

# Not so Great Expectations: A Model of Growth and Informational Frictions\*

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## ABSTRACT

I develop a model of asset markets with dispersed private information in a continuous-time, macroeconomic setting where firms learn from financial prices when making their investment decisions. I derive a tractable equilibrium that highlights a feedback loop between investor trading behavior and firm real investment. While the strength of real signals for the expectations of firms and investors is procyclical, financial signals are strongest during economic transitions. Through this channel, contamination in price signals during financial crises can distort expectations to be more pessimistic, and lead to deeper recessions and more gradual recoveries. I explore the welfare and asset pricing implications of my model, as well as its empirical implications for real investment.

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# I. Introduction

In this paper, I introduce a tractable, dynamic framework for studying the feedback loop in learning that occurs between financial markets and firms when financial markets aggregate investor private information about the productivity of real investment. Through this informational channel, financial market prices are more important for learning than real activity at the trough of business cycles, and are most informative as signals about investment productivity during downturns and recoveries. My analysis establishes a link between recessions with financial origins and slow recoveries by illustrating how financial crises during downturns can delay recoveries by distorting firm expectations, which depresses real investment and feeds back into the incentives for financial market participants to trade on their private information.

Two observations motivate my investigation. The first is that market prices aggregate the private information of investors about macroeconomic and financial conditions, and that firms, in making their real decisions, respond to this useful information.<sup>1</sup> Since the mid-1980's, however, the rapid growth of the market-based financial system (Pozsar et al 2012), especially from 2002-2007 (Philippon (2008)), has increased financial opacity, as intermediaries extended credit and diversified risk through securitization and the OTC derivatives markets that arose in the wake of LTCM.<sup>2</sup> This heightened opacity has made it difficult for economic agents and policymakers to assess not only the depth of financial distress once a bust occurs, but also its distribution across the financial sector. This was particularly relevant in the recent recession, as regulators scrambled to map out the cross-party linkages of the unregulated financial system in late 2008 (FCIC 2011). As a result, market prices have become noisier signals about the strength of the economy, and economic actors, both real and financial, face more severe informational frictions.

That asset prices contain useful information about the macroeconomy has been well-documented in the literature.<sup>3</sup> Both during and in the aftermath of the financial crisis,

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<sup>1</sup>See, for instance, Luo (2005), Chen, Goldstein, and Jiang (2007) and Bakke and Whited (2010). For evidence that firms learn from their own profit realizations, the other key signal in our model, see, for instance, Moyen and Platikanov (2013).

<sup>2</sup>Former FRBNY President and Treasury Secretary Timothy Geitner, in fact, made it part of his agenda before the financial crisis to move the OTC derivatives market onto exchanges to increase transparency.

<sup>3</sup>For stock prices, for instance, see Fama (1981), Barro (1990), and Beaudry and Portier (2006), while for credit spreads, see Gertler and Lown (1999), Gilchrist, Yankov, and Zakrasjek (2009), Gilchrist and Zakrajsek (2012), and Ng and Wright (2013), and for a wide cross-section of asset classes, see Stock and Watson (2003).

many viewed the dramatic fall in asset prices as a signal that the US economy was entering a recession potentially as deep as the Great Depression.<sup>4</sup> When the stock market bottomed out in March 2009, in fact, the Michigan Survey of Consumers "fear of a prolonged depression" question had its lowest score since the 1991 recession.

The second observation is that recessions with financial origins appear to be deeper and have slower recoveries. A salient feature of the recent US experience, for instance, is the anemic economic recovery compared to previous cycles, especially in GDP, lending, and productivity (Haltmaier (2012), Reifschneider et al (2013)). While there is growing evidence that financial crises lead to deeper recessions, however, it is less clear if, and how, they also slow recoveries.<sup>5</sup> My model provides a framework for addressing conceptual questions about business cycles and uncertainty that explicitly incorporates a financial sector, and can also help explain why financial shocks can have asymmetric impacts over the business cycle (Aizenman et al (2012)).<sup>6</sup> The uncertainty I consider here that distorts investment arises from learning, and is therefore different from that in Bloom (2009), which focuses on shocks to firm fundamental volatility.

Informational frictions can lead firms to voluntarily withdraw from investment because of weak expectations about the state of the economy, rather than from uncertainty itself, a phenomenon which can help explain several stylized facts. First, the FRB Senior Loan Officer Survey cites weak credit demand as a reason for the low level of C&I loans until the end of 2010. Second, since the recession, firms have increased the cash and cash-like instruments on their balance sheets and saved their income as retained earnings rather than investing (Baily and Bosworth (2013), Sanchez and Yurgadul (2013), Kliesen (2013)).<sup>7</sup> Third, firms have appeared reluctant to fill vacancies, as studies such as Daly et al (2012) and Leduc and Liu (2013) find a potential shift in the Beveridge Curve after the recent recession, which reflects

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<sup>4</sup>For evidence regarding the fall in the stock market, see, for instance, Robert Barro's March 2009 WSJ Article "What Are the Odds of a Depression?" that accompanies Barro and Ursúa (2009), and Gerald Dwyer's September 2009 article, "Stock Prices in the Financial Crisis" from FRB Atlanta's Notes from the Vault.

<sup>5</sup>While studies like Reinhart and Rogoff (2009a,b, 2011), Ng and Wright (2013), and Jorda, Schularick and Taylor (2015), for instance, argue that financial crises result in slower recoveries, others such as Haltmaier (2012) and Stock and Watson (2012) find little difference, and those such as Bloom (2009), Muir (2014), and Bordo and Haubrich (2012) predict faster upswings following financial crises.

<sup>6</sup>For instance, while the S&L crisis and the bursting of the housing bubble accompanied recessions that had slow recoveries, the collapse of Long-Term Capital Management (LTCM) in 1998, arguably an event that almost led to the meltdown of the whole financial system, had no significant impact on the real economy.

<sup>7</sup>Pinkowitz, Stulz, and Williamson (2013) provide evidence that this increase in cash holdings is driven by perceived low investment opportunities by firms, since it is concentrated among the highly profitable firms in their sample.

a higher vacancy rate compared to the unemployment rate, while Davis, Faberman, and Haltiwanger (2013) document a fall in recruiting intensity. Though, for simplicity, my model will only involve capital, the same forces depressing real investment would also depress labor market demand in a more general framework. This evidence suggests that the slow recovery may, at least in part, have been driven by firms choosing to delay investment because of a persistent poor economic outlook.

To study the implications of learning in the presence of informational frictions for financial market trading and real activity, I integrate the classic information aggregation framework of Grossman and Stiglitz (1980) and Hellwig (1980) into a standard, general equilibrium macroeconomic model in continuous-time. This setting allows me not only to examine the dynamic, real consequences of informational frictions when there is a feedback loop between real activity and financial markets, but also to depart from the CARA-normal and risk-neutral-normal frameworks, which are less desirable for addressing macroeconomic questions, and to study agents with log utility without the need for approximation. Both tasks have posed a well-known and substantial challenge in the information aggregation literature, and separate strands have developed to examine feedback in each direction. A finance literature, including Albagi (2010), Goldstein, Ozdenoren, and Yuan (2013), and Subrahmanyam and Titman (2013), examines how asset prices impact real activity through the learning channel, while a macroeconomic literature, including Tinn (2010) and Angeletos, Lorenzoni, and Pavan (2012), investigates how real investment decisions are distorted by the ability to manipulate asset prices in the presence of informational frictions. I am able to make progress by appealing to the local linearity inherent in working in continuous-time, as well as to a standard assumption about the information structure of households and a convenient functional form for firm real investment.

The model presented herein features a continuum of non-overlapping generations of households that trade riskless debt and claims to the assets of firms in centralized financial markets. Households here represent the hedge funds, financial analysts, intermediaries, and other investors that participate in financial markets. Households each possess a private signal regarding the underlying strength of the economy when they trade, and have access to trees in a garden that are traded in a decentralized market. Asset prices in my economy aggregate the private information of agents, and aggregate shocks to these trees represent a source of noise that prevents them from being fully revealing to both households and firms. To avoid

both the infinite regress problem of Townsend (1983) and a time-varying correlation between the wealth of households and the persistence of their beliefs, I follow Allen, Morris, and Shin (2006), Bacchetta and van Wincoop (2008), and Straub and Ulbricht (2013) and assume that, though households in each generation pass along their wealth to their children, they do not pass along their private information. This assumption of investor myopia is necessary to maintain tractability in learning by helping me avoid these two issues.

Perfectly competitive, identical firms in my economy produce output and use financial prices, which aggregate private information dispersed among households, and real signals from production to form their expectations about the underlying state of the economy when making investment decisions. This introduces a channel for financial market shocks to feed into real activity by distorting the expectations of firms, since the impact of financial shocks on prices cannot be fully disentangled from informational trading. By affecting the returns on their securities and the informativeness of real economic signals through their investment choices, firms, in turn, impact the incentives of investors to trade on their private information to take advantage of the uncertain economic environment. This can lead to an adverse feedback loop that exacerbates real shocks to the economy during contractions that can deepen and lengthen recessions.

With these ingredients, I derive a tractable, linear noisy rational expectations equilibrium that offers several insights about learning from real and financial signals over the business cycle when there is this feedback loop. First, time-varying second moments are important for macroeconomic dynamics even without the real-options "wait-and-see" channel featured in Bloom (2009) for investment. In most environments with learning and asymmetric information, the conditional variance of beliefs is either constant or deterministically converging toward a (possibly trivial) limit. In my setting, this conditional variance varies stochastically with the level of investment, and this gives rise to countercyclical uncertainty in the economy. The second insight is that, while real signals about the macroeconomy are procyclical in their informativeness in learning, similar to the mechanism in Van Nieuwerburgh and Veldkamp (2006), financial signals are strongest during economic transitions. This feature arises because households have dispersed information and trade more aggressively against each other when there is uncertainty about the state of the economy, and this increase in trading leads more of their private information to be incorporated into prices. The strength of the financial signal trades off the return to investment with the level of uncertainty in the

economy, and these two quantities are negatively correlated over the business cycle. Finally, nonlinearity in investment slows recoveries since the informativeness of real and financial signals is tied to real investment. As investment falls, both real and financial signals weaken, which leads uncertainty to remain high and persistent until investment recovers. Real signals flatten because firms are less active, and financial signals flatten because the value of household private information anchors on the return to real investment.

I next offer an explanation of the slow US recovery in the context of my mechanism as stemming from confusion in financial price signals brought about by the financial crisis. This confusion led real investment to fall further during the recession and real and financial signals to flatten, which made it more difficult for agents to act on the recovery. I characterize welfare in the economy and identify a role for policy in improving the provision of public information about current economic conditions, since investors and firms do not fully internalize the benefit of the information that their activities produce.

Lastly, I turn to some of the empirical implications of my framework. I illustrate how informational frictions give rise to an informational component in risk premia. This component has predictive power for future returns and real activity, which varies with the level of uncertainty and investment in the economy. It also gives rise to business cycle variation in trading volume based on informational trading. I then conclude by discussing empirical implications for real investment.

## II. Related Literature

I view my amplification mechanism from feedback in learning as playing a contributing role in transmitting financial shocks to the real US economy to bring about deeper recessions and anemic recoveries. I frame it as being complementary to other channels highlighted in the macroeconomics literature linking recessions and financial crises, such as the balance sheet channels of He and Krishnamurthy (2013) for intermediaries and Mian and Sufi (2012) for households, and the collateral channels of Moreira and Savov (2013) and Gorton and Ordoñez (2014). My paper is part of several literatures on asymmetric information and the real consequences of asset prices. I discuss my relation to each of these literatures in turn.

My work is related to the literature on dynamic models of asymmetric information, such as Wang (1994), Foster and Viswanathan (1996), He and Wang (1995), and Allen, Morris and Shin (2006), which do not have real sectors and feature static economic environments

where the asset's fundamental is fixed. Foster and Viswanathan (1996) models strategic, dynamic trading between investors with private information and a market maker in a static informational environment, while He and Wang (1995) examines the impact on trading volume when investors trade on public signals and dynamic private information in the presence of persistent noise supply shocks. Allen, Morris, and Shin (2006) and Bacchetta and van Wincoop (2006, 2008) investigate the role of higher-order expectations introduced by dispersed information in the determination of asset prices. My study focuses on the impact on asset prices and real activity when agents learn not only from endogenous information in prices generated by dispersed information, but also from the endogenous information in the return process governing the asset's time-varying fundamentals. To my knowledge, my work is also one of the first studies to study the long-run implications of a dynamic model of asymmetric information.

While my work exploits the local linearity of continuous-time and a short-lived investor informational structure to help maintain tractability in a locally linear noisy rational expectations equilibrium, the literature has developed other settings of information aggregation that deliver tractable equilibria outside of the CARA-Normal paradigm. Albagi, Hellwig, and Tsyvinski (2015), for instance, construct an equilibrium with log-concavity and an unboundeness assumption on the distribution of private signals that delivers a sufficient statistic for the market price as the private signal of the marginal trader. Albagi, Hellwig, and Tsyvinski (2014, 2015) employ risk-neutral agents with normally-distributed asset fundamentals and position limits to deliver tractable nonlinear equilibria in a static setting. Other papers like Sockin and Xiong (2015) develop analytic log-linear equilibria in a static setting by exploiting Cobb-Douglas utility with fundamentals that have log-normal distributions. Straub and Ulbricht (2013) makes use of a conjugate prior framework with one period-lived, risk-neutral agents to maintain tractability in learning in a dynamic setting.

My work also contributes to the literature on informational frictions and the macroeconomy, which include Woodford (2003), Van Nieuwerburgh and Veldkamp (2006), Lorenzoni (2009), Angeletos and La'O (2013), Straub and Ulbricht (2013), Hassan and Mertens (2014a,b), and David, Hopenhayn, and Venkateswaran (2014). Only Straub and Ulbricht (2013), Hassan and Mertens (2014a,b), and David, Hopenhayn, and Venkateswaran (2014) consider the real consequences of informational frictions with centralized asset market trading to aggregate information. Informational frictions are, however, static in Hassan and Mertens

(2014a,b), because of the assumption of perfect consumption insurance across agents, and in David, Hopenhayn, and Venkateswaran (2014), who focus on resource misallocation across firms from imperfect information, because firms observe their fundamentals after revenue is realized each period.<sup>8</sup> Straub and Ulbricht (2013) explore the feedback loop between learning and the collateral channel, which destroys information during busts when agents become financially constrained because of a decline in the value of collateral with an exogenous, but hidden fundamental. My focus instead is on the feedback between asset prices and real investment that arises through the persistent distortion of the beliefs that govern real investment. In contrast to models like Albagi (2010) and Straub and Ulbricht (2013), my learning mechanism does not arise because of financial frictions, but only informational frictions, which implies that relieving credit conditions for firms will have little effect in my setting in improving economic conditions.

Finally, my paper also relates to the growing literature on the real effects of asset prices, which includes Albagi (2010), Tinn (2010), Goldstein, Ozdenoren, and Yuan (2011, 2013), Angeletos, Lorenzoni, and Pavan (2012), Ordoñez (2012), Subrahmanyam and Titman (2013), Albagi, Hellwig, and Tsyvinski (2014), and Sockin and Xiong (2015).<sup>9</sup> Goldstein, Ozdenoren, and Yuan (2013) explores the coordination motive among financial investors when stock prices inform real investment decisions, while Albagi (2010) examines the distortion to real investment that occurs when financial market participants face funding constraints. Angeletos, Lorenzoni, and Pavan (2012) investigates the distortion to real investment and financial prices in a sequential game when entrepreneurs make investment decisions before claims are sold to the market to rationalize the dot com bubble. Albagi, Hellwig, and Tsyvinski (2014) highlights the inefficiency that asymmetric information introduces into real investment when existing shareholders extract informational rent by making investment decisions before selling shares to imperfectly-informed capital markets. My dynamic model features feedback both from real investment to the beliefs and trading incentives of financial market participants, as in Tinn (2010) and Angeletos, Lorenzoni, and Pavan (2012), Ordoñez (2012), and Albagi, Hellwig, and Tsyvinski (2014), and from financial markets back to real investment, as in Albagi (2010), Goldstein, Ozdenoren, and Yuan (2013), and Sockin and

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<sup>8</sup>In my setting, firms face more severe information frictions than in Hassan and Mertens (2014a,b) and David, Hopenhayn, and Venkateswaran (2014) because they neither observe private signals nor the past history of the realized fundamental. As a result, learning occurs more slowly and uncertainty about the fundamental fluctuates endogenously over time.

<sup>9</sup>See Bond, Edmans, and Goldstein (2012) for a survey of this literature.



Xiong (2015). In contrast to these studies, my focus is on the dynamic consequences for real activity of learning from endogenous real and financial signals.

### III. A Model of Informational Frictions

#### A. The Environment

I consider an infinite-horizon production economy in continuous-time on a probability triple  $(\Omega, \mathbf{F}, \mathcal{P})$  equipped with a filtration  $\mathcal{F}_t$ . There are three fundamental, aggregate shocks in the economy  $\{Z_t^\theta, Z_t^k, Z_t^\xi\}$  which are standard independent Weiner processes. To focus on the impact of informational frictions in financial markets on real activity, I turn off the conventional channels for financial markets to feed back to real activity through financial frictions in borrowing and lending.

There are perfectly competitive, identical firms in the economy that manage capital  $K_t$  for households with which they produce output  $Y_t$  according to

$$Y_t = aK_t,$$

for  $a > 0$ . Firms are able to grow capital according to

$$\frac{dK_t}{K_t} = (I_t\theta_t - \delta) dt + \sigma_k dZ_t^k, \quad (1)$$

where  $I_t$  is investment per unit of capital,  $\theta_t$  is the productivity of real investment in installing new capital, similar to the investment-specific technology shock of Greenwood, Hercowitz, and Krusell (1997, 2000),  $\delta$  is depreciation, and  $Z_t^k$  is a Total Factor Productivity (TFP) shock to existing capital.<sup>10</sup> Importantly, both the productivity of real investment  $\theta_t$  and the TFP shock  $Z_t^k$  are unobservable to firms and households in the economy. A natural consequence of informational frictions with this specification for capital accumulation is that

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<sup>10</sup>Capital  $K_t$  can more precisely be viewed as a measure of capital efficiency. To see this, let me decompose  $K_t$  as  $K_t = e_t k_t$ , where  $e_t$  is a TFP shock and  $k_t$  is physical capital with laws of motion  $de_t = e_t \sigma_k dZ_t^k$  and  $dk_t = (I_t \theta_t - \delta) k_t dt$ , respectively. Then, as given above,

$$\frac{dK_t}{K_t} = \frac{d(e_t k_t)}{e_t k_t} = (I_t \theta_t - \delta) dt + \sigma_k dZ_t^k.$$

it will naturally lead firms and households to try to learn about  $\theta_t$  from observing the growth of capital.<sup>11</sup>

The productivity of real investment evolves according to an Ornstein-Uhlenbeck (OU) process

$$d\theta_t = \lambda_\theta (\bar{\theta} - \theta_t) dt + \sigma_\theta dZ_t^\theta, \quad (2)$$

which has the known solution, found by applying Itô's Lemma to  $e^{\lambda_\theta t} \theta_t$  and integrating from 0 to  $t$ ,

$$\theta_t = \theta_0 e^{-\lambda_\theta t} + \bar{\theta} (1 - e^{-\lambda_\theta t}) + \int_0^t \sigma_\theta e^{\lambda_\theta(s-t)} dZ_s^\theta. \quad (3)$$

The OU process is the continuous-time analogue of an AR(1) process in discrete-time with finite long-run mean  $\bar{\theta}$ , rate of mean-reversion  $\lambda_\theta$ , and iid shocks with (local) instantaneous variance  $\sigma_\theta^2$ .<sup>12</sup>

Households consume the output of firms and invest in two assets traded in centralized markets in the economy: claims to the cash flows of firm capital, which have price  $q_t$ , and (locally) riskless debt, which is an inside asset traded only among households, with instantaneous interest rate  $r_t$ . Importantly, since firm equity and riskless debt are traded in centralized asset markets, their prices are observable to both households and firms when they form their expectations about  $\theta_t$ . In addition, each household can invest in trees in a garden that are in fixed, unit supply that bear aggregate fruit  $A_t$ . These trees are traded through a farmer in a decentralized market with current price  $P_t$ . In contrast to firm equity and riskless debt, the price of trees and their fruit are only observed by the current generation of households, and their price is not recorded into public knowledge.

## B. Households

There is a continuum  $\mathcal{I} = [0, 1]$  of risk-averse households that are part of a non-overlapping generational structure that each live for an instant  $\Delta t$ . I index time for households as  $t, t + \Delta t, t + 2\Delta t$  and consider the continuous-time limit when  $\Delta t$  is of the order  $dt$ . Each generation of household  $i$  inherits wealth  $w_t(i)$  from its parents, which it uses before it dies to consume

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<sup>11</sup>In reality, firms also have idiosyncratic components to their capital growth process that prevent them from learning  $\theta_t$  perfectly by choosing not to invest. I abstract from firm heterogeneity for simplicity.

<sup>12</sup>Theoretically, it is possible for  $\theta_t$  to take negative values, similar to dividends in Wang (1993), though one can choose parameter values so that this occurs with negligible probability. Since beliefs over  $\theta_t$  must be absolutely continuous with respect to the true distribution, such restrictions would apply to the posterior for  $\theta_t$  as well.

