	0.15%

Table 7: The fair market values, dollar Deltas, and dollar Rhos estimated from 500 representative VA policies.

	Fair Market Value	Dollar Delta	Dollar Rho
MC	3,003,947,180	(8,150,275,955)	(9,736,358)
KP	3,090,778,168	(8,438,870,353)	(10,043,320)
LHS	2,999,971,555	(8,172,049,778)	(9,716,086)
KP-MC	86,830,989	(288,594,398)	(306,961)
LHS-MC	(3,975,625)	(21,773,823)	20,272
(KP-MC)/MC	2.89%	3.54%	3.15%
(LHS-MC)/MC	-0.13%	0.27%	-0.21%

Table 8: The fair market values, dollar Deltas, and dollar Rhos estimated from 1000 representative VA policies.

In our seventh test, we used the two methods to select 1000 representative VA policies. The results of this test case are shown in Table 8, from which we see than the results from the clustering method improve from the previous test cases. However, the results from the LHS method are similar to those from previous test cases.

In our last test, we used the two methods to select 2000 representative VA policies from the whole portfolio. The results of this test case are shown in Table 9. From this table we see that the results from the clustering method improve a lot from the previous test cases. The differences from the clustering method are around 1%. However, the results from the LHS method are still similar to those from previous test cases.

From the above eight test cases we have the following observations:

- For the clustering method, the more number of representative VA poli-

	Fair Market Value	Dollar Delta	Dollar Rho
MC	3,003,947,180	(8,150,275,955)	(9,736,358)
KP	3,024,267,213	(8,233,152,864)	(9,805,786)
LHS	3,013,409,471	(8,172,960,693)	(9,763,927)
KP-MC	20,320,033	(82,876,909)	(69,427)
LHS-MC	9,462,291	(22,684,738)	(27,568)
(KP-MC)/MC	0.68%	1.02%	0.71%
(LHS-MC)/MC	0.31%	0.28%	0.28%

Table 9: The fair market values, dollar Deltas, and dollar Rhos estimated from 2000 representative VA policies.

cies, the more accurate the estimation.

- For the LHS method, a small number (e.g., 50, 100, 200, 300, 400) of representative VA policies can lead to accurate estimation.
- When the same number of representative VA policies is used, the LHS method is superior to the clustering method (i.e., the k-prototypes method).

To achieve the same accuracy of estimation, a smaller set of representative VA policies is preferable to a larger one because the smaller the set of representative policies, the faster the estimation process. In this sense, the LHS method is preferable to the clustering method for selecting representative VA policies.

Table 10 shows the time used by the two methods to select various numbers of representative VA policies from the portfolio of 200,000 policies. From the table we see that the LHS method is faster than the clustering method when used to select a small number of representative VA policies. However, the LHS method is slower than the clustering method when used to select a large number of representative VA policies. Since the quality of the representative VA policies selected by the LHS method is better than that of the representative VA policies selected by the clustering method, we can use the LHS method to select small sets of representative policies.

	Clustering	LHS
50	16.65	3.72
100	16.21	5.05
200	12.44	7.93
300	12.81	11.95
400	11.71	15.79
500	12.68	20.47
1000	9.93	55.21
2000	9.71	152.60

Table 10: Time used by the clustering method and the LHS method to generate different number of representative VA policies from a portfolio of 200,000 policies. The numbers are in seconds.

5. Concluding Remarks

For an insurance company that has a big VA portfolio, a major challenge in risk management of the VA business is to calculate the fair market value and the Greeks of the VA portfolio in an efficient way. Gan (2013) proposed a method based on data clustering and machine learning to address the computational problem from the perspective of mathematical modeling instead of hardware. The idea behind this method is to first select a small set of representative policies, then price the representative policies, and finally estimate the value of the whole portfolio. The method is efficient in that only a small set of representative policies is required to be priced by the time-consuming Monte Carlo simulation model. However, the method can be improved in many ways.

In this paper, we proposed a simple LHS method to select representative VA policies, which are used by the ordinary kriging method to estimate the fair market value and the Greeks (e.g., dollar Delta, dollar Rho) of a large portfolio of VA policies. Latin hypercube sampling was a statistical method proposed by McKay et al. (1979) for generating plausible design points to construct computer experiments. There are many ways to implement the Latin hypercube sampling method (Viana, 2013). The Latin hypercube sampling method implemented here is based on the maximin criterion.

We conducted several test cases to compare the quality of the representative VA policies selected by the LHS method and the k-prototypes algorithm initially used in (Gan, 2013). Our test results show that the LHS method is superior to the clustering method in terms of accuracy. In addition, for small sets of representative VA policies, the LHS method is also faster than the clustering method. In particular, our test results show that a set of 100 representative VA policies selected by the LHS method generates better results than a set of 2000 representative VA policies selected by the clustering method does.

The method proposed by Gan (2013) consists of several steps and components, which can be improved in one way or another. In future, we would like to test other Latin hypercube sampling methods (Viana, 2013), the distance function, and the kriging model. The distance used in (Gan, 2013) and this paper is Euclidean distance. We would like to test other distances such as the Manhattan distance (Gan et al., 2007; Gan, 2011). Since the kriging model used in this paper and (Gan, 2013) is a simple ordinary kriging model (Isaaks and Srivastava, 1990), we would like to test other sophisticated kriging models such as the universal kriging model (Caballero et al., 2013).

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