$c_t(i)$  and to invest on behalf of its children in asset markets. Each household invests a fraction  $x_t^k(i)$  of its wealth  $w_t(i)$  in firm equity, which is perfectly divisible,  $x_t^A(i)$  in trees, and  $1 - x_t^k(i) - x_t^A(i)$  in riskless debt. Households have log utility over flow consumption  $\log c_t(i)$  and subjective discount rate  $\rho$  over the bequest utility  $v_{t+\Delta t}(i)$  that they leave to future generations.<sup>13</sup>

Similar to Wang (1994), I assume households can invest in the trees of a garden that are in fixed, unit supply. I will refer to this garden as a Lucas orchard since the amount of fruit it bears over time will be exogenous. One can think of trees in the orchard, for instance, as representing a portfolio of mortgages or private loans made to outside entrepreneurs that each household holds on its balance sheet. Specifically, household  $i$  can invest in trees with price  $P_t$  that yield a dividend of the aggregate fruit  $A_t$  at time  $t$ . Households have heterogeneous abilities in managing trees. From tending to their field, household  $i$  gets paid for growing the fruit of its trees  $A_t(i)$  according to the stochastic differential equation

$$\frac{dA_t(i)}{A_t} = \xi_t(i) dt + \sigma_A dZ_t^A,$$

where  $Z_t^A$  is a Wiener process independent from  $Z_t^\theta$  and  $Z_t^\xi$ . The household gets paid according to the fraction of trees it tends to at the rate at which it grows them. Aggregating across the field, the fruit of the orchard grows according to  $dA_t = \int_0^1 \frac{x_t^A(i)w_t(i)}{P_t} dA_t(i) di$ .

The expected growth rate in fruit  $\xi_t(i)$  has a component common across households  $\xi_t$  and an idiosyncratic component  $\sigma_e Z_t^e(i)$  specific to household  $i$ ' skill such that

$$\xi_t(i) = \xi_t + \sigma_e Z_t^e(i),$$

where  $Z_t^e(i)$  is a standard  $N(0, 1)$  random variable that is independent across  $(i, t)$  and independent from  $Z_t^\theta$ ,  $Z_t^k$ ,  $Z_t^\xi$ , and  $Z_t^A \forall (t, i)$ .<sup>14</sup> Since each agents ability to grow fruit is private information, agents cannot credibly contract or sell their information on  $\xi_t(i)$  or  $Z_t^A(i)$ . Households are part of a continuum and, as such, there is no aggregate risk from their idiosyncratic shocks  $\sigma_e Z_t^e(i)$  to the growth rate of fruit in the sense that their sum converges

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<sup>13</sup>From Genotte (1986), general homothetic preferences with incomplete information introduce a negative dynamic hedging term in addition to agents' myopic demand. Brown and Jennings (1989) provides a numerical analysis of the impact on investor trading that this additional hedging term introduces with dispersed information.

<sup>14</sup>One can model this Gaussian process, for instance, as a time-change Wiener process.

to zero in the  $\mathcal{L}^2 - norm$ .<sup>15</sup> Since  $A_t$  will be independent of any individual households' skill, allowing households to tend to trees despite their skill provides diversification for the orchard.<sup>16</sup>

The common component of the expected growth rate of fruit  $\xi_t$  is another OU process that follows the law of motion

$$d\xi_t = \lambda_\xi (\bar{\xi} - \xi_t) dt + \sigma_\xi dZ_t^\xi,$$

where  $\lambda_\xi$  the rate of mean-reversion,  $\bar{\xi}$  finite the long run mean, and  $\sigma_\xi$  the (local) instantaneous variance of its iid innovations. I assume that  $\xi_t$  is initialized from its stationary distribution  $\xi_0 \sim \mathcal{N}\left(\bar{\xi}, \frac{\sigma_\xi^2}{2\lambda_\xi}\right)$ . This common component  $\xi_t$  adds a source of noise  $Z_t^\xi$  to prices that prevents them from fully revealing the fundamental  $\theta_t$ . This innovation  $Z_t^\xi$ , consequently, can be thought of as a pure financial shock to prices. Later, when I consider the impact of a financial shock in my economy, it will be the response of the economy to a unit innovation in  $Z_t^\xi$ , which will feed into real activity through prices through the learning channel. This allows me to focus on the informational effect of one feature of financial crises: asset firesales that depress prices.

Households in the economy have private information about the unobserved productivity of real investment  $\theta_t$ . At each date  $t$ , household  $i$  receives news about  $\theta_t$  through a private signal  $s_t(i)$

$$s_t(i) = \theta_t + \sigma_s Z_t^s(i),$$

where  $Z_t^s(i)$  is a standard  $N(0, 1)$  random variable that represents household  $i$ 's idiosyncratic signal noise that is independent across  $(i, t)$  and independent from  $Z_t^\theta$ ,  $Z_t^\xi$ , and  $Z_t^e(i) \forall (t, i)$ . Households at  $t = 0$  have a common Gaussian prior  $\theta_0 \sim N(\hat{\theta}_0, \Sigma_0)$ .

To simplify my analysis, and to focus on the feedback between the real sector and financial markets from learning, I assume that, while parents in a generation pass along their wealth to their children within a household, they do not pass along their private information, which includes their own private signal, their ability to grow fruit, and their initial wealth. As

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<sup>15</sup>Since convergence of stochastic objects in continuous-time is in the  $\mathcal{L}^2 - norm$ , there is little reason to think about convergence in an *a.s.* sense. There do, however, exist Fubini extensions of the Lebesgue measure for the index of agents such that the convergence is *a.s.* See, for instance, Sun and Zhang (2009).

<sup>16</sup>Alternatively, I could have assumed a common growth rate of fruit  $\xi_t$  and given households private signals  $\xi_t(i)$  about this growth rate. This would complicate the portfolio choice problem for households in choosing their holdings of trees without affecting the results for learning.

discussed in the introduction, models of information aggregation even in static settings are very difficult to solve, and I make this common, simplifying assumption so that learning by households and firms remains tractable. This lets me avoid both the infinite regress problem of Townsend (1983), where market prices partially reveal a moving-average representation of the investment productivity  $\theta_t$ , and a time-varying correlation between the persistence of wealth of households and the persistence of their private beliefs.<sup>17</sup> In addition to making learning intractable, it would also render the equilibrium no longer Markovian.

This assumption about the information structure, however, is not material for the main qualitative insights of my analysis. Relaxing it would introduce an additional component to the riskless rate that reflects that optimistic households tend to be wealthier during expansions and poorer during recessions, similar to Detemple and Murphy (1994) and Xiong and Yan (2010) for heterogeneous beliefs. This effect, however, is not likely to be significant given the nature of uncertainty in the economy. The lower uncertainty at business cycle peaks mitigates the correlation between wealth and beliefs at peaks and during busts because households hold similar beliefs about investment productivity. This dampens the increased price volatility that this interaction would normally introduce.

In addition to their private signal  $s_t(i)$  and expected growth rate of fruit  $\xi_t(i)$  of trees they tend to, all households in a generation observe the history of firm capital in the economy  $K_t$ , investment  $I_t$ , the price of firm equity  $q_t$ , and the riskless rate  $r_t$ .<sup>18</sup> While private information is known by an individual, and would have to be remembered and passed along to progeny, historical public information is kept in public records and is readily available. Let the common knowledge, or public, filtration  $\mathcal{F}_t^c$  be the minimal sigma-algebra generated by these public signals  $\{K_u, I_u, q_u, r_u\}_{u \leq t}$ . Define  $\hat{\theta}_t^c = E[\theta_t | \mathcal{F}_t^c]$  to be the conditional expectation of  $\theta_t$  under the public filtration, where  $E[\cdot | \mathcal{F}_t^c]$  is the conditional expectations operator with respect to the information set  $\mathcal{F}_t^c$ , and  $\Sigma_t = E\left[\left(\theta_t - \hat{\theta}_t^c\right)^2 | \mathcal{F}_t^c\right]$  the conditional variance.

Households form rational expectations about the underlying state  $\theta_t$  by Bayes' Rule given their information set  $\mathcal{F}_t^i = \mathcal{F}_t^c \vee \{w_t(i), A_t, p_t, s_t(i), \xi_t(i)\}$ , which is the public filtration  $\mathcal{F}_t^c$  augmented with the household's private wealth, signal, and private shock to growing fruit. One can thus interpret the information structure of my economy as all households entering the current period with a common, time-varying prior based on the full history of

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<sup>17</sup>Nimark (2012) instead takes the approach of having traders with long-lived private information but static wealth to break the time-varying correlation between trader wealth and private beliefs.

<sup>18</sup>Since output is related to asset growth by  $y_t = aK_t$ , observing asset growth is the same as observing output.

public information  $\mathcal{F}_t^c$ , and then each updates its prior based on its private signal  $s_t(i)$ , wealth  $w_t(i)$ , and private shock to growing fruit  $\xi_t(i)$ . Similar to the public belief, define  $\hat{\theta}_t(i) = E[\theta_t | \mathcal{F}_t^i]$  to be the conditional expectation of  $\theta_t$  of household  $i$ , and  $\Sigma_t(i) = E\left[\left(\theta_t - \hat{\theta}_t^c\right)^2 | \mathcal{F}_t^i\right]$  the conditional variance.

Households in each generation choose their consumption and investment to maximize their utility and their utility bequest to future generations  $v_{t+\Delta t}(i)$ , according to

$$0 = \sup_{\{c_t(i), x_t^k(i), x_t^A(i)\}} \left\{ \rho \Delta t \log c_t(i) + (1 - \rho \Delta t) E[v_{t+\Delta t}(i) | \mathcal{F}_t^i] - v_t(i) \right\}, \quad (4)$$

subject to the law of motion of their wealth  $w_t(i)$  derived below. All households have the same initial wealth  $w_0$ . The optimization problem is solved under household  $i$ 's filtration  $\mathcal{F}_t^i$  which incorporates household  $i$ 's private beliefs about investment productivity  $\theta_t$ .<sup>19</sup>

### C. Firms

I keep the model of firms as simple as possible. There is a continuum of perfectly competitive, identical firms in the economy who issue equity claims to their capital to households and invest  $I_t K_t$  to grow their assets  $K_t$  according to equation (1). They face frictions in adjusting their level of investment and can only imperfectly control it by choosing effort  $g_t$ , so that  $I_t$  evolves according to

$$dI_t = g_t I_t dt.$$

Firms incur a linear cost (or revenue) $\frac{1}{\rho}g_t$  for this adjustment per unit of current investment  $I_t K_t$ . The cost is meant to slow the adjustment of real investment and captures that real investment, in practice, is sluggish. For technical reasons, I assume that firms must maintain a minimal level of investment  $\underline{I}$  such that  $I_t \geq \underline{I}$ . This prevents the signals about investment productivity  $\theta_t$  in the economy from fully flattening, since, if  $I = 0$ , then neither firms nor households care about the productivity of real investment. This condition imposes that  $g_t \geq 0$  if  $I_t = \underline{I}$ , and consequently  $\underline{I}$  is a reflecting boundary for  $I_t$ . For clarity of exposition, I assume that the linear cost  $\frac{1}{\rho}I_t g_t$  is rebated back to firms as a subsidy  $\tau_t K_t$  to simplify calculations, though this is neither necessary for tractability nor for my qualitative results.

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<sup>19</sup>Since households have log utility, and are therefore myopic, their optimal policies for consumption and investment, as well as the pricing kernel implied by their marginal utilities, will be the same regardless of whether they are part of a non-generational structure or long-lived. One can, therefore, also think of the economy as being populated by a continuum of long-lived households that are bounded rationally in their ability to remember private information.

As will be shown, if firms could choose the level of investment  $I_t$  directly, then  $I_t$  would have a well-defined solution between  $\underline{I}$  and  $a$  because the value of their equity  $q_t$  is pinned down by household risk aversion, and is decreasing in  $I_t$ . I choose the linear cost structure so that, with this slow adjustment, firms will have this same optimal  $I_t$  that they are slowly trying to adjust to by varying  $g_t$ . Therefore, the policies with and without the technical restrictions are qualitatively similar.

The choice of functional form for the capital accumulation equation (1) makes transparent the impact of firm beliefs on real investment and uncertainty in the economy, as well as shuts down any variation in the second moment of firm capital accumulation because of investment to turn off the real-options channel featured in Bloom (2009). While this law of motion will give rise to a stark relationship between capital growth and the signal strength of real investment, similar to the production function of firms in Van Nieuwerburgh and Veldkamp (2006), the interaction between investment and the level of uncertainty in determining the behavior of market prices, which is the focus of my analysis, will be an equilibrium outcome. The insights about the relationships explored here will hold more generally as long as firms care about the current, hidden state of the economy when they invest, and that there is more information from real signals when real activity is high.<sup>20</sup>

Households that hold firm equity receive a payment  $D_t$  of the residual cash flow from operations and investment

$$D_t = \left( a - I_t - \frac{1}{\rho} g_t I_t + \tau_t \right) K_t,$$

Firms finance their investment  $I_t K_t$  from their cash flow from operations, the shortfall of which is made up by households through the sale of additional equity. Since financial markets are frictionless, they do not need to hold cash reserves.

For simplicity, firms do not have access to the private information of households and choose investment using only public information. While, in reality, firms are likely to have private information about the idiosyncratic component of their businesses or industries, they still have imperfect knowledge of general macroeconomic trends.<sup>21</sup> For firms to have access

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<sup>20</sup>One may notice that learning from capital accumulation would be strong during recessions as well as expansions if real investment became largely negative. Since aggregate US private nonresidential fixed investment historically has been nonnegative, I abstract from this artifact of the specification of the capital accumulation process.

<sup>21</sup>My mechanism is robust to firms having private information as long as they do not have superior information to households, in which case they would not need to learn from prices. See, for instance, David,

only to public information, they cannot observe the pricing kernels of their investors or their investors' ownership stakes in the firm. If they did, then firms would know the identity of the marginal buyer of its equity, which would allow it to infer information about investment productivity  $\theta_t$ . Given that firms make use of only public information, their investment strategies must be adapted to the common knowledge filtration  $\mathcal{F}_t^c$ .

I assume that firms attempt to maximize shareholder value. Though they must choose their investment policies from "behind the veil", since they do not know the composition of their shareholders, their policies in equilibrium will be robust to this uncertainty. Let  $\Lambda_t$  be the pricing kernel of its shareholders, formed by aggregating the pricing kernels of individual shareholders  $\Lambda_t(i)$  weighted by their stake in the firm  $\frac{x_t^k(i)w_t(i)}{q_t K_t}$ , and  $E_t$  the value of firm equity. Firms then solve the optimization problem

$$E_0 = \sup_{\{g_t\}_{s \geq 0}} E \left[ \int_0^\infty \frac{\Lambda_s}{\Lambda_0} D_s ds \mid \mathcal{F}_0^c \right], \quad (5)$$

subject to the transversality condition

$$\lim_{T \nearrow \infty} E[\Lambda_T E_T \mid \mathcal{F}_0^c] = 0.$$

Since firms are perfectly competitive and atomistic, they take the pricing kernel of their shareholders as given. Though I restrict my attention to equity, it is worth mentioning that, since households have superior information compared to firms about general macroeconomic trends, firms could find it optimal to issue additional securities in order to have more signals from which to learn about the underlying state  $\theta_t$ . Such a richer setting would introduce additional complexity, since instruments like risky debt have nonlinear payoff structures, without adding much additional insight.

#### D. Market Clearing

Household  $i$  takes the net position firm equity  $x_t(i)w_t(i) - q_t k_t(i)$ , where  $q_t k_t(i)$  is its initial holdings. Aggregating over all households then imposes the market clearing condition

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Hopenhayn, and Venkateswaran (2014) for a setting in which firms also observe noisy private signals about their fundamentals.



for the market for firm equity

$$\int_0^1 (x_t(i) w_t(i) - q_t k_t(i)) di = \int_0^1 x_t(i) w_t(i) di - q_t K_t = 0,$$

where  $K_t = \int_0^1 k_t(i) di$  is the total assets of the firm at time  $t$ . Market clearing in the market for riskless debt additionally imposes that

$$\int_0^1 (1 - x_t^k(i) - x_t^A(i)) w_t(i) di = 0,$$

and for the Lucas orchard

$$\int_0^1 (x_t^A(i) w_t(i) - P_t) di = 0.$$

Finally, there is market clearing in the market for output

$$\int_0^1 c_t(i) di + I_t K_t = a K_t + \int A(i) di$$

I search for a recursive competitive noisy rational expectations equilibrium.

### *E. Recursive Competitive Noisy Rational Expectations Equilibrium*

Let  $\omega$  be a state vector of publicly observable objects. A recursive competitive equilibrium for the economy is a list of policy functions  $c(w(i), \hat{\theta}(i), \xi(i), \omega)$ ,  $x(w(i), \hat{\theta}(i), \xi(i), \omega)$ , and  $g(\omega)$ , value functions  $v(w(i), \hat{\theta}(i), \xi(i), \omega)$  and  $E(\omega)$ , and a list of prices  $\{q(\omega), r(\omega), P(\omega)\}$  with  $q(\omega), P(\omega) \geq 0$  such that

- **Household Optimization:** For every  $\omega$  and  $i$ , given prices  $\{q(\omega), r(\omega), P(\omega)\}$ , policies  $c(w(i), \hat{\theta}(i), \xi(i), \omega)$ ,  $x^k(w(i), \hat{\theta}(i), \xi(i), \omega)$ , and  $x^A(w(i), \hat{\theta}(i), \xi(i), \omega)$  solve each household's problem (4) and deliver value  $v(w(i), \hat{\theta}(i), \xi(i), \omega)$
- **Firm Optimization:** For every  $\omega$ , given prices  $\{q(\omega), r(\omega)\}$ , policy  $g(\omega)$  solves the firm's problem (5) and delivers value  $E(\omega)$

- Market Clearing: The markets for output, firm equity, trees, and riskless debt clear

$$: \int_0^1 c \left( w(i), \hat{\theta}(i), \xi(i), \omega \right) di + I(\omega) K = aK \quad (\text{output}) \quad (6)$$

$$: \int_0^1 x^k \left( w(i), \hat{\theta}(i), \xi(i), \omega \right) w(i) di = qK \quad (\text{firm equity}) \quad (7)$$

$$: \int_0^1 \left( x^A \left( w(i), \hat{\theta}(i), \xi(i), \omega \right) w(i) - P \right) di = 0 \quad (\text{trees}), \quad (8)$$

$$: 0 = \int_0^1 \left( 1 - x^k \left( w(i), \hat{\theta}(i), \xi(i), \omega \right) \right) w(i) di$$

$$: \quad - \int_0^1 \left( x^A \left( w(i), \hat{\theta}(i), \xi(i), \omega \right) \right) w(i) di \quad (\text{riskless debt}), \quad (9)$$

- Consistency:  $w(i)$  follows its law of motion  $\forall i \in [0, 1]$ , household  $i$  forms its expectation about  $\theta$  based on its information set  $\mathcal{F}^i$ , and firms form their expectation about  $\theta$  based on their information set  $\mathcal{F}^c$  according to Bayes' Rule

and the transversality conditions are satisfied.

## IV. The Equilibrium

I first state the main proposition of the section and then build up to this proposition in a sequence of key steps. Following the noisy rational expectations literature, I will conjecture locally linear processes for asset prices in the economy, and then verify that this constitutes an equilibrium.

**Proposition 1** *There exists a locally linear noisy rational expectations equilibrium in which the riskless return  $r$  is given by*

$$r = \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta(I, \Sigma) \left( \theta - \hat{\theta}^c \right) + \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi + \rho + \frac{\sigma_A^2}{\sigma_A^2 + \sigma_k^2} \left( I \hat{\theta}^c - \frac{I}{a - I} g - \delta - \sigma_k^2 \right),$$

where  $R_\theta(I, \Sigma)$  solves the cubic equation

$$R_\theta(I, \Sigma)^3 - \frac{\sigma_A^2}{\sigma_k^2} I R_\theta(I, \Sigma)^2 + \left( \frac{1}{\sigma_s^2} + \frac{1}{\Sigma} \right) \sigma_e^2 R_\theta(I, \Sigma) - \frac{\sigma_A^2 \sigma_e^2}{\sigma_k^2 \sigma_s^2} I = 0. \quad (10)$$

each household's investment in firm equity  $x^k(i)$  and in trees  $x^A(i)$  is

$$\begin{bmatrix} x^k(i) \\ x^A(i) \end{bmatrix} = \begin{bmatrix} \frac{\rho - \frac{I}{a-I}g + I\hat{\theta}(i) - r - \delta}{\sigma_k^2} \\ \frac{\rho + \xi(i) - r}{\sigma_A^2} \end{bmatrix}.$$

and optimal investment by firms  $g$  satisfies

$$g = \rho \left( q\hat{\theta}^c - 1 \right) \mathbf{1} \left\{ I > \underline{I} \cup \hat{\theta}^c \geq \frac{\rho}{a - \underline{I}} \right\}.$$

Similar to He and Wang (1995), individual households take a position in firm equity that can be decomposed into a component common to all households  $\frac{\rho + I\hat{\theta}^c - \frac{I}{a-I}g - r - \delta}{\sigma_k^2}$  and a term that reflects their informational advantage based on their private information  $\frac{I}{\sigma_k^2} \left( \hat{\theta}(i) - \hat{\theta}^c \right)$ . This informational advantage term reflects disagreement among households about the Sharpe Ratio of investing in firm capital. In contrast to He and Wang (1995), and other financial models of dispersed information like Foster and Viswanathan (1996) and Allen, Morris, and Shin (2006), the intensity with which households trade on their private information is influenced by real factors in the economy. Though private information is static, since the private information of households is short-lived because of the non-generational structure, the intensity with which households trade on their private information now displays business cycle variation because the environment in which they trade is time-varying.

As is common in general equilibrium models of production, interest rates adjust until all wealth is invested in firm equity and trees. Focusing on the interaction between financial markets and real investment necessitates the adoption of such a setting that has this feature. In models of heterogeneous beliefs, such as Detemple and Murphy (1994) and Xiong and Yan (2010), the riskless rate  $r$ , which is the price at which relative pessimists are willing to offer leverage to relative optimists to hold all firm equity in equilibrium, reflects the disagreement among households about real investment productivity  $\theta_t$ . In my setting, it serves to aggregate their private information. This riskless rate falls during recessions to raise the expected excess return to firm equity, and shift down the level of optimism of the marginal buyer so that enough households purchase firm equity for asset markets to clear. Similarly, it rises during booms to shift up the level of optimism of the marginal buyer to curb the high demand of households for firm equity because of limited supply.

The market clearing condition for riskless debt effectively pins down the risk premium

on firm equity required for asset markets to clear. As such, one can view the excess return on equity as being the relevant market rate that aggregates information. Alternatively, one could interpret the interest rate in my stylized setting as being a composite market rate that arises from the trading of a well-diversified portfolio of securities. In the empirical discussion, I focus on the excess return to firm equity, or the spread between the return to firm equity and this riskless interest rate, to try to avoid taking a stance on which market rates have informative content.

The first step in solving the equilibrium is to conjecture functional forms for asset prices  $q$ ,  $r$ , and  $P(i)$ . I anticipate that the price of firm equity  $q$  will be a linear function of real investment  $I$ . I also conjecture that the riskless rate  $r$  will be a linear function of the tracking error about real investment productivity of the public belief  $\theta - \hat{\theta}^c$ , with a coefficient that varies with real investment  $I$  and the conditional variance of the public belief  $\Sigma$ , and the common component of expected growth in fruit  $\xi$ . Similarly, I conjecture that the price of trees  $P$  will be a linear function of its aggregate fruit  $A$ , so that

$$\begin{aligned} q &= q_I I, \\ r &= r_0 + r_\theta(I, \Sigma) \left( \theta - \hat{\theta}^c \right) + r_\xi \xi, \\ P &= p_A A, \end{aligned} \tag{11}$$

where  $r_\theta(I, \Sigma) \in \mathcal{F}^c$  since  $(I, \Sigma) \in \mathcal{F}^c$ . I assume that  $|r_\xi|^{-1} > 0$  and that  $r_\theta(I, \Sigma)$  is uniformly bounded and nonvanishing *a.s.* Since the fruit from the garden is independent of an individual households' skill, it follows that the price of each tree can only be a function of public state variables and the fruit  $A$ .

The next step toward solving the equilibrium is to solve for the consumption and portfolio choice of household  $i$  given its information set  $\mathcal{F}^i$ . I now derive the law of motion of the wealth of household  $i$   $w(i)$ . Define the vector  $X(i) = \left[ x^k(i) \quad x^A(i) \right]'$ . Applying Itô's Lemma to  $K$  and  $P$ , the wealth of household  $i$   $w(i)$  then evolves according to

$$dw(i) = (rw(i) - c(i)) dt + w(i) X(i)' \left[ \begin{array}{c} \frac{(a-I)Kdt + Kdq + qdK}{qK} - rdt \\ \frac{Adt + dP}{P} - rdt \end{array} \right],$$

which can be expanded to yield

$$dw(i) = (rw(i) - c(i)) dt + w(i) X(i)' \begin{bmatrix} \left(\frac{a-I}{q} - r\right) dt + \left(\frac{dq}{q} + \frac{dK}{K}\right) \\ \left(\frac{A}{P} - r\right) dt + \frac{dP}{P} \end{bmatrix}, \quad (12)$$

and is irrespective of the measure. The variance term for  $\frac{dK}{K}$  is irrespective of the measure because of diffusion invariance. Intuitively, it is easier to estimate variances than means of processes, so that even if two households disagreed on the drift of a process, they cannot disagree on its variance. The dividend  $a - I$  reflects the dividend  $D$  after the rebate for the adjustment cost.

To make progress in solving household  $i$ 's problem, I analyze each household's problem (4) in the limit as  $\Delta t \searrow dt$ . Since uncertainty over  $\theta_t$  represents a compound lottery for households over the uncertainty in the change in  $\theta_t$ , I can separate their filtering from their optimization problem and treat  $\hat{\theta}_t(i) = E[\theta_t | \mathcal{F}_t^i]$  with conditional variance  $\Sigma_t(i) = E\left[\left(\theta_t - \hat{\theta}_t(i)\right)^2 | \mathcal{F}_t^i\right]$  as  $\theta_t$  in their optimization problem.

Given that households have log preferences over consumption, households will optimally consume a fixed fraction of their wealth at each date  $t$ . Furthermore, they will also choose a myopic portfolio in the sense that it maximizes the Sharpe Ratio of their investment and ignores market incompleteness. This is summarized in the following proposition.

**Proposition 2** *The household's value function takes the form  $v(w(i), \hat{\theta}(i), \xi(i), h) = \frac{1}{\rho} \log w(i) + f(\hat{\theta}(i), \xi(i), h)$ , where  $h_t$  is a vector of general equilibrium objects. Furthermore, the household's optimal consumption and portfolio choice take the form*

$$\begin{aligned} c(i) &= \rho w(i), \\ X(i) &= \begin{bmatrix} \frac{\frac{a-I}{q} + \frac{\partial I^q}{q} I g + I \hat{\theta}(i) - r - \delta}{\frac{\sigma_k^2}{\frac{1}{p_A} + \xi(i)} - r} \\ \frac{\sigma_A^2}{\sigma_A^2} \end{bmatrix}. \end{aligned}$$

Furthermore, define  $\Lambda_t(i) = e^{-\rho t} \frac{1}{w_t(i)}$  to be the pricing kernel of household  $i$ . Then the riskless rate and risky firm equity satisfy  $\forall i$

$$\begin{aligned} r &= -\frac{1}{dt} E \left[ \frac{d\Lambda(i)}{\Lambda(i)} \mid \mathcal{F}^i \right], \\ \vec{0}_{2 \times 1} &= \begin{bmatrix} \frac{a-I}{q} dt + E \left[ \frac{d(\Lambda(i)qK)}{\Lambda(i)qK} \mid \mathcal{F}^i \right] \\ \frac{1}{p_A} dt + E \left[ \frac{d(\Lambda(i)P)}{\Lambda(i)P} \mid \mathcal{F}^i \right] \end{bmatrix}. \end{aligned}$$

An immediate observation is that, similar to Detemple (1986), a separation principle applies in my noisy rational expectations equilibrium: the optimal consumption and investment policies are chosen independent of the learning process. Intuitively, since households are fully rational and update their beliefs with Bayesian learning, I can separate the filtering problem faced by households from their consumption choices and portfolio optimization.

Given the optimal choice of consumption  $c(i) = \rho w(i)$  from the proposition, it follows that the law of motion of  $w(i)$  can be written as

$$\frac{dw(i)}{w(i)} = (r - \rho) dt + X(i)' \left[ \begin{array}{l} \left( \frac{a-I}{q} + \frac{\partial I q}{q} I g - r \right) dt + \frac{dK}{K} \\ \left( \frac{1}{p_A} - r \right) dt + \frac{dA(i)}{A} \end{array} \right], \quad (13)$$

which is also irrespective of the measure because of diffusion invariance.

From the market clearing conditions for the market for firm equity, trees, and riskless inside debt (7), (8), and (9), total wealth in the economy is given by

$$W = qK + P. \quad (14)$$

Equation (14) states that, in equilibrium, the total wealth in the economy  $W$  is equal to the total value of firm assets  $qK$  and trees  $P$ . Substituting  $c(i) = \rho w(i)$  and equation (14) into the market clearing condition for output (6), it follows that

$$qK + P = \frac{a - I}{\rho} K + \frac{A}{\rho}. \quad (15)$$

The total value of assets in the economy is equal to the output from firm capital and trees divided by the common household discount rate  $\rho$ . Since the dividends  $A_t$  from trees are independent of the output from capital, it follows that

$$P = \frac{A}{\rho},$$

so that households consume all of the dividends of trees in equilibrium, since  $\rho P = A$ . Consequently, from equation (15), the price of firm equity is then

$$q = \frac{a - I}{\rho}, \quad (16)$$

from which follows that  $\frac{a-I}{q} = \rho$  and  $\frac{1}{p_A} = \frac{A}{P} = \rho$ , and households, in equilibrium, receives a

constant dividend yield from their assets.

I now derive the conditional beliefs of households and firms about  $\theta_t$  with respect to the common knowledge filtration  $\mathcal{F}^c$  and households' private information sets  $\mathcal{F}^i$ . The public signals that households have available for forming their expectations are  $\log K$ ,  $q$ ,  $I$ , and  $r$ . Since firms only have access to public information, it must be the case that firm investment  $I \in \mathcal{F}^c$ . Consequently, there is no additional information contained in  $I$ , or  $q$  given equation (16), once households have formed their beliefs. I can then generate the public filtration  $\mathcal{F}^c$  with these two public signals  $\mathcal{F}^c = \sigma(\{\log K_u, r_u\}_{u \leq t})$ .

Given equation (11), one can construct the public signal  $S$

$$S = \frac{r - r_0 + r_\theta(I, \Sigma) \hat{\theta}^c}{r_\xi} = R_\theta(I, \Sigma) \theta + \xi. \quad (17)$$

Further assuming that  $R_\theta$  is a process of finite total variation, applying Itô's Lemma to  $S$ ,  $S$  follows the law of motion

$$\begin{aligned} dS &= \left( \partial_\Sigma R_\theta \frac{d\Sigma}{dt} + \partial_I R_\theta I g \right) \theta dt + R_\theta \lambda_\theta (\bar{\theta} - \theta) dt + \lambda_\xi (\bar{\xi} - S + R_\theta \theta) dt \\ &\quad + R_\theta \sigma_\theta dZ^\theta + \sigma_\xi dZ^\xi. \end{aligned}$$

Given these arguments, I can construct the vector of public signals  $\zeta = \left[ \log K \quad S \right]'$  whose history, along with initial household wealth  $w_0$  and firm assets  $K_0$ , generate the information set  $\mathcal{F}^c$ .<sup>22</sup> Assuming that households and firms using only the history of the public signals have a normal prior about  $\theta_t$ , then after observing the two conditionally normal signals from  $\zeta_t$ , their optimal updating rule for their beliefs about  $\theta_t$  is linear, and their posterior belief about  $\theta_t$  will also be conditionally normal. In continuous-time, these updating rules characterize the laws of motion for the conditional expectation and variance of these beliefs,  $\hat{\theta}^c = E[\theta | \mathcal{F}^c]$  and  $\Sigma = E\left[\left(\theta - \hat{\theta}^c\right)^2 | \mathcal{F}^c\right]$ , respectively. Households then update these public estimates with their conditionally normally distributed private signals,  $s_t(i)$  and  $\xi_t(i)$ , following another linear updating rule.<sup>23</sup> I then have the following result.

**Proposition 3** *The conditional belief of households using only public information is Gaussian*

<sup>22</sup>In addition,  $\zeta$  contains  $\hat{\theta}^c$ ,  $\Sigma$ , and  $I$ , which are all publicly observable, though I suppress these arguments from the vector for simplicity since they do not contain new information about  $\theta_t$ .

<sup>23</sup>Since household private wealth  $w_t(i)$  contains the unobserved history of fundamental and private shocks  $\{Z_u^\theta, Z_u^k, Z_u^\xi, Z_u^s(i), Z_u^a(i), Z_u^e(i)\}_{u \leq t}$ , there is little information to be gleaned from observing it and households can safely ignore it in their learning.

with conditional expectation  $\hat{\theta}^c = E[\theta \mid \mathcal{F}^c]$  and conditional variance  $\Sigma = E\left[\left(\theta - \hat{\theta}^c\right)^2 \mid \mathcal{F}^c\right] \in \left[0, \frac{\sigma_\theta^2}{2\lambda_\theta}\right]$  that follow the laws of motion

$$d\hat{\theta}^c = \lambda_\theta \left(\bar{\theta} - \hat{\theta}^c\right) dt + \sigma_{\hat{\theta}^k}(I, \Sigma) d\tilde{Z}^k + \sigma_{\hat{\theta}^r}(I, \hat{\theta}^c, \Sigma) d\tilde{Z}^r,$$

where

$$\begin{aligned} \sigma_{\hat{\theta}^k}(I, \Sigma) &= I \frac{\Sigma}{\sigma_k}, \\ \sigma_{\hat{\theta}^r}(I, \hat{\theta}^c, \Sigma) &= \frac{R_\theta(I, \Sigma) \sigma_\theta^2 + (\partial_\Sigma R_\theta(I, \Sigma) \frac{d\Sigma}{dt} + \partial_I R_\theta(I, \Sigma) I g + (\lambda_\xi - \lambda_\theta) R_\theta(I, \Sigma)) \Sigma}{\sqrt{(R_\theta(I, \Sigma) \sigma_\theta)^2 + \sigma_\xi^2}}, \end{aligned}$$

and

$$\frac{d\Sigma}{dt} = -\frac{B}{2A} + \frac{1}{2A} \sqrt{2B + 4A \left( \sigma_\theta^2 - 2\lambda_\theta \Sigma - I^2 \frac{\Sigma^2}{\sigma_k^2} \right) - 1}$$

with

$$\begin{aligned} A &= \frac{(\partial_\Sigma R_\theta(I, \Sigma) \Sigma)^2}{(R_\theta(I, \Sigma) \sigma_\theta)^2 + \sigma_\xi^2}, \\ B &= 1 + 2\partial_\Sigma R_\theta(I, \Sigma) \Sigma \frac{R_\theta(I, \Sigma) \sigma_\theta^2 + (\partial_I R_\theta(I, \Sigma) I g + (\lambda_\xi - \lambda_\theta) R_\theta(I, \Sigma)) \Sigma}{(R_\theta(I, \Sigma) \sigma_\theta)^2 + \sigma_\xi^2}. \end{aligned}$$

and

$$\begin{aligned} d\tilde{Z}^k &= \frac{d \log K + \left(\frac{1}{2}\sigma_k^2 + \delta - I\hat{\theta}^c\right) dt}{\sigma_k}, \\ d\tilde{Z}^r &= \frac{dS - \left(\lambda_\xi \left(\bar{\xi} - S + R_\theta(I, \Sigma) \hat{\theta}^c\right) + (\partial_\Sigma R_\theta(I, \Sigma) \frac{d\Sigma}{dt} + \partial_I R_\theta(I, \Sigma) I g) \hat{\theta}^c\right) dt}{\sqrt{(R_\theta(I, \Sigma) \sigma_\theta)^2 + \sigma_\xi^2}} \\ &\quad - \frac{R_\theta(I, \Sigma) \lambda_\theta}{\sqrt{(R_\theta(I, \Sigma) \sigma_\theta)^2 + \sigma_\xi^2}} \left(\bar{\theta} - \hat{\theta}^c\right) dt, \end{aligned}$$

form a vector of standard Wiener processes with respect to  $\mathcal{F}^c$ .

The conditional expectation of  $\theta$  of household  $i$   $\hat{\theta}(i) = E[\theta \mid \mathcal{F}^i]$  and the conditional



variance  $\Sigma(i) = E \left[ \left( \theta - \hat{\theta}(i) \right)^2 \mid \mathcal{F}^i \right]$  are related to the public estimates  $\hat{\theta}^c$  and  $\Sigma$  by

$$\begin{aligned} \hat{\theta}(i) &= \hat{\theta}^c + \frac{\Sigma \sigma_e^2}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} (s(i) - E[s(i) \mid \mathcal{F}^c]) \\ &\quad - \frac{\sigma_s^2 R_\theta(I, \Sigma) \Sigma}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} (\xi(i) - E[\xi(i) \mid \mathcal{F}^c]), \\ \Sigma(i) &= \frac{\sigma_s^2 \sigma_e^2}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} \Sigma. \end{aligned}$$

The conditional mean of the public or common knowledge belief  $\hat{\theta}^c$  is derived from the endogenous public signals  $\log K$  and  $r$ , while each household's private belief  $\hat{\theta}(i)$  is a linear combination of this public belief and their private signal  $s_t(i)$  and expected growth of fruit  $\xi_t(i)$ . This first moment of public beliefs  $\hat{\theta}^c$  is an important state variable because it survives the aggregation of the beliefs of households, and because it is the belief about real investment productivity  $\theta$  of firms. Similar to the Kalman Filter in discrete-time, the loadings on the normalized innovations  $d\tilde{Z}^k$  and  $d\tilde{Z}^r$  formed from the real investment and market signals,  $\sigma_{\hat{\theta}^k}$  and  $\sigma_{\hat{\theta}^r}$ , respectively, represent the Kalman Gains of the public signals. Changes in the first moment of public beliefs  $\hat{\theta}^c$  are a linear combination of a term capturing the deterministic mean-reversion of investment productivity,  $\lambda_\theta (\bar{\theta} - \hat{\theta}^c) dt$ , and a stochastic component related to the news from the innovations to the public signals,  $\sigma_{\hat{\theta}^k}(I, \Sigma) d\tilde{Z}^k + \sigma_{\hat{\theta}^r}(I, \hat{\theta}^c, \Sigma) d\tilde{Z}^r$ .

The law of motion of the second moment of public beliefs  $\Sigma$ , in contrast, is (locally) deterministic and is the continuous-time analogue of the Ricatti equation for the Kalman filter.<sup>24</sup> To help interpret it, I rewrite it as

$$\frac{d\Sigma}{dt} = \sigma_\theta^2 - 2\lambda_\theta \Sigma - \sigma_{\hat{\theta}^k}(I, \Sigma)^2 - \sigma_{\hat{\theta}^r}(I, \hat{\theta}^c, \Sigma)^2,$$

where  $\sigma_{\hat{\theta}^r}(I, \hat{\theta}^c, \Sigma)$  is a function of  $\frac{d\Sigma}{dt}$ . The first term  $\sigma_\theta^2$  reflects the increase in uncertainty from the new unobservable innovation to  $\theta$ , while the second term  $-2\lambda_\theta \Sigma$  captures the fall in uncertainty because the effect of past shocks fade because of mean-reversion. The last two terms  $-\sigma_{\hat{\theta}^k}(I, \Sigma)^2 - \sigma_{\hat{\theta}^r}(I, \hat{\theta}^c, \Sigma)^2$  represents the decrease in uncertainty from observing

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<sup>24</sup>The law of motion in Proposition 3 suggests it is possible for a solution to  $\frac{d\Sigma}{dt}$  not to exist. Nonexistence can occur because learning from market prices leads to the simultaneous determination of the change in the conditional variance  $\frac{d\Sigma}{dt}$  and the strength of the market signal  $\sigma_{\hat{\theta}^r}$ , which is a function of  $\frac{d\Sigma}{dt}$ . In all of my numerical simulations, nonexistence has never been an issue.

the real investment and market signals. The conditional variance of beliefs  $\Sigma$  is stochastic unconditionally because it is a function of the first moment of public beliefs  $\hat{\theta}^c$  through real investment  $I$ . As a result, it can never settle down to a steady-state since  $\hat{\theta}^c$ , which tracks the true investment productivity  $\theta$ , can never settle down.

An important feature of the optimal filter is that the conditional variance of public beliefs  $\Sigma$  is time-varying over the business cycle, and fluctuates endogenously according to its law of motion given in Proposition 3, which depends on its current value, the filtered investment productivity  $\hat{\theta}^c$ , and the level of investment by firms  $I$ . The stochastic time-variation in  $\Sigma$  is in contrast to dynamic models of asymmetric information like Wang (1993) that focus on the steady-state solution for the conditional variance of beliefs to which the economy tends deterministically. In this setting,  $\Sigma$  influences the quantity of private information households have, and how they trade on it in financial markets. As a result of shutting down the "wait and see" channel of Bloom (2009) for uncertainty to feed into firm investment behavior, firm investment decisions are indirectly influenced by  $\Sigma$  through how it affects the informativeness of the real and financial signals. Since  $\Sigma$  is time-varying, it is part of the state vector, along with  $I$  and  $\hat{\theta}^c$ , that summarizes the current state of the economy.

In addition to their private signals, households learn about the underlying strength of the economy  $\theta$  from the growth of firm assets  $\log K$ , whose informativeness (signal-to-noise ratio) is increasing in the level of real investment  $I$ , and from the riskless rate, whose informativeness  $R_\theta(I, \Sigma)$  is also influenced by  $I$ . This link from the investment choices of firms to the learning process of households represents one part of the feedback loop between real activity and asset markets that I wish to highlight. The ability of real investment decisions to distort investor expectations is similar to the channel explored in Angeletos, Lorenzoni, and Pavan (2012) to rationalize the tech bubble of the early 2000's.

I now turn to the problem faced by firms. Given that firms only have access to public information, their conditional expectation of  $\theta$  when making their investment decision  $g$  is  $\hat{\theta}^c$ . Furthermore, since the price of firm equity is pinned down by market clearing  $q = \frac{a-I}{\rho}$ , it must be the case that the optimal choice of  $g$  under the pricing kernel of investors confirms this price.

**Proposition 4** *The value of firm equity is given by  $E = qK$ , and the optimal level of investment is given by*

$$g = \rho \left( q\hat{\theta}^c - 1 \right) \mathbf{1}_{\{I > I \cup \hat{\theta}^c \geq \frac{\rho}{a-I}\}}. \quad (18)$$

From the functional form of the optimal investment policy, it is apparent that  $I = \underline{I}$  and  $I = a$  are reflecting boundaries, since when  $I = a$ , then  $q = 0$  and  $g < 0$ . As a result, the price of firm equity can never be negative. Similarly, when  $I = \underline{I}$  and  $\hat{\theta}^c \leq \frac{\rho}{a-\underline{I}}$ , then  $dI = 0$  and investment stays at  $\underline{I}$  until  $g$  becomes positive. Since  $I$  has finite variation, its sample paths are continuous in time, and  $I$  will approach its two boundaries continuously.

To see how investment in my setting compares to one in which I allow firms to freely choose  $I$ , it is easy to see that the FOCs for the firm's problem would then be

$$-1 + q\hat{\theta}^c \leq 0,$$

with equality when  $\hat{\theta}^c \geq \frac{\rho}{a-\underline{I}}$  since  $q = \frac{a-I}{\rho}$ , from which it follows that  $I^{opt} = \underline{I} + \left(a - \frac{\rho}{\hat{\theta}^c} - \underline{I}\right) \mathbf{1}_{\{\hat{\theta}^c \geq \frac{\rho}{a-\underline{I}}\}}$ . Since firms choose bang-bang policies, the price of capital  $q$  adjusts to make them indifferent to the optimal level of investment  $I^{opt}$ . Notice that when  $I = a - \frac{\rho}{\hat{\theta}^c}$  when  $I$  can only be slowly adjusted, then  $I = I^{opt}$  and  $g = 0$ . If  $I$  were above its optimal value  $I > I^{opt}$ , then  $g > 0$ , and similarly  $g < 0$  when  $I$  is below its optimal value  $I < I^{opt}$ . Thus  $g$  tries to adjust  $I$  toward the optimal level the firm would choose if  $I$  could be chosen freely. This is the sense in which investment is sluggish.

Given the solution to the optimal investment strategy of firms,  $q$  has the interpretation of being Tobin's  $q$ . Investment by firms aims to equate the perceived productivity of real investment  $\hat{\theta}^c$  to  $1/q$ , the book-to-market value of its assets. Thus informational frictions distort firm real investment by creating a misperception about the value of their assets. This highlights a key difference between my channel for firm beliefs to distort real activity and that of Straub and Ulbricht (2013). In their setting, entrepreneurs are never confused about the optimal level of production, but rather about the value of the collateral they must pledge to workers because of financial frictions. In my setting, firms optimally choose a level of production that is distorted because of their beliefs about investment productivity. Also, in contrast to models of uncertainty like Bloom (2009), investment in my economy declines because of shocks to the first moment of productivity rather than from shocks to the second moment through a "real-options" channel.

Learning by firms introduces a channel through which the first moment of beliefs about investment productivity  $\hat{\theta}^c$  influences the second moment  $\Sigma$ . From Proposition 3, the change in uncertainty  $\Sigma \frac{d\Sigma}{dt}$  depends on the change in investment  $g$ , which is a function of firm beliefs  $\hat{\theta}^c$ . Thus the filtering equations for  $\hat{\theta}^c$  and  $\Sigma$  are coupled because there is feedback

from second moments to first moments, which is a natural feature of the optimal nonlinear filter, and from first moments to second moments, because learning by firms determines their investment decisions, which influences the informativeness of the two public signals.

Since household trading behavior impacts the riskless rate  $r$ , from which both households and firms learn, the riskless rate acts as a channel for financial market shocks to feed into real investment decisions by influencing firm expectations. This mechanism for asset prices to distort firm investment is similar to Goldstein, Ordozen, and Yuan (2013). Along with the impact of investment decisions on household learning discussed above, these two forces characterize the feedback loop in learning between financial markets and real activity.

To derive the functional form for the riskless rate  $r$ , I must aggregate the wealth-weighted private expectations of all households, which will reveal the current true  $\theta_t$  and the signal noise of households. Given that the private beliefs of each household are uncorrelated with their wealth share because households do not pass along their private information to later generations, the Law of Large Numbers will cause the aggregation of idiosyncratic signal noise to vanish. Let  $W = \int_0^1 w(i) di$  be the total wealth of all households. Then I obtain the following result.

**Proposition 5** *Aggregating the wealth-weighted deviation in the conditional expectation  $\theta$  of household  $i$   $\hat{\theta}(i)$  from the common knowledge expectation  $\hat{\theta}^c$  yields a.s.*

$$\int \frac{w(i)}{W} \left( I \left( \hat{\theta}(i) - \hat{\theta}^c \right) - \xi(i) \right) di = I \frac{\Sigma \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} \left( \theta - \hat{\theta}^c \right) - \xi,$$

and the convergence  $\forall t$  is in the  $\mathcal{L}^2$  - norm.

By aggregating the beliefs of individual households, the riskless rate  $r$  partially reveals the true real investment productivity  $\theta$  and the common knowledge estimate  $\hat{\theta}^c$ .<sup>25</sup> Given this aggregation result, the noisy rational expectations equilibrium and the riskless rate  $r$  then satisfy the main theorem of the section, and it follows that the state vector  $\omega$  for the economy is  $\omega = \left[ \theta, \xi, \hat{\theta}^c, I, \Sigma \right]$ .

Having characterized the equilibrium, I now highlight several properties of  $R_\theta$ .

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<sup>25</sup>A caveat to this result is that it relies on households being symmetrically informed. If, instead, households had different signal precisions  $\sigma_s(i)$ , then the wealth distribution of households would matter for prices. Asymptotically, however, one would expect households with superior information to eventually drive out the less well-informed households. This would lead to a degenerate wealth distribution in which wealth once again does not matter.

**Proposition 6** Suppose  $\sigma_s < 1$ , then it is sufficient for  $\sigma_e > \frac{a}{2} \frac{\sigma_A^2}{\sigma_k^2} \sqrt{18 + \sigma_s^2}$  for  $R_\theta \in \frac{\sigma_A^2}{\sigma_k^2} I \left[ \frac{\Sigma}{\Sigma + \sigma_s^2}, 1 \right]$  in equation (10) to be unique. In addition,  $R_\theta$  has the following properties: 1)  $R_\theta$  is (weakly) increasing in  $I$  and  $\Sigma$ , and (weakly) decreasing in  $\sigma_s$  and  $\sigma_e$ , 2)  $R_\theta \searrow \frac{\sigma_A^2}{\sigma_k^2} I \frac{\Sigma}{\Sigma + \sigma_s^2}$  as  $\sigma_e \nearrow \infty$  and  $R_\theta \searrow 0$  as  $\sigma_s \nearrow \infty$ , 3)  $R_\theta \nearrow \frac{\sigma_A^2}{\sigma_k^2} I$  as  $\min \{ \sigma_s, \sigma_e \}$  approaches 0, and 4) if  $\sigma_e \geq a \frac{\sigma_A^2}{\sigma_k^2} \sqrt{\frac{\sigma_\theta^2}{\lambda_\theta} + \sigma_s^2}$ , then  $\partial_I R_\theta \leq \frac{\sigma_A^2}{\sigma_k^2}$ .

Proposition 6 provides a sufficient condition for  $R_\theta$  to be unique, as well as several of its comparative statics and limiting properties.<sup>26</sup> The proposition also reveals that the expected return to trees  $\xi(i)$  acts as an additional signal for households about  $\theta$  insofar as households have private information about  $\theta$ ,  $s(i)$ , that is aggregated in the riskless rate  $r$ . As the idiosyncratic variance in  $\xi(i)$ ,  $\sigma_e$ , becomes arbitrarily large, households can ignore the information content in  $\xi(i)$  without significant effect. This is similar to the assumption of large idiosyncratic shocks in Diamond and Verrechia (1981). In this limit, where  $R_\theta = \frac{\sigma_A^2}{\sigma_k^2} I \frac{\Sigma}{\Sigma + \sigma_s^2}$ , it is transparent how the information content in the riskless rate  $R_\theta$  is formed. While the intensity with which households trade on their private information is procyclical, since  $\frac{I}{\sigma_k^2} \hat{\theta}(i)$  in  $x^k(i)$  is increasing in real investment by firms  $I$ , the information content in the market price is also increasing in uncertainty about  $\theta$ , measured by  $\Sigma$ , because market prices aggregate the private information of households to partially reveal  $\theta$ . These two forces interact so that asset prices will be strongest, in the sense that the variation in  $\hat{\theta}^c$  driven by the market signal is largest, when  $I$  and  $\Sigma$  are in an intermediate range.

In addition to the strength of the market signal, the loading that households and firms place on the market signal when updating their beliefs also displays business cycle variation. Figure 1 in Appendix A plots, as a numerical example, the loading of the market signal  $\sigma_{\hat{\theta}^c} \left( I, \hat{\theta}^c, \Sigma \right)$  for a fixed level of filtered investment productivity  $\hat{\theta}^c$  for the parameters listed in Table 1. The plot reveals the rich interaction between investment  $I$  and uncertainty  $\Sigma$  in determining the weight households and firms put on the market signal. Numerical experiments suggest that  $\sigma_{\hat{\theta}^c} \left( I, \hat{\theta}^c, \Sigma \right)$  is increasing in the perceived investment productivity  $\hat{\theta}^c$ . These observations illustrate that more of the variation in the beliefs in households and firms is driven by the market signal when  $I$  and  $\Sigma$  are in an intermediate range.

This last point merits some emphasis. While it is well-appreciated that risk premia in financial markets are countercyclical, it is less appreciated that the strength of asset prices

<sup>26</sup>Though sufficient, the condition in Proposition 6 is not necessary. I find that the solution to  $R_\theta(I, \Sigma)$  is unique for a wider range of parameter values.

as signals of economic strength also exhibits business cycle asymmetries. This asymmetry arises because the incentives for investors to trade on their private information anchors on both the level of real investment and uncertainty in the economy.

## V. The Impact of Feedback in Learning

To understand the role of informational frictions in influencing the beliefs of households and firms, I first write two representations for the riskless rate  $r$  with Proposition 1, since  $r \in \mathcal{F}^c \subseteq \mathcal{F}$ ,

$$\begin{aligned} r &= \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta(I, \Sigma) (\theta - \hat{\theta}^c) + \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi + \rho + \frac{\sigma_A^2 \left( I\hat{\theta}^c - \frac{I}{a-I}g - \delta \right) - \sigma_k^2 \sigma_A^2}{\sigma_A^2 + \sigma_k^2} \\ &= \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \hat{\xi}^c + \rho + \frac{\sigma_A^2 \left( I\hat{\theta}^c - \frac{I}{a-I}g - \delta \right) - \sigma_k^2 \sigma_A^2}{\sigma_A^2 + \sigma_k^2}. \end{aligned}$$

Equating these two expressions for  $r$ , it follows that

$$\hat{\xi}^c - \xi = R_\theta(I, \Sigma) (\theta - \hat{\theta}^c). \quad (19)$$

Equation (19) reveals how beliefs are formed from financial prices. There is a trade-off in relative optimism about firm real investment productivity  $\hat{\theta}^c - \theta$  and the aggregate expected return to trees  $\hat{\xi}^c - \xi$ . When households and firms perceive the aggregate expected return to trees to be higher than its true value,  $\hat{\xi}^c > \xi$ , then by equation (19) it must be the case (since  $R_\theta(I, \Sigma) \geq 0$ ) that the economy underestimates real investment productivity  $\theta$ ,  $\theta > \hat{\theta}^c$ . Financial prices distort real production because perceptions about  $\theta$  impact the real investment decisions of firms. How far  $\hat{\theta}^c$  can be from the true  $\theta$  in the  $\mathcal{L}^2$ -sense at any given point in time,  $\Sigma$ , is time-varying.

To understand how the disparity between the true investment productivity and beliefs evolve over time, define  $\varepsilon = \theta - \hat{\theta}^c$  to be the tracking error of the common knowledge estimate of  $\theta$ . Suppressing arguments, by Itô's Lemma  $\varepsilon_t$  follows the law of motion under the physical

measure  $\mathcal{P}^{27}$

$$\begin{aligned}
d\varepsilon = & - \left( \lambda_\theta + \frac{I\sigma_{\hat{\theta}K}}{\sigma_k} + \sigma_{\hat{\theta}r} \frac{\sigma_{\hat{\theta}r} - R_\theta\sigma_\theta^2}{\Sigma} \right) \varepsilon dt + \left( 1 - \frac{\sigma_{\hat{\theta}r}R_\theta}{\sqrt{(R_\theta\sigma_\theta)^2 + \sigma_\xi^2}} \right) \sigma_\theta dZ^\theta \\
& - \sigma_{\hat{\theta}K} dZ^k - \frac{\sigma_{\hat{\theta}r}\sigma_\xi}{\sqrt{(R_\theta\sigma_\theta)^2 + \sigma_\xi^2}} dZ^\xi.
\end{aligned} \tag{20}$$

From equation (20), the misperception of economic agents about  $\theta$ ,  $\varepsilon$ , has a mean-reverting drift that is related to the natural mean-reversion of shocks to  $\theta$ ,  $\lambda_\theta$ , and to corrections that arrive as news from the two public signals,  $\log K$  and  $r$ , captured in the latter two terms. The contribution from the arrival of news varies over the business cycle, allowing for misperceptions to persist more when the strengths of the two public signals are weaker.

To assess further the impact of feedback in learning, I characterize the equilibrium in two benchmark economies, one with perfect information and one in which only households have perfect information, as helpful anchors for my analysis. The first benchmark gives us insight into how the economy behaves in the absence of any informational frictions, while the second will help to clarify the role that dispersed information among households plays in influencing the behavior of the market signal. I then explain the slow US recovery in the context of this feedback loop.

#### A. Two Benchmarks

Suppose that  $\theta_t$  is observable to all households and firms. Then all households will allocate identical fractions of their portfolios to risky projects and the riskless asset. In this benchmark setting, it is sufficient to solve the equilibrium for the aggregate state variables, since the wealth of households will only differ in their history of preference shocks. The following proposition summarizes the recursive competitive equilibrium that the recursive noisy rational expectations equilibrium tends to, in the aggregate, as informational frictions vanish for all agents.

**Proposition 7** *When  $\theta$  is observable to all households and firms, a) the price of firm equity is given by*

$$q = \frac{a - I}{\rho},$$

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<sup>27</sup>In contrast, since  $E[\varepsilon | \mathcal{F}^c] \equiv 0$ , it follows that  $\varepsilon$  under the common knowledge measure  $\mathcal{P}^c$  is a martingale. This must be the case since under the optimal filter, expectational errors are not forecastable.

b) the riskless return  $r$  satisfies

$$r = \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi + \rho + \frac{\sigma_A^2 (I\theta - \frac{I}{a-I}g - \delta - \sigma_k^2)}{\sigma_A^2 + \sigma_k^2}$$

when  $I > \underline{I}$ , c) optimal consumption and portfolio investment weights  $X(i)$  by households satisfy

$$\begin{aligned} c(i) &= \rho w(i), \\ X(i) &= \begin{bmatrix} \frac{\rho - \frac{I}{a-I}g + I\theta - r - \delta}{\sigma_k^2} \\ \frac{\rho + \xi(i) - r}{\sigma_A^2} \end{bmatrix}, \end{aligned}$$

and d) optimal investment by firms is given by

$$g = \rho(q\theta - 1) \mathbf{1}_{\{I > \underline{I} \cup \theta \geq \frac{\rho}{a-I}\}}.$$

The equilibrium with perfect information appears similar to the one with informational frictions, except that the riskless rate no longer reflects the wedge between the beliefs of agents and the true underlying productivity of real investment  $\theta$  because households and firms are now perfectly informed. The economy is isomorphic to one with a representative agent household who owns and manages all assets in the economy, and chooses the riskless rate so that it invests all its resources in assets given its preference shock. In this setting, there is no role for noise from preference shocks  $\xi$  to transmit to real investment decisions because firms do not learn from prices. Financial market activity has no consequence for the business cycle at all.

The second benchmark provides an intermediate case between the informational frictions economy of the previous section and the perfect-information benchmark. Though households behave identically when they have perfect information, there is still feedback from financial market noise  $\xi$  to real investment decisions because firms still must learn about  $\theta$  from market prices. The behavior of this economy is summarized in the next proposition.

**Proposition 8** *When  $\theta$  is observable to all households, a) the price of firm equity is given by*

$$q = \frac{a - I}{\rho},$$



b) the riskless return  $r$  satisfies

$$r = \frac{\sigma_A^2}{\sigma_A^2 + \sigma_k^2} I\theta + \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi + \rho - \frac{\sigma_A^2 \left( \frac{I}{a-I} g + \delta + \sigma_k^2 \right)}{\sigma_A^2 + \sigma_k^2},$$

c) optimal consumption and portfolio investment weights  $X(i)$  by households satisfy

$$\begin{aligned} c(i) &= \rho w(i), \\ X(i) &= \begin{bmatrix} \frac{\rho - \frac{I}{a-I} g + I\theta - r - \delta}{\frac{\sigma_k^2}{\sigma_A^2} - r} \\ \frac{\rho + \xi(i) - r}{\sigma_A^2} \end{bmatrix}, \end{aligned}$$

and d) optimal investment by firms is given by

$$g = \rho \left( q\hat{\theta}^c - 1 \right) \mathbf{1}_{\{I > \underline{I} \cup \hat{\theta}^c \geq \frac{\rho}{a-I}\}}.$$

Furthermore, beliefs, prices, and optimal policies in the economy with informational frictions approach their representative agent benchmark values as  $\sigma_s \searrow 0$ .

In this intermediate case, firms must still learn from both the growth of firm assets and market prices. Noise from market prices from preference shocks  $\xi$  can potentially feed back into firm learning, and therefore their investment decisions, yet there is an important distinction from the NREE equilibrium. Since households have perfect information, the level of uncertainty in the economy  $\Sigma$  does not affect their trading behavior, and consequently it has a smaller role in determining the influence and strength of the market signal. This can be seen from the difference in the loadings on the tracking error  $\theta - \hat{\theta}^c$  in the expressions for  $r$  in Propositions 1 and 8. In the NREE economy, the signal-to-noise ratio is  $R_\theta(I, \Sigma)$ , while in this representative agent setting it is  $R_\theta(I) = \frac{\sigma_A^2}{\sigma_k^2} I$ . This implies that the market signal in the representative agent setting mimics much of the cyclical behavior of the real investment signal (though it is not redundant because the noise in the two signals are conditionally independent of each other). The market signal  $S = \left( 1 + \frac{\sigma_A^2}{\sigma_k^2} \right) (r - \rho) + \frac{\sigma_A^2}{\sigma_k^2} \left( \frac{I}{a-I} g + \delta \right) + \sigma_A^2$  for households and firms then has the law of motion

$$dS = R_\theta(I) (g + \lambda_\xi - \lambda_\theta) \theta dt + R_\theta(I) \sigma_\theta dZ^\theta + \sigma_\xi dZ^\xi,$$

where  $R_\theta(I)$  is increasing in  $I$  and independent of uncertainty  $\Sigma$ . The market signal is, consequently, strongest when during booms investment is high.

This setting consequently highlights the importance of dispersed information for the mechanism of the NREE economy: aggregation of dispersed information gives the market signal much of its countercyclical behavior because the quantity of private information  $\Sigma$  matters for how households trade on their private information. There is a drastic difference, then, in the predictions of how an economy with a representative household behaves compared to an economy with households with heterogeneous information.<sup>28</sup>

### B. *Explaining the Slow US Recovery*

My analysis highlights a potential channel by which recessions with financial origins can have deeper recessions and slower recoveries, and can help explain how the financial crisis of late 2008 may have contributed to the anemic US recovery. Economic agents rely more on price signals for helpful guidance about the state of the economy as the economy enters a downturn. Financial crises during downturns distort these price signals and, as a result of severe informational frictions, investors and firms interpret part of the collapse in asset prices as a signal of severe economic weakness. This further depresses real activity, causing both real and financial signals to flatten, which increases uncertainty and causes it to remain elevated. This makes it harder for private agents to act on signs of a recovery. Despite evidence of economic improvement, and a rebounding of financial markets, the heightened level of uncertainty makes it difficult for a recovery to gain traction and stifles growth.

To illustrate this story, I construct a continuous-time analogue of the Impulse Response functions commonly employed in the discrete-time macroeconomics literature. This nonlinear function is derived from the Clark-Ocone-Haussman version of the (local) Martingale Representation Theorem, in a manner similar to Borovicka, Hansen, and Scheinkman (2014).<sup>29</sup> Define  $\vec{Z}$  to be the vector Wiener process  $\vec{Z} = \begin{bmatrix} Z^\theta & Z^k & Z^\xi \end{bmatrix}'$ .

**Proposition 9** *Consider the vector  $V = \begin{bmatrix} \theta & \hat{\theta}^c & I & \Sigma \end{bmatrix}'$  with law of motion*

$$dV_t = b(V_t) dt + \sigma(V_t) d\vec{Z}.$$

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<sup>28</sup>In principle, one could also consider a benchmark with a representative household that receives a noisy signal instead of having perfect information. Since the noise in the household's private signal would not vanish from market prices, however, it is less clear how the informativeness of the market signal would change over the cycle, since the noise in the price from the household's private signal would also increase as  $I$  and  $\Sigma$  increased.

<sup>29</sup>See Fournié et al (1999) for a relevant primer.

The Impulse Response function for initial condition  $V_0 = v_0$  at time  $T$  to a unit shock to the linear combination  $\vec{\eta}'(v_0) \vec{Z}$ ,  $\phi_v(\vec{\eta}(v_0), V_T)$ , is given by

$$\phi_v(\vec{\eta}(v_0), V_T) = \vec{\eta}'(v_0)' E[H_T | V_0 = v_0] \sigma(v_0),$$

where  $H_t = \frac{\partial V_T}{\partial v_0}$  is the first-variation process with law of motion

$$dH_t = b'(V_t) H(t) dt + \sigma_{H\theta}(V_t) H_t dZ_t^\theta + \sigma_{Hk}(V_t) H_t dZ_t^k + \sigma_{H\xi}(V_t) H_t(V_t) dZ_t^\xi,$$

with coefficient matrices defined in Appendix B, and  $\sigma(v_0)$  is given by

$$\sigma(v_0) = \begin{bmatrix} \sigma_\theta & 0 & 0 \\ \sigma_{\hat{\theta}r}(v_0) \frac{R_\theta(v_0)\sigma_\theta}{\sqrt{(R_\theta(v_0)\sigma_\theta)^2 + \sigma_\xi^2}} & \sigma_{\hat{\theta}k}(v_0) & \sigma_{\hat{\theta}r}(v_0) \frac{\sigma_\xi}{\sqrt{(R_\theta(v_0)\sigma_\theta)^2 + \sigma_\xi^2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Impulse Response function  $\phi_v(\vec{\eta}(v_0), V_T)$  has a natural interpretation and can be decomposed into two pieces: a static effect  $\sigma(v_0)$  that represents the immediate impact of a unit shock to  $\vec{Z}$  to the state variables in the economy  $V_t$ , and a (nonlinear) dynamic multiplier  $E[H_T | V_0 = v_0]$  that measures how the initial shock propagates in expectation over time. The latter piece is found by recognizing that the sensitivity of the state vector to initial conditions  $\frac{\partial V_t}{\partial v_0}$  is itself a diffusion process whose coefficient loadings are the (time-varying) differentiated coefficient loadings of  $V_t$ .<sup>30</sup> Since the system is nonlinear, the Impulse Response Function  $\phi_v(\vec{\eta}(v_0), V_T)$  depends on the initial conditions  $v_0$ . In what follows, I pin down part fo the state vector by initializing investment at its steady-state value in the absence of additional aggregate shocks,  $I_0 = \left(a - \frac{\rho}{\theta_0^c}\right) \mathbf{1}_{\{\hat{\theta}_0^c \geq \frac{\rho}{a-I}\}}$ , and assuming that there is no initial misperceptions about real investment productivity,  $\hat{\theta}_0^c = \theta_0$ .

An important feature of  $\phi_v(\vec{\eta}(v_0), V_T)$  is that it is constructed under the physical prob-

<sup>30</sup>To gain intuition for  $\phi_v(\vec{\eta}(v_0), V_T)$ , I can compute  $\phi_v(\vec{e}_1, V_T)$ , the Impulse Response for  $\theta_t$ , explicitly since  $\theta_t$  follows an exogenous OU process. The first-variation process for  $\theta_t$ ,  $\frac{\partial \theta_t}{\partial \theta_0}$ , has law of motion

$$d\left(\frac{\partial \theta_t}{\partial \theta_0}\right) = -\lambda_\theta \frac{\partial \theta_t}{\partial \theta_0} dt,$$

from which follows that  $\phi_v(\vec{e}_1, V_T) = \frac{\partial \theta_T}{\partial \theta_0} \sigma_\theta = e^{-\lambda_\theta(T-t)} \sigma_\theta$ . Since  $\theta_t$  follows a linear AR(1) process, the immediate impact of a unit shock to  $Z^\theta$  is to raise  $\theta_0$  by  $\sigma_\theta$ , and this contribution to future  $\theta_t$  decays exponentially over time with rate of mean-reversion  $\lambda_\theta$ .

ability measure  $\mathcal{P}$  rather than, for instance, the measure implied by common knowledge  $\mathcal{P}^c$ . This allows me to investigate how the true aggregate structural shocks  $\vec{Z}$  propagate through the economy, rather than their filtered counterparts  $\begin{bmatrix} \tilde{Z}^k & \tilde{Z}^r \end{bmatrix}$ , which are endogenous outcomes from learning.<sup>31</sup> From equation (20), the filtered estimate  $\hat{\theta}^c$  has a drift under the physical measure that reflects the information revelation from public signals. In the presence of informational frictions, firms decompose  $\theta$  and  $\xi$  instead into their filtered counterparts,  $\hat{\theta}^c$  and  $\hat{\xi}^c$ , respectively. It is through this channel that financial shocks propagate to the real economy by distorting firm expectations about real investment productivity.

Given that uncertainty  $\Sigma$  is negatively correlated with filtered real investment productivity  $\hat{\theta}^c$  and real investment  $I$  in the stationary distribution, I consider a boom to be a high  $\hat{\theta}^c$ , low  $\Sigma$  state, and a bust to be a low  $\hat{\theta}^c$ , high  $\Sigma$  state. Figures 2 and 3 in Appendix A illustrate the impact of a negative unit shock to the aggregate expected growth of trees  $Z^\xi$  during an expansion  $(\hat{\theta}^c, \Sigma) = (0.3, 0.002)$  and during a contraction  $(\hat{\theta}^c, \Sigma) = (0.2, 0.003)$ , respectively. The Impulse Responses are plotted for the parameters listed in Table 1, with the exception that the rates of mean-reversion,  $\lambda_\theta$  and  $\lambda_\xi$ , are set to 0 to focus on the propagation of the shocks through the learning channel.

In both economies, the negative shock to prices lowers the conditional estimates of real investment productivity  $\hat{\theta}^c$  through learning. This, in turn, lowers real investment  $I$  by firms, which raises uncertainty  $\Sigma$  because real and financial signals are now weaker. As a result of learning, however, this initial fall in  $\hat{\theta}^c$  dampens as economic agents learn the initial shock was actually to trees and not investment productivity. This raises real investment, creating an endogenous downturn and recovery, but uncertainty is now permanently higher because the depressed investment obscured information about new shocks to  $\hat{\theta}^c$ . Finally, output, which compounds past investment choices through capital accumulation, is permanently lower because of the shock to trees through learning.

As a result of informational frictions, a shock to asset prices has a more pronounced and persistent impact on investment and output during contractions, when uncertainty is higher and real signals are weaker. This mechanism can possibly help explain how a negative financial shock during a recession can further depress real activity and lead to a more gradual recovery, while financial events like the LTCM crisis that occurred during an expansion had a more attenuated effect on the real economy. Key to this result is that uncertainty is time-

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<sup>31</sup>The difference between the two Impulse Response functions highlights the fundamental invertibility issue that economic agents face in disentangling the sources of aggregate fluctuations in real and financial markets.

varying, with a law of motion given in Proposition 3, and countercyclical. When uncertainty is higher, noise in financial prices that is interpreted as bad news perpetuates low investment. This, in turn, perpetuates high uncertainty and allows the distortion to beliefs from the noise in financial prices to persist.

The above analysis consequently identifies a potential benefit of unconventional monetary policy in the presence of informational frictions. By buying treasury and mortgage-backed securities through Quantitative Easing (QE), the US government supported financial prices and provided financing for investors to purchase assets from riskier asset classes, such as equities and speculative-grade debt. This injection of capital may have lessened the noise that investor balance sheet impairment introduced into financial prices during the financial crisis that distorted the expectations of private agents about the strength of the US economy. In continuing QE in its various forms of QE1-QE3 until late 2014, however, the buoying of financial markets may have later contributed to the noise in financial prices. The April 2011 WSJ article "Is the Market Overvalued?", for instance, discusses how market participants and economists could not disentangle signs of strong corporate profitability from the effects of QE behind the high valuations in the stock market.

## VI. Welfare

I now turn to the welfare implications of my analysis. The economy with informational frictions may be constrained inefficient because households and firms do not fully internalize the benefit of the public information they produce by trading in asset markets and engaging in real investment, which motivates a role for welfare-improving policies. In this spirit, I consider several thought experiments that augment the provision of public information in the economy to highlight this potential externality. In contrast to Angeletos and Pavan (2009), the economy featured here has endogenous stochastic volatility that interacts with production and informational frictions.

I begin this section by characterizing ex-ante welfare in the economy. I adopt a utilitarian weighting scheme to aggregate utility across the heterogeneous households, normalizing welfare to initial household consumption to remove the level effect of initial conditions. This helps me construct a measure of welfare in the economy that has a stationary distribution conducive to conducting thought experiments. Informational frictions impact welfare through two channels: a distortion to real investment and household trading, and a cost

that comes from the inequality in household wealth that arises because of the dispersion of private beliefs. This is summarized in the following proposition.

**Proposition 10** *Ex-ante utilitarian welfare in the economy with informational frictions is given by*

$$\begin{aligned}
U = & \frac{1}{\rho} E \left[ \int_0^\infty e^{-\rho t} \left( \frac{q_t K_t}{q_t K_t + P_t} \cdot \underbrace{\left( I_t \theta_t - \frac{I_t}{a - I_t} g_t - \delta \right)}_{\text{Efficiency of Real Investment}} + \frac{P_t}{q_t K_t + P_t} \xi_t \right) dt \mid \mathcal{F}_0 \right] \\
& - \underbrace{\frac{1}{2\rho\sigma_k^2} E \left[ \int_0^\infty e^{-\rho t} \left( \frac{I_t \Sigma_t \sigma_s \sigma_e \sqrt{\sigma_e^2 + \sigma_s^2 R_\theta (I_t, \Sigma_t)^2}}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I_t, \Sigma_t)^2 \Sigma_t} \right)^2 dt \mid \mathcal{F}_0 \right]}_{\text{Cross-Sectional Inequality}} \\
& + \frac{1}{2\rho^2} \frac{\sigma_e^2}{\sigma_A^2} - \frac{1}{\rho} E \left[ \int_0^\infty e^{-\rho t} \left( \left( \frac{q_t K_t}{q_t K_t + P_t} \right)^2 \sigma_k^2 + \left( \frac{P_t}{q_t K_t + P_t} \right)^2 \sigma_A^2 \right) dt \mid \mathcal{F}_0 \right],
\end{aligned}$$

Then, under this welfare criterion, there exists a representative household in the economy who holds all firm equity and trees, and whose wealth  $w$  evolves according to

$$\begin{aligned}
\frac{dw}{w} = & \left( \frac{qK}{qK + P} \left( I\theta - \frac{I}{a - I} g - \delta \right) + \frac{P}{qK + P} \xi \right) dt + \frac{qK}{qK + P} \sigma_k dZ^k + \frac{P}{qK + P} \sigma_A dZ^A \\
& - \frac{1}{2} \left( \left( \frac{P}{qK + P} \right)^2 \sigma_A^2 + 2 \frac{\sigma_e^2}{\sigma_A^2} + \frac{1}{\sigma_k^2} \left( \frac{I \Sigma \sigma_s \sigma_e \sqrt{\sigma_e^2 + \sigma_s^2 R_\theta (I, \Sigma)^2}}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I, \Sigma)^2 \Sigma} \right)^2 \right) dt.
\end{aligned}$$

From Proposition 10, informational frictions introduce two distortions to welfare: 1) a loss to the efficiency of real investment, since firms invest based on imperfect information, and 2) an increase in cross-sectional inequality from households trading on their private information. This loss to efficiency also impacts the aggregate wealth of the economy by distorting the market value of firms through firms' investment decisions. As a result of inequality that arises because of informational frictions and differences in the returns of trees, the representative household under this welfare criterion is different from a representative household who holds all firm equity and trees. This distinction is absent from representative agent models, and the quasilinear case studied in Angeletos and Pavan (2009), and comes from the aggregation of flow utility  $\log c(i)$  rather than consumption  $c(i)$  in the utilitarian welfare function. The effects of the distortion show up as a tax on the representative household, and consequently

one can think of the transfer of wealth from shocks to trees and informational frictions as imposing a tax on the economy. The contribution to this tax from uncertainty vanishes when households have identical beliefs, which occurs in the limiting cases when  $\sigma_s \rightarrow \{0, \infty\}$  or  $\Sigma \searrow 0$ .

Having derived ex-ante utilitarian welfare to understand the forces that impinge on household utility, I construct a measure of expected welfare using only public information once the economy has reached its stationary distribution, and initial conditions no longer matter, as a sensible measure for conducting my thought experiments. To target household and firm investing behavior, I introduce a proportional position cost  $\tau^r$  on households, similar to the proportional tax  $\tau$  of Angeletos, Lorenzoni, and Pavan (2012), and a linear subsidy on firm real investment  $\tau^I$ , similar to Angeletos and Pavan (2009).<sup>32</sup> I construct these instruments so that the extracted revenue is returned to households as lump-sum transfers that households view as being proportional to their wealth. The position cost lets me manipulate households' trading decisions while the real investment subsidy lets me manipulate firms' investment decisions.

Solving for household's optimal investment in the presence of the position cost, it is straightforward to see from Proposition 2 that household  $i$  invests fractions  $X(i) = \begin{bmatrix} x^k(i) & x^A(i) \end{bmatrix}'$

$$X(i) = \begin{bmatrix} \frac{1}{1-\tau^r} \frac{\rho - \frac{I}{a-I}g + I\hat{\theta}(i) - r - \delta}{\sigma_k^2} \\ \frac{\rho + \xi(i) - r}{\sigma_A^2} \end{bmatrix},$$

of its wealth in firm equity and trees. Then, by similar arguments to those in Section IV, one can arrive at the form for the riskless rate  $r$

$$r = \frac{(1-\tau^r)\sigma_k^2}{(1-\tau^r)\sigma_k^2 + \sigma_A^2} R_\theta(I, \Sigma) (\theta - \hat{\theta}^c) + \frac{(1-\tau^r)\sigma_k^2}{(1-\tau^r)\sigma_k^2 + \sigma_A^2} \xi + \rho \\ + \frac{\sigma_A^2 \left( I\hat{\theta}^c - \frac{I}{a-I}g - \delta \right) - (1-\tau^r)\sigma_A^2\sigma_k^2}{(1-\tau^r)\sigma_k^2 + \sigma_A^2}$$

where  $R_\theta(I, \Sigma)$  now satisfies

$$R_\theta(I, \Sigma)^3 - \frac{\sigma_A^2}{\sigma_k^2} \frac{I}{1-\tau^r} R_\theta(I, \Sigma)^2 + \left( \frac{1}{\sigma_s^2} + \frac{1}{\Sigma} \right) \sigma_e^2 R_\theta(I, \Sigma) - \frac{\sigma_A^2 \sigma_e^2}{\sigma_k^2 \sigma_s^2} \frac{I}{1-\tau^r} = 0.$$

<sup>32</sup>I impose the position cost on investment in equity rather than trees since equity is traded on a centralized platform and the cost could, in principle, be implemented.

Consequently, one has for  $\tau^r \in [0, 1]$  that  $\partial_{\tau^r} R_\theta(I, \Sigma) \geq 0$ , and therefore the position cost has the property that it induces households to take larger positions in the risky asset based on their private information. This happens because households in continuous-time can rebalance their portfolios instantaneously to take a large enough position to offset the impact of the cost. Since the cost is returned lump-sum, however, it introduces a distortion to household wealth. A higher position cost  $\tau^r$  increases the amount of public information in the price by causing households to take a larger position based on their private information, but it also introduces more wealth inequality as a result of trading on private information. There is then a tradeoff for welfare to increasing  $\tau^r$  to increase the provision of public information.

It is also straightforward to see from Proposition 4 that the real investment subsidy induces the firm to choose a growth rate for real investment  $g$

$$g = \left( (a - I) \hat{\theta}^c - (1 - \tau^I) \rho \right) \mathbf{1}_{\left\{ I > I \cup \hat{\theta}^c \geq \frac{(1 - \tau^I) \rho}{a - I} \right\}}.$$

With these instruments in place, I now search for the probability law of the economy once it has reached its stationary distribution  $p(I, \Sigma, \hat{\theta}^c, \hat{\xi}^c)$ , if it exists. I derive the Kolmogorov Forward Equation (KFE) which summarizes the (instantaneous) transition of the probability law of the economy  $p_t(I, \Sigma, \hat{\theta}^c, \hat{\xi}^c)$  and characterize the conditions under which  $\partial_t p_t(I, \Sigma, \hat{\theta}^c, \hat{\xi}^c) = 0$ . This reduces to solving the appropriate boundary value problem for a second-order elliptic partial differential equation, summarized in the following proposition.<sup>33</sup>

**Proposition 11** *The stationary distribution of the economy  $p(I, \Sigma, \hat{\theta}^c, \hat{\xi}^c)$  satisfies the Kolmogorov Forward Equation*

$$\begin{aligned} 0 = & -\partial_{\hat{\theta}^c} \left\{ p \lambda_\theta (\bar{\theta} - \hat{\theta}^c) \right\} - \partial_I \left\{ p I \left( (a - I) \hat{\theta}^c - (1 - \tau^I) \rho \right) \right\} \mathbf{1}_{\left\{ I > I \cup \hat{\theta}^c \geq \frac{(1 - \tau^I) \rho}{a - I} \right\}} - \partial_\Sigma \left\{ p \frac{d\Sigma}{dt} \right\} \\ & - \partial_{\hat{\xi}^c} \left\{ p \lambda_\xi (\bar{\xi} - \hat{\xi}^c) \right\} + \frac{1}{2} \partial_{\hat{\xi}^c \hat{\xi}^c} p \left( R_\theta^2 \sigma_{\theta k}^2 + \frac{(\sigma_\xi^2 - R_\theta (\partial_\Sigma R_\theta \frac{d\Sigma}{dt} + \partial_I R_\theta I g + (\lambda_\xi - \lambda_\theta) R_\theta) \Sigma)^2}{(R_\theta \sigma_\theta)^2 + \sigma_\xi^2} \right) \\ & + \frac{1}{2} \partial_{\hat{\theta}^c \hat{\theta}^c} \left\{ p (\sigma_{\theta k}^2 + \sigma_{\theta r}^2) \right\} + \partial_{\hat{\theta}^c \hat{\xi}^c} \left\{ p \left( R_\theta \sigma_{\theta k}^2 + \frac{\sigma_\xi^2 - R_\theta (\partial_\Sigma R_\theta \frac{d\Sigma}{dt} + \partial_I R_\theta I g + (\lambda_\xi - \lambda_\theta) R_\theta) \Sigma}{\sqrt{(R_\theta \sigma_\theta)^2 + \sigma_\xi^2}} \sigma_{\hat{\theta} r} \right) \right\} \end{aligned}$$

with boundary conditions given in Appendix B.

The KFE that defines the stationary distribution is a conservation of mass law that has

<sup>33</sup>See Internet Appendix C for details on the numerical implementation scheme for the KFE.



an intuitive interpretation. It states that the sum of the flows of probability through a cube in the  $(I, \Sigma, \hat{\theta}^c, \hat{\xi}^c)$  space must be zero for the probability mass of the cube to be conserved over time. The stochastic components of  $\hat{\theta}^c$  and  $\hat{\xi}^c$  introduce second-order terms in the KFE related to their volatility since the high variability of Wiener processes has a first-order effect on the laws of motion of  $\hat{\theta}^c$  and  $\hat{\xi}^c$ . In the case where  $\sigma_s \nearrow \infty$ , the economy is analogous to that of Van Nieuwerburgh and Veldkamp (2006) in which only a real investment signal provides information.

Given the KFE, I now construct my welfare measure. Let  $U_p^c$  be utilitarian welfare in the economy, normalized to initial wealth, and  $E^p[\cdot]$  be the expectation operator with respect to the stationary distribution. Then I have the following corollary.

**Corollary 1:** *Expected utilitarian welfare under the stationary distribution  $U_p^c$  with position costs and real investment subsidy  $\tau^r$  and  $\tau^l$ , respectively, is given by*

$$\begin{aligned}
U_p^c = & \frac{1}{\rho^2} E^p \left[ \frac{\frac{1}{2} \left( I_0 \hat{\theta}_0^c - \frac{I_0}{a-I_0} g_0 \right)^2 + (\sigma_A^2 - \delta + \tau^r) \left( I_0 \hat{\theta}_0^c - \frac{I_0}{a-I_0} g_0 \right) - \left( I_0 \hat{\theta}_0^c - \frac{I_0}{a-I_0} g_0 \right) \hat{\xi}_0^c}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2} \right] \\
& + \frac{1}{\rho^2} E^p \left[ \frac{I_0 \Sigma_0 R_\theta (I_0, \Sigma_0)}{\sigma_A^2} - \frac{(1 - \tau^r)^2 \sigma_k^4 \Sigma_0 R_\theta (I_0, \Sigma_0)^2}{2 \sigma_A^4 \sigma_A^2 + (1 - \tau^r) \sigma_k^2} \right] \\
& - \frac{1}{2\rho^2 (1 - \tau^r)^2 \sigma_k^2} E^p \left[ \left( \frac{\sigma_s \sigma_e I_0 \Sigma_0 \sqrt{\sigma_e^2 + \sigma_s^2 R_\theta (I_0, \Sigma_0)^2}}{(\Sigma_0 + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I_0, \Sigma_0)^2 \Sigma_0} \right)^2 \right] + \frac{1}{2\rho^2} \frac{\sigma_e^2}{\sigma_A^2} \\
& + \frac{1}{2\rho^2} \frac{\delta^2 + \bar{\xi}^2 + \frac{\sigma_\xi^2}{2\lambda_\xi} + 2(\delta + \sigma_k^2 - \tau^r) \bar{\xi} - 2\delta \sigma_A^2 - (1 - \tau^r) \sigma_A^2 \sigma_k^2 - 2\tau^r (\delta + \sigma_A^2)}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2}.
\end{aligned}$$

The first expectation consists of terms related to the return to real investment, with the third term reflecting the covariance between the filtered expected return to trees  $\hat{\xi}_0^c$  and the time-varying piece of the expected return to equity  $I_0 \hat{\theta}_0^c - \frac{I_0}{a-I_0} g_0$  induced through the riskless rate by learning. The second expectation is composed of variance terms from uncertainty, and the third expectation captures the cross-sectional inequality among households that arises because of asymmetric information. The direct contribution to welfare from this last piece is unambiguously negative, and it is unlikely that informational frictions can improve real investment efficiency since firms can only be distorted away from the level of investment they would choose with perfect-information. For the parameters listed in Table

1 of Appendix A, welfare is about 0.714% lower than in the perfect-information benchmark economy and 0.512% lower than in the economy analogous to that of Van Nieuwerburgh and Veldkamp (2006) where households do not aggregate private information in financial markets. This model loss reflects the tradeoff between the increased informativeness of public signals, relative to the additional noise introduced by financial prices, and the cross-sectional inequality induced by households trading on their private information.

To highlight the presence of information externalities in the economy, I conduct several illustrative thought experiments varying the position cost and real investment subsidy. I report the gain in welfare in consumption equivalent  $\alpha$  in the tradition of Lucas (1987). To try to capture the incremental impact of both instruments on welfare through the informational channel, I subtract out expected welfare under the perfect-information benchmark  $U_p^{perf}$ , since the position cost and investment subsidy will mechanically impact welfare by raising the average level of equity trading and real investment in the economy, respectively. It is easy to derive the analogous KFE for the perfect-information benchmark economy

$$-\partial_{\theta} \{p\lambda_{\theta} (\bar{\theta} - \theta)\} - \partial_I \{pI ((a - I)\theta - (1 - \tau^c)\rho)\} \mathbf{1}_{\{I > I \cup \theta \geq \frac{(1 - \tau^I)\rho}{a - I}\}} + \frac{1}{2}\sigma_{\theta}^2 \partial_{\theta\theta} P = 0,$$

which has similar boundary conditions.<sup>34</sup>

From Table 3, the position cost mitigates the loss in welfare from informational frictions. The intuition for this is that the gain in informational provision by having households take larger positions is larger than the cost of increasing inequality by having households trade more on their private information. Since better public information lowers the average level of uncertainty in the economy, however, this mitigates the rise in inequality.

$\tau^r$	0.001	0.002	0.003	0.004	0.005
$\alpha$	0.0145	0.0291	0.0441	0.0592	0.0747

Table 3: Position Cost Experiment

To see if subsidizing real investment improves welfare by improving the informational content of public information, I give firms a proportional investment subsidy  $\tau^I$  whenever

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<sup>34</sup>One may notice that expected welfare under perfect-information appears to be computed under a different probability measure. By the Law of Iterated Expectations, however, expected welfare under the common knowledge measure when agents have perfect-information is the same as expected welfare under perfect-information up to uncertainty about unobservable state variables. As a result of the functional form for real investment, the only relevant state variable is the level of real investment, which is common knowledge. Consequently, the welfare calculations under the two measures coincide.

investment falls below 0.10, which is below its mean of 0.1125 in the stationary distribution. This has the interpretation of being a countercyclical real investment subsidy.

$\tau^I$	0.001	0.002	0.003	0.004	0.005
$\alpha$	0.0040	0.0081	0.0122	0.0163	0.0204

Table 4: Investment Subsidy Experiment

Table 4 reveals that the investment subsidy also mitigates the loss in welfare from informational frictions. Since the subsidy increases real investment, which raises the average position households take in asset markets, it also has a similar effect to the position cost.

These two thought experiments illustrate that there is a role for welfare-improving policies that address an information externality that arises because of decentralization. If instead there were only one trader or one firm in the economy, such an agent would internalize its impact on the formation of the endogenous public signals when choosing its investment policies. While also likely to be present in static settings of incomplete information, this externality has a dynamic dimension because households and firms learn from signals formed inefficiently because of decentralization in the past. These thought experiments motivate a more systematic analysis of policy interventions to address such information externalities within an optimal policy framework.

## VII. Empirical Implications

In this section, I explore several empirical implications of my framework that build off the observation that financial prices provide useful signals about the state of the economy, and that the strength of these signals is strongest during downturns and recoveries. I first discuss the asset pricing implications of my analysis, and then turn to conceptual links my framework provides between financial markets and real investment.

### A. *Implications for Asset Pricing*

In this section, I characterize the business cycle implications of macroeconomic uncertainty in financial markets for asset risk premia and asset turnover. My analysis illustrates that, in the presence of informational frictions, there is an additional component to asset expected returns and turnover that reflects uncertainty about the state of the economy. This informational piece arises because households have heterogeneous private information and

the degree to which they have heterogeneous beliefs increases as uncertainty rises about real investment productivity. Furthermore, it gives asset returns predictive power for future returns and macroeconomic growth. The strength of this predictive power, however, varies over the business cycle, and I show that this variation is related to the behavior of asset turnover from informational trading.

### A.1. Risk Premia

When the true state of the economy is known, then from Proposition 7 households earn a risk premium on holding firm equity

$$RP_{perf} = \rho - \frac{I}{a-I}g + I\theta - \delta - r = \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \left( \rho \frac{a}{a-I} - \delta \right) + \underbrace{\frac{\sigma_k^2 \sigma_A^2}{\sigma_A^2 + \sigma_k^2}}_{\text{variance risk}} - \underbrace{\frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi}_{\text{expected return to trees}},$$

which compensates households for variance risk and the aggregate risk to the expected return to trees. From Proposition 1, however, in the presence of informational frictions this risk premium includes an additional piece

$$RP_{NREE} = \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \left( \rho \frac{a}{a-I} - \delta \right) + \frac{\sigma_k^2 \sigma_A^2}{\sigma_A^2 + \sigma_k^2} - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi + \underbrace{\left( I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta \right) (\theta - \hat{\theta}^c)}_{\text{informational risk}}.$$

that compensates investors for informational risk. This piece arises because households overreact to trees and capital quality shocks, and underreact to news about real investment productivity, driving a wedge between  $\theta$  and  $\hat{\theta}^c$ . Since  $R_\theta \leq \frac{\sigma_A^2}{\sigma_k^2} I$ , it is the case that  $I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta \geq 0$ . In what follows, I will assume that  $\sigma_e \geq a \frac{\sigma_A^2}{\sigma_k^2} \sqrt{\frac{\sigma_\theta^2}{\lambda_\theta} + \sigma_s^2}$ , so that, by Proposition 6,  $I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta$  is also increasing in the level of real investment  $I$ . This excess return for informational risk then is increasing in the level of investment  $I$  by firms and in the "average" pessimism of economic agents  $\theta - \hat{\theta}^c$ , and decreasing in the level of uncertainty in the economy  $\Sigma$  (since  $\partial_\Sigma R_\theta \geq 0$ ). Investors earn risk compensation not only because of variation in the expected return to trees and variance risk, but also because of distorted beliefs.

Similar to the speculative risk premium in Nimark (2012), this additional informational piece is, by construction, orthogonal to all public information, since  $E \left[ \left( I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta \right) (\theta - \hat{\theta}^c) \mid \mathcal{F}^c \right] = 0$ . Unlike the conditional mean, however, the conditional variance of this informational piece  $CV = E \left[ \left( I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta \right)^2 (\theta - \hat{\theta}^c)^2 \mid \mathcal{F}^c \right] = \left( I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta \right)^2 \Sigma$  is, in principle, measurable

by the econometrician. Since  $I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta \geq 0$ , this conditional variance is increasing in real investment  $I$ . It is also increasing in the conditional variance of beliefs  $\Sigma$  since, after substituting for  $\partial_\Sigma R_\theta$  with Proposition 9 and subsequently for  $\frac{\sigma_e^2}{\Sigma} R_\theta$  with Proposition 1, I arrive at

$$\frac{dCV}{d\Sigma} = \left( I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta \frac{3 \frac{\sigma_A^2 \sigma_e^2}{\sigma_k^2 \sigma_s^2} I + \frac{\sigma_A^2}{\sigma_k^2} I R_\theta^2 - 2 \frac{\sigma_e^2}{\sigma_s^2} R_\theta}{\left( 2R_\theta - \frac{\sigma_A^2}{\sigma_k^2} I \right) R_\theta^2 + \frac{\sigma_A^2 \sigma_e^2}{\sigma_k^2 \sigma_s^2} I} \right) \left( I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta \right).$$

When  $\Sigma \searrow 0$ , then  $R_\theta \searrow 0$  and  $\frac{dCV}{d\Sigma} > 0$ , while when  $\Sigma \nearrow \infty$ , then  $R_\theta \nearrow \frac{\sigma_A^2}{\sigma_k^2} I$  and  $\frac{dCV}{d\Sigma} = \left( \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} I \right)^2 > 0$ . Thus  $\frac{dCV}{d\Sigma} > 0$ . Since  $I$  and  $\Sigma$  are negatively correlated, this informational risk premium contributes most to the time-variation in risk premia during economic transitions, when  $I$  and  $\Sigma$  are both in an intermediate range.

To see how this informational component of risk premia affects the predictive power of asset prices for output,  $Y_t = aK_t$ , I integrate equation (1) from  $t$  to  $s \geq t$  to find that output growth  $\log \frac{Y_s}{Y_t}$  is given by

$$\log \frac{Y_s}{Y_t} = \int_t^s I_u \theta_u du + \sigma_k (Z_s^k - Z_t^k).$$

Using only public information, the covariance between output growth and expected excess returns in asset prices is

$$\begin{aligned} Cov \left[ \log \frac{Y_s}{Y_t}, RP_{NREEt} \mid \mathcal{F}_t^c \right] &= \left( I_t - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta (I_t, \Sigma_t) \right) Cov \left[ \int_t^s I_u \theta_u du, \theta_t - \hat{\theta}_t^c \mid \mathcal{F}_t^c \right] \\ &\quad - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} Cov \left[ \int_t^s I_u \theta_u du, \xi_t \mid \mathcal{F}_t^c \right]. \end{aligned}$$

Since the riskless rate  $r_t$  is observable,  $r_t \in \mathcal{F}_t^c$ , I substitute for  $\frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi_t$  with  $r_t$  from Proposition 1 to find

$$Cov \left[ \log \frac{Y_s}{Y_t}, RP_{NREEt} \mid \mathcal{F}_t^c \right] = I_t Cov \left[ \int_t^s I_u \theta_u du, \theta_t - \hat{\theta}_t^c \mid \mathcal{F}_t^c \right].$$

In the absence of informational frictions, then, the covariance between risk premia and output growth is zero, since there is no misperception among firms or investors about  $\theta_t$ , so  $\hat{\theta}_t^c \equiv \theta_t$ .

In the presence of informational frictions, however, this covariance is nonzero. Informational frictions introduce a short-run positive correlation between output growth and current risk premia since the true future investment productivity  $\theta_u$   $u \geq t$  and investment are pos-

itively correlated at short-horizons with the true current investment productivity  $\theta_t$ .<sup>35</sup> At longer horizon, the correlation weakens because of the mean-reversion in investment productivity  $\theta_t$ . Since uncertainty  $\Sigma_t$  is countercyclical, the covariance also weakens around the peaks of business cycles, contributing to the countercyclical properties of asset price predictability for output growth. Similar insights hold for the relationship between expected returns and the growth in real investment.<sup>36</sup>

Substituting with  $r_t$  from Proposition 1, and recognizing that  $\xi_s$  and  $\theta_t - \hat{\theta}_t^c$  are correlated only insofar as  $\theta_t - \hat{\theta}_t^c$  is correlated with  $\xi_t$ , I also find that

$$\begin{aligned} & Cov \left[ \int_t^s RP_{NREEu} du, RP_{NREEt} \mid \mathcal{F}_t^c \right] \\ = & I_t Cov \left[ \int_t^s \left( I_u - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta(I_u, \Sigma_u) \right) (\theta_u - \hat{\theta}_u^c) du, \theta_t - \hat{\theta}_t^c \mid \mathcal{F}_t^c \right] \\ & + \frac{\rho \sigma_k^2}{\sigma_A^2 + \sigma_k^2} I_t Cov \left[ \int_t^s \frac{I_u}{a - I_u} du, \theta_t - \hat{\theta}_t^c \mid \mathcal{F}_t^c \right] + \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \frac{1 - e^{-\lambda_\xi(s-t)}}{\lambda_\xi} I_t \Sigma_t R_\theta(I_t, \Sigma_t), \end{aligned}$$

from which follows that  $Cov \left[ \int_t^s RP_{NREEu} du, RP_{NREEt} \mid \mathcal{F}_t^c \right]$  is positive. The correlation weakens at longer horizons because  $\theta_t$  and  $\hat{\theta}_t^c$  are mean-reverting.

Though there is this persistence in returns, households do not trade to eliminate this predictability. By the Law of Total Covariance, I can manipulate  $Cov \left[ \int_t^s RP_{NREEu} du, RP_{NREEt} \mid \mathcal{F}_t^c \right]$  to arrive at

$$\begin{aligned} & E \left[ Cov \left[ \int_t^s RP_{NREEu} du, RP_{NREEt} \mid \mathcal{F}_t^i \right] \mid \mathcal{F}_t^c \right] \\ = & Cov \left[ \int_t^s RP_{NREEu} du, RP_{NREEt} \mid \mathcal{F}_t^c \right] \\ & - Cov \left[ E \left[ \int_t^s RP_{NREEu} du \mid \mathcal{F}_t^i \right], E \left[ RP_{NREEt} \mid \mathcal{F}_t^i \right] \mid \mathcal{F}_t^c \right], \end{aligned}$$

from which it is apparent that the "average" perceived covariance of expected returns by household  $i$   $Cov \left[ \int_t^s RP_{NREEu} du, RP_{NREEt} \mid \mathcal{F}_t^i \right]$  differs from the "average" covariance of

<sup>35</sup>That future investment  $I_u$  and  $\theta_t$  are positively correlated follows since the growth of investment  $I_u$  is increasing  $\hat{\theta}_u^c$  from Proposition 4, and  $\hat{\theta}_u^c = \theta_u + \varepsilon_u$  for some  $\varepsilon_u$  such that  $E[\varepsilon_u \mid \mathcal{F}_u^c] = 0$ , since  $\hat{\theta}_u^c$  is an unbiased estimator of  $\theta_u$ .

<sup>36</sup>My focus here is on conditional covariances. It is less clear that the signs and strengths of these covariances also hold unconditionally, since for random variables  $X, Y$ , and  $Z$ , by the Law of Total Covariance

$$Cov[X, Y] = E(Cov[X, Y \mid Z]) + Cov[E(X \mid Z), E(Y \mid Z)].$$

This implies that empirical tests would ideally focus on these conditional relationships.

expected returns  $Cov \left[ \int_t^s RP_{NREEu} du, RP_{NREEt} \mid \mathcal{F}_t^c \right]$  because of heterogeneous information. Consequently, households differ not only in their beliefs about expected returns, but also in their beliefs about the persistence of returns, which gives them incentive to trade without eliminating the predictability found with only public information.

This exercise illustrates that, in the presence of informational frictions, asset risk premia inherently contain an informational component that reflects uncertainty over current macroeconomic conditions above and beyond the correlation between real and financial shocks (since  $\xi$  may, in practice, be correlated with  $\theta$ ). Such a positive relationship between returns and future real activity, which arises because of the underreaction of investors to changes in the prospects of firms, is consistent, for instance, with the findings of Barro (1990), Fama (1990), and Schwert (1990). Moreover, this additional informational component exhibits countercyclical behavior, since uncertainty about investment productivity is countercyclical in the economy, and larger when assets on financial investor's balance sheets are weak (larger, negative  $\xi$  shocks which depress  $\hat{\theta}^c$ ). This may help explain why studies such as Stock and Watson (2003) and Ng and Wright (2013) find that the predictive power of asset prices for macroeconomic outcomes is somewhat episodic over business cycles, since the informational content of asset prices displays business cycle variation.

Figure 4 in Appendix A illustrates a sample path of the economy in the special case that real investment productivity  $\theta_t$  and the aggregate expected return to trees  $\xi_t$  are random walks. The risk premia from informational risk is countercyclical because, on average, firms underreact to slowdowns in  $\theta_t$  because of learning and the sluggishness of investment. Similarly, firms, on average, underreact to the growth in  $\theta_t$  during recoverings. Consequently, at the peak of business cycles, the filtered estimate  $\hat{\theta}_t^c$  is expected to be above the true  $\theta_t$  in the future, while at the trough it is expected to be below.

In addition to providing a measure of the strength of investor balance sheets  $\xi$ , financial prices reflect the average expectations of market participants about the strength of the economy. This provides a strong empirical prediction that asset returns have predictive power for future returns and macroeconomic aggregates that varies with the business cycle, which is strongest during economic transitions, and motivates more tests of asset pricing predictability that take this explicitly into account. Henkel, Martin, and Nardari (2011) and Dangl and Halling (2012), for instance, provide evidence of business cycle asymmetries in stock market return predictability.

Given the risk premia from the firm's perspective  $RP_{NREE}$ , one can construct the risk premium demanded by an individual household to hold firm equity

$$RP_{NREE}(i) = RP_{NREE} + \left( I - \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} R_\theta \right) (\hat{\theta}^c - \hat{\theta}(i)).$$

Since  $RP_{NREE}(i)$  is increasing in the pessimism of household  $i$ , lower  $\hat{\theta}(i)$  relative to the average  $\hat{\theta}^c$ , it follows that more pessimistic households demand higher compensation to hold firm equity, and for sufficient pessimism instead sit on their capital by investing it in the riskless asset. This pattern is consistent with the tightening of lending standards seen in the FRB Senior Loan Officer Survey during the recent recession and recovery. In support of this prediction, the survey respondents often cited a poor economic outlook, along with bank competition, as a key factor in shaping their lending standards.

## A.2. Trading Volume

An advantage of a heterogeneous agent framework is that I can explore business cycle implications for trading volume. Though trading volume and asset turnover have been studied extensively in the literature, relatively little attention has been given to their business cycle properties.<sup>37</sup> My analysis aims to help understand how differential and incomplete information influences trading volume over the business cycle and provides new empirical predictions.

To explore these issues, I derive a measure  $\mathcal{V}$  on trading volume from informational trading in debt markets at any given instant in the economy. To do so, I recognize that households trade for informational and market-making reasons, and each invests a fraction of its wealth

$$1 - \bar{\mathbf{1}}'_{2 \times 1} X(i) = -\frac{1}{\sigma_k^2 (\Sigma + \sigma_s^2)} \frac{I \Sigma \sigma_e^2 \sigma_s}{\sigma_e^2 + \sigma_s^2 R_\theta^2 \Sigma} Z^s(i) - \left( 1 - \frac{\frac{\sigma_A^2}{\sigma_k^2} I \Sigma R_\theta \sigma_s^2}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta^2 \Sigma} \right) \frac{\sigma_e}{\sigma_A^2} Z^e(i),$$

in riskless debt, where I have substituted for the riskless rate from Proposition 1. Intuitively, informational and market-making households take the offsetting position against the demand for trees plus a directional bet on the prospects of the economy based on the noise in their private signals. I now construct pseudo market-making traders that start with wealth  $W$ ,

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<sup>37</sup>See Lo and Wang (2009) for a survey of this literature.



where  $W = qK + \int p(i) di$  is the aggregate wealth in the economy, and always receive the same signal noise  $Z^s(i)$  and idiosyncratic shock to the growth rate of trees  $Z^e(i)$ . For simplicity, I assume that the volatility  $\sigma_e$  of the idiosyncratic piece of the expected return to trees,  $\xi(i)$ , is sufficiently large that households can safely ignore it as a signal about real investment productivity  $\theta_t$ . From Proposition 6, in this limit the signal-to-noise ratio in the riskless rate  $R_\theta(I, \Sigma)$  approaches  $\frac{\sigma_A^2}{\sigma_k^2} I \frac{\Sigma}{\Sigma + \sigma_s^2}$ .

This construction of pseudo traders is meant to mitigate the trading that arises because of the generational structure of households, which mechanically leads to large changes in individual trader positions. I do not view the simplification as material for my results since I am abstracting from changes in positions that occur because of large changes in beliefs resulting from the myopic nature of households, and large changes that occur because of fluctuations in the expected return of trees, which are static effects over the business cycle.

The informational and market-making traders each enter the market with a position  $X_I(i) = \left(1 - \vec{1}'_{2 \times 1} X(i)\right) W$ , where  $\mathbf{1}_{2 \times 1}$  is the  $2 \times 1$  unit vector. The martingale component of the position they will trade to have is then

$$dX_I(i) - E[dX_I(i) \mid \mathcal{F}] = - \left( \frac{1}{\sigma_k^2} \frac{I \Sigma \sigma_s}{\Sigma + \sigma_s^2} Z^s(i) + \frac{\sigma_e}{\sigma_A^2} Z^e(i) \right) qK \sigma_k dZ^k.$$

Following the insights of Xiong and Yan (2010), I aggregate the local volatility of these position changes as a measure of trading volume, since the price of riskless debt is always 1,<sup>38</sup>

$$\frac{1}{dt} E[v \mid \mathcal{F}] = (qK)^2 \int_0^1 \left( (1 - \mathbf{1}'_{2 \times 1} X(i)) \sigma_k \right)^2 di.$$

Substituting for  $1 - \vec{1}'_{2 \times 1} X(i)$  and applying the weak LLN, I arrive at

$$\frac{1}{dt} E[v \mid \mathcal{F}] = (qK)^2 \left( \left( \frac{I \Sigma}{\Sigma + \sigma_s^2} \frac{\sigma_s}{\sigma_k} \right)^2 + \left( \frac{\sigma_e \sigma_k}{\sigma_A^2} \right)^2 \right).$$

In the absence of informational frictions, this expression reduces to

$$\frac{1}{dt} E[v^* \mid \mathcal{F}] = (qK)^2 \left( \frac{\sigma_e \sigma_k}{\sigma_A^2} \right)^2,$$

which represents the level of pseudo trading volume not driven by information. Thus the

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<sup>38</sup>Xiong and Yan (2010) motivates this measure by recognizing that the absolute value of realized position changes over small intervals is finite and increasing, on average, in the volatility of the position change.

difference  $\frac{1}{dt}E[v | \mathcal{F}] - \frac{1}{dt}E[v^* | \mathcal{F}]$  delivers me my measure of trading volume from informational trading  $\mathcal{V}$  in debt markets

$$\mathcal{V} = \left( \frac{I\Sigma}{\Sigma + \sigma_s^2} \frac{\sigma_s}{\sigma_k} qK \right)^2.$$

When there is no asymmetric information among households, either  $\sigma_s \searrow 0$  and all households are equally informed,  $\sigma_s \nearrow \infty$  and all households are equally naive, or  $\Sigma \searrow 0$ , and there is no uncertainty about  $\theta$ , then informational trading volume  $\mathcal{V}$  is zero. Intuitively, households trade when they have heterogeneous information on which to speculate against each other.

Trading volume  $\mathcal{V}$  from informational trading is increasing in the level of uncertainty  $\Sigma$  and hump-shaped in real investment  $I$  (since  $q = \frac{a-I}{\rho}$ ). Similar to Xiong and Yan (2010), this measure of trading volume is increasing in the disagreement among investors, as measured by  $\Sigma(i)$ , since when  $\sigma_e$  is arbitrarily large,  $\Sigma(i) = \frac{\sigma_s^2}{\Sigma + \sigma_s^2} \Sigma$  is increasing in  $\Sigma$ . Li and Li (2014) provide evidence that belief dispersion about macroeconomic conditions positively correlates with stock market turnover. This pattern can help explain why market prices are most informative about investment productivity during economic transitions, which is when a negative financial shock can be particularly devastating. Market prices have their highest information content during these parts of the business cycle because they are when households are trading intensely on their private information.

### B. Real Investment and Tobin's $Q$

Information frictions give rise to business cycle variation in the relationship between stock prices and real investment.<sup>39</sup> In my model, the instantaneous realized return to holding firm equity  $dR_t^k$  is

$$dR_t^k = \frac{D + d(qK)}{qK} = \left( \frac{a\rho}{a-I} - \delta + I\sqrt{\Sigma}e \right) dt + \sigma_k dZ^k,$$

where  $e = \frac{\theta - \hat{\theta}^c}{\sqrt{\Sigma}} \sim \mathcal{N}(0, 1)$ . Consequently, the predictability of investment for stock returns varies over time and depends on whether firms are over or underreacting to  $\theta$ . This conditional  $q$ -theory relationship arises because 1) firms act on their expectations rather than the true, hidden real investment productivity, and 2) households trade more aggressively on their private information the higher  $I$  and  $\Sigma$  are in the economy.

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<sup>39</sup>I thank Eric Loualiche for pointing this out to me.

Information frictions also introduce business cycle variation in the link between Tobin's  $Q$  measured from firms' market value and the cost of installing new capital relative to its replacement cost. From Proposition 4, firms adjust their level of real investment until the market value of each unit of capital,  $q$ , is equal to the inverse of their conditional expectation of real investment productivity  $\hat{\theta}^c$ , or  $q = \frac{1}{\hat{\theta}^c}$ . Since  $\frac{1}{\theta}$  represents the true cost for firms of installing new capital, where  $\theta$  is the true, hidden real investment productivity, while 1 is its replacement cost,  $q - \frac{1}{\theta}$  represents a time-varying wedge between the market's valuation of firms' assets and their recorded value. Assuming the process for  $\theta$  is such that  $E\left[\frac{1}{\theta} \mid \mathcal{F}^c\right] < \infty$ , it is straightforward to compute that

$$E\left[q - \frac{1}{\theta} \mid \mathcal{F}^c\right] = \frac{1}{\hat{\theta}^c} E\left[\frac{1}{\theta} (\theta - \hat{\theta}^c) \mid \mathcal{F}^c\right] = \frac{1}{\hat{\theta}^c} Cov\left[\frac{1}{\theta}, \theta \mid \mathcal{F}^c\right] < 0.$$

Consequently, on average, the market undervalues the prospects of firms than in the absence of uncertainty. Since firms are atomistic and behave competitively, they cannot exploit this systematic undervaluation to collectively disinvest. As a result, there is, on average, overinvestment in equilibrium, as the numerical exercise from Section VI suggests. For the parameters listed in Table 1 of Appendix A, average real investment in the perfect-information benchmark economy is approximately 0.1112, while with informational frictions it is 0.1125.

## VIII. Conclusion

In this paper, I develop a dynamic model of information aggregation in financial markets in a macroeconomic setting where both financial investors and firms learn about the productivity of real investment from market prices. My dynamic framework features a feedback loop between investor trading behavior and firm real investment decisions by which noise in financial prices can feed into real investment through learning by firms, and then feed back into financial prices through the impact of learning and investment on the trading incentives of market participants. This feedback loop highlights a possible amplification mechanism through which the financial crisis of 2008 contributed to the deep recession and anemic recovery in the US by distorting firm expectations about the strength of the US economy.

While the strength of signals from real activity is procyclical, that of financial signals is strongest during economic transitions. This occurs because the value of private information that financial investors have increases with uncertainty about real investment productivity,

which is countercyclical, and more information is aggregated into prices as investors speculate against each other on their private information. As a result, noise in financial signals during recessions can have a persistent impact on investment and output through the learning channel.

I then explore the welfare and empirical implications of my model. Informational frictions introduce a role for policy to provide guidance to economic agents about the current state of the economy. As an empirical prediction of my model, informational frictions also give rise to an informational component in asset risk premia that has predictive power for future returns and real activity. This predictive power is greatest during economic transitions when trading volume from informational trading is highest. Finally, informational frictions also give rise to a conditional relationship between investment and Tobin's Q.

In addition to depressed recoveries, the feedback loop explored in this paper generates endogenous boom and bust cycles for investment as a result of noise in financial prices. This mechanism may help explain other economic phenomenon such as, for instance, the internet or housing bubble. Noise in housing prices from the expansion of credit during the early 2000's may have distorted expectations about the industry's profitability and, as the housing sector grew, home buyers and builders learned about the true prospects of the housing market, which triggered a bust. A central theme to this learning mechanism is that there is heightened risk during recessions of depressed recoveries and endogenous boom-bust cycles as a result of noise in financial prices.

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## Appendix A: Tables and Figures

In the numerical experiments that follow, I treat one time unit (t.u) as a year. I set the subjective discount rate  $\rho$  to be 0.02 and depreciation  $\delta$  to be 0.10 following the literature. I choose  $a = 0.2$  to match the average ratio of annual US GDP to the US capital stock from 1973 to 2011. I choose the mean-reversion parameters of  $\theta$  and  $\xi$ ,  $\lambda_\theta$  and  $\lambda_\xi$ , respectively, to be 0.06, corresponding to an annual AR(1) persistence of about 0.94. Following He and Krishnamurthy (2014), I set the volatility of the capital quality shock  $\sigma_k$  to be 0.03, and I choose the long-run mean of real investment productivity  $\bar{\theta}$  to be 0.23 to match the average level of real investment  $I$  in my model's stationary distribution to the historical US investment-capital ratio from 1973 to 2010 of about 0.11. I set the minimum level of investment  $\underline{I}$  to be  $1 \times 10^{-6}$  so that the investment constraint never binds in equilibrium.

Given the stylized structure of my model, and that financial fundamentals tend to be more volatile than their real counterparts, I choose reasonable values for the remaining parameters.

$a$	0.20	$\rho$	0.02	$\delta$	0.10
$\lambda_\theta$	0.06	$\sigma_\theta$	0.02	$\bar{\theta}$	0.23
$\lambda_\xi$	0.06	$\sigma_\xi$	0.10	$\bar{\xi}$	0.00
$\sigma_k$	0.03	$\sigma_A$	0.10	$\sigma_s$	0.05
$\sigma_e$	0.10	$\underline{I}$	$1 \times 10^{-6}$		

Table 1: Parameter Values

Table 2 lists several simulated moments from the stationary distribution given the parameter values in Table 1.

$E[I_0]$	0.1125
$E[\Sigma_0]$	0.0025
$Corr[\hat{\theta}_0^c, I_0]$	0.482
$Corr[I_0, \Sigma_0]$	-0.899
$Corr[\hat{\theta}_0^c, \Sigma_0]$	-0.356

Table 2: Simulated Moments

That  $Cov_p(\hat{\theta}_0^c, \Sigma_0)$ ,  $Cov_p(I_0, \Sigma_0) < 0$  is a generic feature across parameter values.

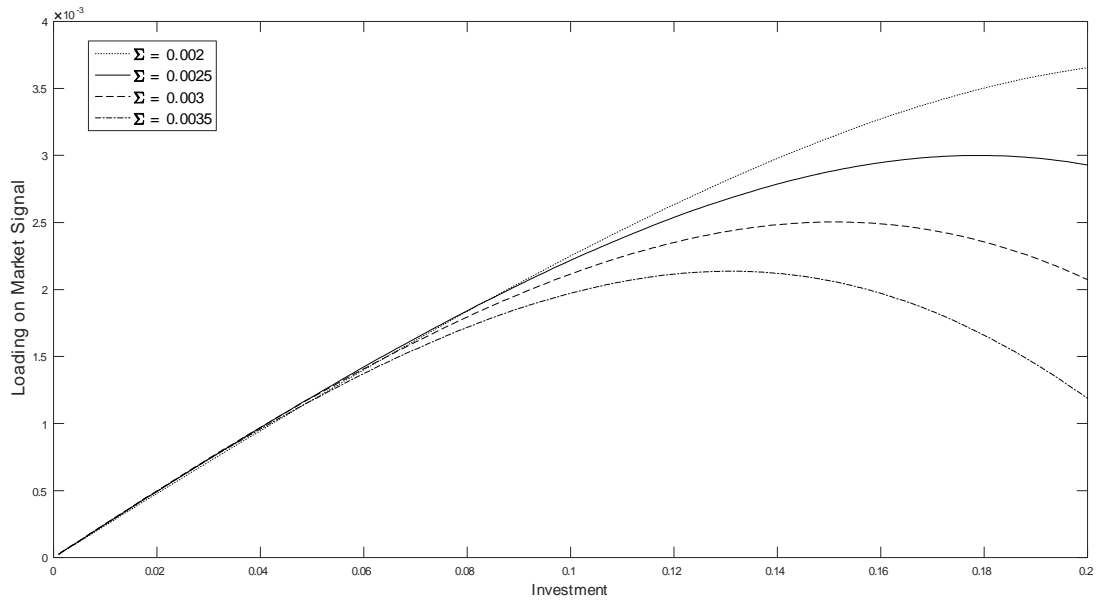


Figure 1: Loading on Market Signal for Filtered Real Investment Productivity  $\hat{\theta}^c = 0.23$

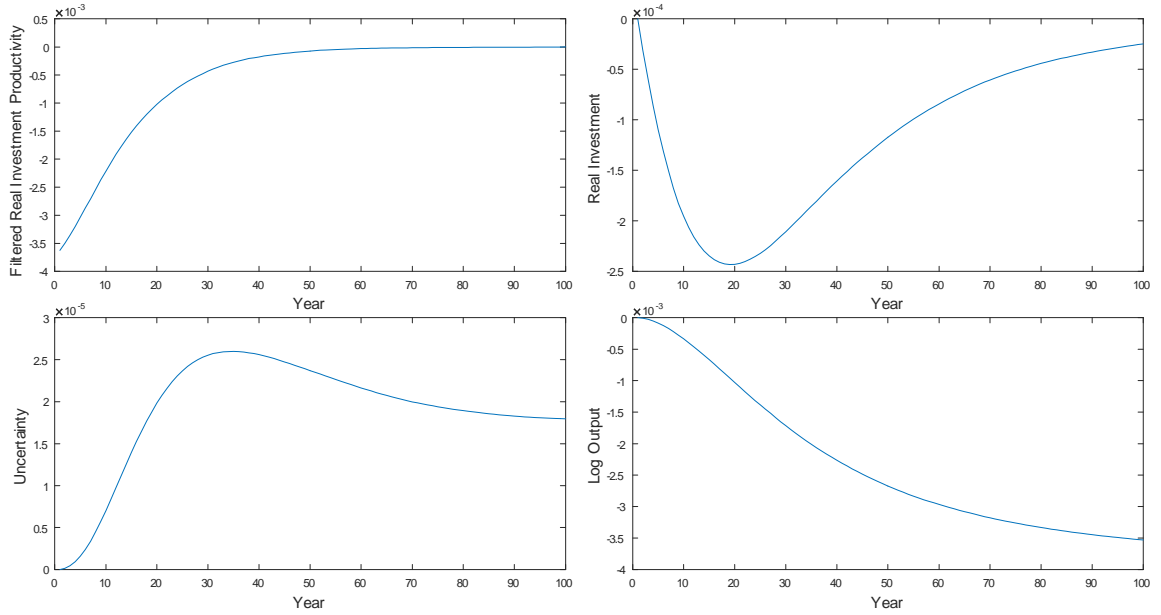


Figure 2: Impulse Response to a Unit Financial Shock During an Expansion

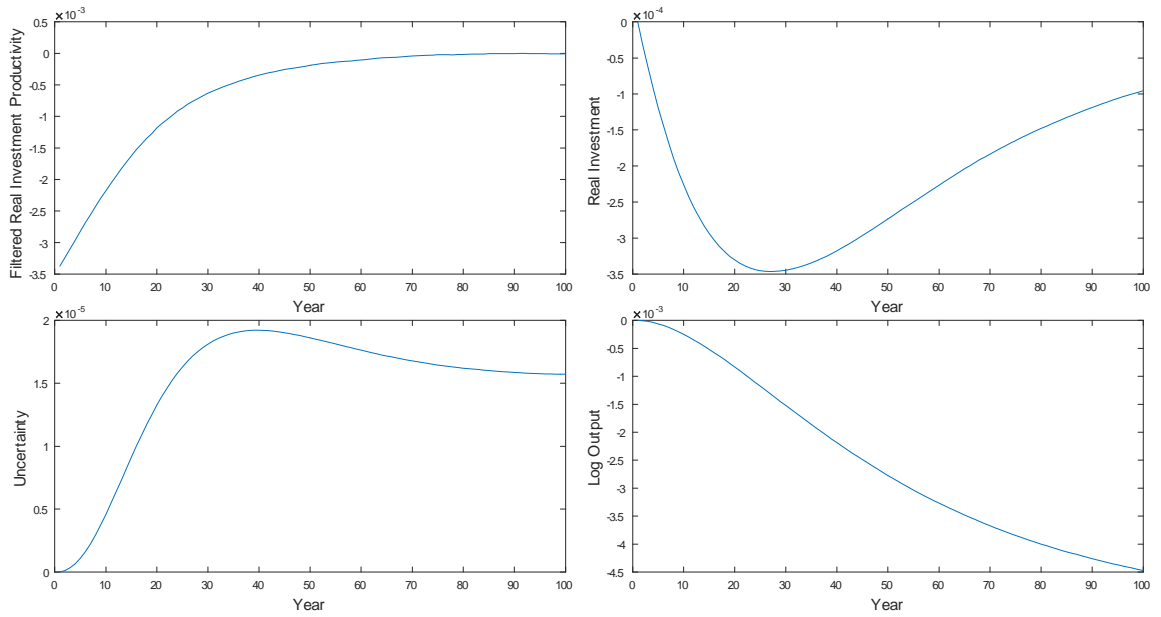


Figure 3: Impulse Response to a Unit Financial Shock During a Contraction

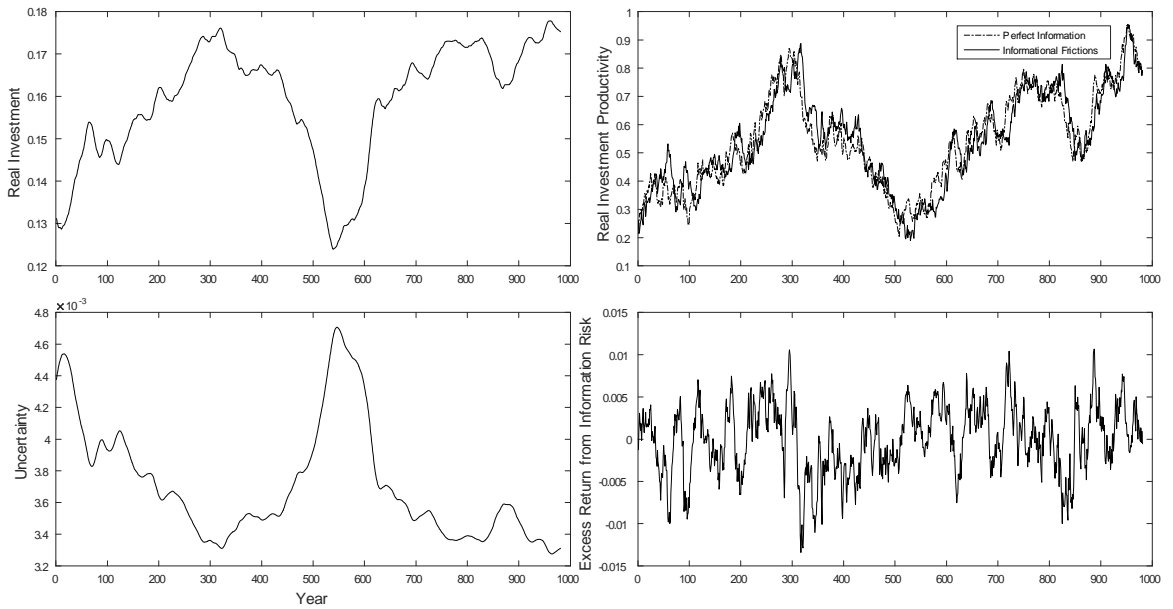


Figure 4: Sample Path of the Economy

## Appendix B: Proofs of Propositions

*Proof of Proposition 2:*

Households solve the optimization problem (4) subject to equation (12). In a recursive competitive equilibrium, all equilibrium objects are functions of the state of the economy from the household's perspective  $(w(i), \hat{\theta}(i), \xi(i), h)$ , where  $h$  is a list of general equilibrium objects including  $\log K$ ,  $q$ , and  $r$ .<sup>40,41</sup>

Taking the limit of problem (4) as  $\Delta t \searrow dt$ , assuming  $v$  is twice differentiable in its arguments, I can differentiate  $v$  and take expectations to find

$$\rho v(i) = \sup_{\{c(i), X(i)\}} \log c(i) + \partial_w v(i) \frac{1}{dt} E [dw(i) \mid \mathcal{F}^i] + \frac{1}{2} \partial_{ww} v(i) \frac{1}{dt} d \langle w(i) \mid \mathcal{F}_t^i \rangle + \frac{1}{dt} \partial_t v(i), \quad (\text{B.1})$$

subject to the law of motion of  $w(i)$  (12), and  $\langle \cdot \mid \mathcal{F}_t^i \rangle$  indicates quadratic variation under the measure  $\mathcal{F}_t^i$ . The  $\partial_t v$  term is meant to capture the additional dependence of the drift of the household's bequest utility  $v$  on the vector of general equilibrium objects  $h$  that the household takes as given. Equation (B.1) is the usual Hamilton-Jacobi-Bellman (HJB) equation for optimal control. Necessity and sufficiency of the FOCs for the optimal controls  $\{c(i), X(i)\}$  follows from the concavity of their programs.

Before deriving the FOCs of the HJB equation (B.1) for households, it is useful to recognize that all Wiener processes  $\tilde{Z}_t^\xi(i)$  and  $\tilde{Z}_t^k(i)$  will be uncorrelated under each household  $i$ 's measure since the true processes are uncorrelated and the change of measure under Girsanov's Theorem is equivalent to a change in drift.

Suppressing arguments for the bequest utility  $v$ , the FOCs of the HJB equation (B.1) are

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<sup>40</sup>Since the household treats prices as exogenous, the price of firm claims  $q$  and the riskless rate  $r$  are additional states for the household. This, however, only affects their optimal consumption and portfolio choices, in which they do not see the dependence of these prices on the Markov states.

<sup>41</sup>By the Martingale Representation Theorem for  $\mathcal{L}^2$  processes, all these objects will be continuous Itô-semimartingales with respect to the smallest filtration on which they are measurable to the household. The Wiener processes to which they are adapted, which will be common to all households, are absolutely continuous with respect to the true processes for real investment productivity  $\theta$ , shocks to growing trees  $\xi$ , and the aggregate diffusion for  $K$ .

given by

$$\begin{aligned}
c(i) &: \frac{1}{c(i)} - \partial_w v \leq 0 \quad (= \text{if } c(i) > 0), \\
X(i) &: 0 = w(i) \partial_w v \begin{bmatrix} \frac{a-I}{q} + \frac{\partial_{Iq}}{q} I g + I \hat{\theta}(i) - r - \delta \\ \frac{1}{p_A} + \xi(i) - r \end{bmatrix} + w(i)^2 \partial_{ww} v X(i)' \begin{bmatrix} \sigma_k^2 & \beta \sigma_k \sigma_A \\ \beta \sigma_k \sigma_A & \sigma_A^2 \end{bmatrix} X(i) \\
&\quad + w(i) \partial_{wh} v \begin{bmatrix} d \langle \tilde{Z}^k(i), h \mid \mathcal{F}^i \rangle \\ d \langle \tilde{Z}^A, h \mid \mathcal{F}^i \rangle \end{bmatrix}
\end{aligned}$$

from which follows that

$$X(i) = - \begin{bmatrix} \sigma_k^2 & 0 \\ 0 & \sigma_A^2 \end{bmatrix}^{-1} \left( \frac{\partial_w v}{w(i) \partial_{ww} v} \begin{bmatrix} \frac{a-I}{q} + \frac{\partial_{Iq}}{q} I g + I \hat{\theta}(i) - r - \delta \\ \frac{1}{p_A} + \xi(i) - r \end{bmatrix} + \begin{bmatrix} \frac{d \langle \tilde{Z}^k(i), h \mid \mathcal{F}^i \rangle}{w(i) \partial_{ww} v} \\ \frac{d \langle \tilde{Z}^A, h \mid \mathcal{F}^i \rangle}{w(i) \partial_{ww} v} \end{bmatrix} \right).$$

While objects in  $h$  like  $r$  all have Itô-semimartingale representations by the Martingale Representation Theorem, I do not expand out the quadratic covariation expressions for brevity.

Given that households have log utility, I conjecture that  $v(w(i), \hat{\theta}(i), \xi(i), h) = C \log w(i) + f(\hat{\theta}(i), \xi(i), h)$ . This conjecture implies that

$$\begin{aligned}
c(i) &= \frac{w(i)}{C}, \\
X(i) &= \begin{bmatrix} \frac{\frac{a-I}{q} + \frac{\partial_{Iq}}{q} I g + I \hat{\theta}(i) - r - \delta}{\sigma_k^2} \\ \frac{\frac{1}{p_A} + \xi(i) - r}{\sigma_A^2} \end{bmatrix}.
\end{aligned}$$

Substituting this conjecture and the controls into the maximized HJB equation

$$\begin{aligned}
\rho v &= \log c + \partial_w v \left( w(i) X(i)' \begin{bmatrix} \frac{a-I}{q} + \frac{\partial_{Iq}}{q} I g + I \hat{\theta}(i) - r - \delta \\ \frac{1}{p_A} + \xi(i) - r \end{bmatrix} + r w(i) - c(i) \right) \\
&\quad + \frac{1}{2} \partial_{ww} v w(i)^2 X(i)' \begin{bmatrix} \sigma_k^2 & 0 \\ 0 & \sigma_A^2 \end{bmatrix} X(i) + \partial_t f(\hat{\theta}(i), \xi(i), h),
\end{aligned}$$

where  $\partial_t f(\hat{\theta}(i), \xi(i), h)$  is shorthand for remaining terms in the HJB equation, it follows

that  $C = \frac{1}{\rho}$ ,  $c(i) = \rho w(i)$ , and that  $f(\hat{\theta}(i), \xi(i), h)$  implicitly satisfies

$$\begin{aligned} \rho f(\hat{\theta}(i), \xi(i), \Pi(i), h) &= \log \rho + \frac{1}{\rho} \left( r - \rho + X(i)' \left[ \begin{array}{c} \frac{a-I}{q} + \frac{\partial I q}{q} I g + I \hat{\theta}(i) - r - \delta \\ \frac{1}{p_A} + \xi(i) - r \end{array} \right] \right) \\ &\quad - \frac{1}{2\rho} X(i)' \left[ \begin{array}{cc} \sigma_k^2 & 0 \\ 0 & \sigma_A^2 \end{array} \right] X(i) + \partial_t f(\hat{\theta}(i), \xi(i), h), \end{aligned}$$

which confirms the conjecture since  $x(i)$  does not depend on  $w(i)$ .

Recognizing that  $v(w(i), \hat{\theta}(i), \xi(i), h) = A \log w(i) + f(\hat{\theta}(i), \xi(i), h)$ , the envelope condition for the maximized HJB equation (B.1) evaluated at the optimal controls takes the form

$$\begin{aligned} \rho \partial_w v &= \partial_{ww} v \left( w(i) X(i)' \left[ \begin{array}{c} \frac{a-I}{q} + \frac{\partial I q}{q} I g + I \hat{\theta}(i) - r - \delta \\ \frac{1}{p_A} + \xi(i) - r \end{array} \right] + r w(i) - c(i) \right) \\ &\quad + \left( \frac{1}{2} \partial_{www} v w(i)^2 + \partial_{ww} v w(i) \right) X(i)' \left[ \begin{array}{cc} \sigma_k^2 & 0 \\ 0 & \sigma_A^2 \end{array} \right] X(i) \\ &\quad + \partial_w v \left( X(i)' \left[ \begin{array}{c} \frac{a-I}{q} + \frac{\partial I q}{q} I g + I \hat{\theta}(i) - r - \delta \\ \frac{1}{p_A} + \xi(i) - r \end{array} \right] + r \right). \end{aligned}$$

Applying Itô's Lemma directly to  $\partial_w v$ , one also has that

$$\begin{aligned} d(\partial_w v) &= \partial_{ww} v \left( w(i) X(i)' \left[ \begin{array}{c} \frac{a-I}{q} + \frac{\partial I q}{q} I g + I \hat{\theta}(i) - r - \delta \\ \frac{1}{p_A} + \xi(i) - r \end{array} \right] + r w(i) - c(i) \right) dt \\ &\quad + \frac{1}{2} \partial_{www} v w(i)^2 X(i)' \left[ \begin{array}{cc} \sigma_k^2 & 0 \\ 0 & \sigma_A^2 \end{array} \right] X(i) + \partial_{ww} v w(i) X(i)' \left[ \begin{array}{c} \sigma_k d\tilde{Z}^k \\ \sigma_A dZ^A \end{array} \right]. \end{aligned}$$

Taking expectations and substituting the envelope condition, it follows that

$$\frac{1}{dt} E \left[ \frac{d(\partial_w v)}{\partial_w v} \mid \mathcal{F}^i \right] = \rho - r - X(i)' \left[ \begin{array}{c} \frac{a-I}{q} + \frac{\partial I q}{q} I g + I \hat{\theta}(i) - r - \delta \\ \frac{1}{p_A} + \xi(i) - r \end{array} \right] - \frac{\partial_{ww} v}{\partial_w v} w(i) X(i)' \left[ \begin{array}{cc} \sigma_k^2 & 0 \\ 0 & \sigma_A^2 \end{array} \right] X(i).$$

Given  $\partial_w v = \frac{1}{w}$ , the solution for  $X(i)$ , and defining  $\Lambda_t(i) = e^{-\rho t} \frac{1}{w_t(i)}$  to be the pricing kernel of household  $i$ , it follows that

$$r = -\frac{1}{dt} E \left[ \frac{d\Lambda(i)}{\Lambda(i)} \mid \mathcal{F}^i \right]. \quad (\text{B.2})$$

From  $\Lambda_t(i) = e^{-\rho t} \frac{1}{w_t(i)}$ , the optimal choice of  $X(i)$ , and equation (B.2), it follows that

$$\begin{bmatrix} \frac{a-I}{q} dt + E \left[ \frac{d\Lambda(i)}{\Lambda(i)} + \frac{d(qK)}{qK} \mid \mathcal{F}^i \right] \\ \frac{1}{p_A} dt + E \left[ \frac{d\Lambda(i)}{\Lambda(i)} + \frac{dP}{P} \mid \mathcal{F}^i \right] \end{bmatrix} = \begin{bmatrix} \sigma_k^2 & 0 \\ 0 & \sigma_A^2 \end{bmatrix} X(i) dt = -Cov \left[ \frac{d(qK)}{qK}, \frac{d\Lambda(i)}{\Lambda(i)} \mid \mathcal{F}^i \right].$$

from which one arrives at

$$\begin{bmatrix} \frac{a-I}{q} dt + E \left[ \frac{d(\Lambda(i)qK)}{\Lambda(i)qK} \mid \mathcal{F}^i \right] \\ \frac{1}{p_A} dt + E \left[ \frac{d(\Lambda(i)P)}{\Lambda(i)P} \mid \mathcal{F}^i \right] \end{bmatrix} = \vec{0}_{2 \times 1},$$

which completes the proof.

*Proof of Proposition 3:*

Define  $\bar{R}_\theta(\zeta_t) = R_\theta(I_t, \Sigma_t)$ , and  $\bar{g}_t(\zeta_t) = g_t$ . Given  $\zeta_t$ , one can express the law of motion of the vector of public signals as

$$\begin{aligned} d\zeta_t = & A_0(\zeta_t) dt + \begin{bmatrix} I_t \\ \partial_\Sigma \bar{R}_\theta(\zeta_t) \frac{d\Sigma_t}{dt} + \partial_I \bar{R}_\theta(\zeta_t) I_t \bar{g}_t(\zeta_t) + (\lambda_\xi - \lambda_\theta) \bar{R}_\theta(\zeta_t) \end{bmatrix} \theta_t dt \\ & + \bar{b}_t(\zeta_t) dZ_t^\theta + \bar{B}_t(\zeta_t) dZ_t, \end{aligned}$$

where  $Z_t = [Z_t^k, Z_t^\xi]'$  and

$$\begin{aligned} A_0(\zeta_t) &= \begin{bmatrix} -\delta - \frac{1}{2}\sigma_k^2 \\ \bar{R}_\theta(\zeta_t) \lambda_\theta \bar{\theta} + \lambda_\xi \left( \bar{\xi} - \begin{bmatrix} 0 & 1 \end{bmatrix} \zeta_t \right) \end{bmatrix} \\ \bar{b}_t(\zeta_t) &= \begin{bmatrix} 0 \\ \bar{R}_\theta(\zeta_t) \sigma_\theta \end{bmatrix}, \\ \bar{B}_t(\zeta_t) &= \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_\xi \end{bmatrix}, \end{aligned}$$

with  $\bar{R}_\theta(\zeta_t)$  uniformly bounded and  $\bar{R}_\theta(\zeta_t) > 0 \forall \zeta_t$ . By Theorem 7.17 of Lipster and Shiryaev (1977), then one can construct the vector of standard Wiener processes  $\tilde{Z} =$



$(\tilde{Z}_t, \mathcal{F}_t^c)$  where  $\tilde{Z}_t = [\tilde{Z}_t^k, \tilde{Z}_t^r]'$  admits the representation

$$\tilde{Z}_t = \int_0^t [\bar{b}_t(\zeta_t) \bar{b}_t(\zeta_t)' + \bar{B}_s(\zeta_s) \bar{B}_s(\zeta_s)']^{-1/2} \times \left( d\zeta_s - A_0(\zeta_t) dt - \begin{bmatrix} I_t \\ \partial_{\Sigma} \bar{R}_{\theta}(\zeta_t) \frac{d\Sigma_t}{dt} + \partial_I \bar{R}_{\theta}(\zeta_t) I_t \bar{g}_t(\zeta_t) - \lambda_{\theta} \bar{R}_{\theta}(\zeta_t) \end{bmatrix} \hat{\theta}_t^c dt \right),$$

where  $\hat{\theta}_t^c = E[\theta_t | \mathcal{F}_t^c]$  is the conditional expectation of  $\theta_t$  w.r.t.  $\mathcal{F}_t^c$ . That  $\tilde{Z}$  are standard Wiener processes can be verified directly from Levy's three properties that uniquely identify Wiener processes. That  $\tilde{Z}$  is a martingale generator for  $\mathcal{F}_t^c$  follows since  $\tilde{Z}$  generates  $K$  and  $r$  trivially, from which the other objects of  $\mathcal{F}_t^c$  can be generated, and Lemma 4.9 guarantees the existence of a representation for the driver (which possibly depends on the unobservable  $\theta_t$ ) in the Martingale Representation Theorem (Theorem 5.8) that is measurable w.r.t  $\zeta_t$  P-a.s.

Given that  $\theta_t$  has the representation

$$\theta_t = \int_0^t \lambda_{\theta} (\bar{\theta} - \theta_s) ds + \int_0^t \sigma_{\theta} dZ_s^{\theta},$$

it follows from similar arguments that lead to the proof of Theorem 12.7 that  $\hat{\theta}_t^c$  has the representation

$$\hat{\theta}_t^c = \int_0^t \left( d \left\langle \begin{matrix} S \\ Q \end{matrix}, \sigma_{\theta} Z^{\theta} \right\rangle_s + Cov \left[ \theta_s, \begin{bmatrix} I_s \\ \partial_{\Sigma} \bar{R}_{\theta}(\zeta_t) \frac{d\Sigma_t}{dt} + \partial_I \bar{R}_{\theta}(\zeta_t) I_s \bar{g}_t(\zeta_t) + (\lambda_{\xi} - \lambda_{\theta}) \bar{R}_{\theta}(\zeta_t) \end{bmatrix} \theta_s \mid \mathcal{F}_s^c \right] \right) \left[ \bar{b}_t(\zeta_t) \bar{b}_t(\zeta_t)' + \bar{B}_s(\zeta_s) \bar{B}_s(\zeta_s) \right]^{-1/2} d\tilde{Z}_s + \int_0^t \lambda_{\theta} (\bar{\theta} - \hat{\theta}_s^c) ds,$$

where  $d \langle \xi, Z^{\theta} \rangle_t$  is the quadratic covariation of  $\xi_t$  and  $Z_t^{\theta}$ . It is easy to see that  $Cov[\theta_s, \theta_s | \mathcal{F}_s^c] = Var[\theta_s | \mathcal{F}_s^c] = \Sigma_s$ . The covariance matrix in equation (B.3) is given by

$$\bar{b}_t(\zeta_t) \bar{b}_t(\zeta_t)' + \bar{B}_s(\zeta_s) \bar{B}_s(\zeta_s)' = \begin{bmatrix} \sigma_k^2 & 0 \\ 0 & (\bar{R}_{\theta}(\zeta_t) \sigma_{\theta})^2 + \sigma_{\xi}^2 \end{bmatrix},$$

from which follows that

$$[\bar{b}_t(\zeta_t) \bar{b}_t(\zeta_t)' + \bar{B}_s(\zeta_s) \bar{B}_s(\zeta_s)']^{-1/2} = \begin{bmatrix} \frac{1}{\sigma_k} & 0 \\ 0 & \frac{1}{\sqrt{(\bar{R}_{\theta}(\zeta_t) \sigma_{\theta})^2 + \sigma_{\xi}^2}} \end{bmatrix}$$

Thus it follows that  $\hat{\theta}_t^c$  follows the law of motion

$$d\hat{\theta}_t^c = \frac{I_t \Sigma_t}{\sigma_k} d\tilde{Z}_t^k + \frac{\bar{R}_\theta(\zeta_t) \sigma_\theta^2 + (\partial_\Sigma \bar{R}_\theta(\zeta_t) \frac{d\Sigma_t}{dt} + \partial_I \bar{R}_\theta(\zeta_t) I_t \bar{g}_t(\zeta_t) + (\lambda_\xi - \lambda_\theta) \bar{R}_\theta(\zeta_t)) \Sigma_t}{\sqrt{(\bar{R}_\theta(\zeta_t) \sigma_\theta)^2 + \sigma_\xi^2}} d\tilde{Z}_t^r + \lambda_\theta (\bar{\theta} - \hat{\theta}_t^c) dt.$$

Given the common Gaussian prior of households  $N(\hat{\theta}_0^c, \Sigma_0)$ , establishing the conditional Gaussianity of the posterior  $\theta_t | \mathcal{F}_t^c$  can be done through similar arguments to those made in Chapter 11 of Lipster and Shiryaev (1977) with the appropriate regularity conditions. Similar to the arguments of Theorem 12.7, one can also establish that the conditional variance of beliefs  $\Sigma_t = Var[\theta_t | \mathcal{F}_t^c]$  follows the deterministic law of motion

$$\frac{d\Sigma_t}{dt} = \sigma_\theta^2 - 2\lambda_\theta \Sigma_t - \left( \frac{I_t \Sigma_t}{\sigma_k} \right)^2 - \frac{(\bar{R}_\theta(\zeta_t) \sigma_\theta^2 + (\partial_\Sigma \bar{R}_\theta(\zeta_t) \frac{d\Sigma_t}{dt} + \partial_I \bar{R}_\theta(\zeta_t) I_t \bar{g}_t(\zeta_t) + (\lambda_\xi - \lambda_\theta) \bar{R}_\theta(\zeta_t)) \Sigma_t)^2}{(\bar{R}_\theta(\zeta_t) \sigma_\theta)^2 + \sigma_\xi^2}, \quad (\text{B.4})$$

which is a second-order polynomial in  $\frac{d\Sigma_t}{dt}$ , from which follows from equation (B.4) that

$$\frac{d\Sigma_t}{dt} = -\frac{B(\zeta_t)}{2A(\zeta_t)} \pm \frac{1}{2A(\zeta_t)} \sqrt{2B(\zeta_t) - 4A(\zeta_t) \left( 2\lambda_\theta \Sigma_t - \sigma_\theta^2 + I_t^2 \frac{\Sigma_t^2}{\sigma_k^2} \right) - 1},$$

where

$$A(\zeta_t) = \frac{(\partial_\Sigma \bar{R}_\theta(\zeta_t) \Sigma_t)^2}{(\bar{R}_\theta(\zeta_t) \sigma_\theta)^2 + \sigma_\xi^2},$$

$$B(\zeta_t) = 1 + 2\partial_\Sigma \bar{R}_\theta(\zeta_t) \Sigma_t \frac{\bar{R}_\theta(\zeta_t) \sigma_\theta^2 + (\partial_I \bar{R}_\theta(\zeta_t) I_t \bar{g}_t(\zeta_t) + (\lambda_\xi - \lambda_\theta) \bar{R}_\theta(\zeta_t)) \Sigma_t}{(\bar{R}_\theta(\zeta_t) \sigma_\theta)^2 + \sigma_\xi^2}.$$

In the absence of a market signal, the conditional variance of beliefs  $\Sigma$  would instead evolve according to

$$\frac{d\Sigma_t}{dt} = \sigma_\theta^2 - 2\lambda_\theta \Sigma_t - I_t^2 \frac{\Sigma_t^2}{\sigma_k^2}.$$

Taking the limit as  $\sigma_s \nearrow \infty$ , and approximating  $\sqrt{1-x}$  with a Taylor series expansion as  $1 - \frac{1}{2}x$ , I obtain this limit when I select the larger of the two roots from the full law of motion of  $\Sigma$ . Selecting the larger root also ensures that  $\Sigma$  reflects back from its boundary as  $\Sigma \searrow 0$ , which prevents  $\Sigma = 0$  from being an absorbing point, and that  $\frac{d\Sigma_t}{dt}$  approaches its

appropriate limit as  $\sigma_s \searrow 0$ .<sup>42</sup>

The conditional variance of beliefs  $\Sigma$  is trivially bounded from below by 0. To find the upper bound, consider the case when all public signals are completely uninformative  $\forall t$ , then  $\Sigma$  follows the law of motion

$$\frac{d\Sigma}{dt} = \sigma_\theta^2 - 2\lambda_\theta \Sigma_t,$$

which has the steady-state solution  $\Sigma_t = \frac{\sigma_\theta^2}{2\lambda_\theta}$ . Since any informativeness of the public signals reduces the conditional variance of beliefs,  $\Sigma_t \leq \frac{\sigma_\theta^2}{2\lambda_\theta}$ .

To find the relationship between  $\hat{\theta}_t^c$  and  $\hat{\theta}_t^c(i)$  for households, I make use of the Law of Iterated Expectations to write

$$\hat{\theta}_t^c(i) = E[\theta_t | \mathcal{F}_t^i] = E[\theta_t^c | s_t(i)],$$

where  $\theta_t^c = \theta_t | \mathcal{F}_t^c$ . Consider the common knowledge estimate  $\hat{\theta}_t^c$ , I can arrive at the estimate of household  $i$   $\hat{\theta}_t^c(i)$  by updating  $\mathcal{F}_t^c$  with the private signals of household  $i$   $\{s_t(i), \xi_t(i)\}$ . Since both the public estimate  $\hat{\theta}_t^c$  and the signals  $\{s_t(i), \xi_t(i)\}$  are jointly Gaussian, which is apparent from the linearity of the Kalman Filter in the data  $\{\zeta_s, \theta_s\}_{s \leq t}$ , the process of updating the conditional mean is an exercise in the updating of two sets of Gaussian random variables.

$$\begin{aligned} s_t(i) &= \theta_t + \sigma_s Z_t^s(i), \\ \xi_t(i) &= \xi_t + \sigma_s Z_t^e(i) = S_t - R_\theta(I_t, \Sigma_t) \theta_t + \sigma_e Z_t^e(i). \end{aligned}$$

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<sup>42</sup>In numerical experiments, the smaller root for  $\frac{d\Sigma_t}{dt}$  tends to be unstable and informational frictions dissipate quickly.

Since  $Z^s(i) \perp Z^e(i)$ , it follows then that

$$\begin{aligned}
\hat{\theta}_t(i) &= \hat{\theta}_t^c + Cov \left[ \theta_t, \begin{bmatrix} s_t(i) - E[s_t(i) | \mathcal{F}_t^c] \\ \xi_t(i) + E[\xi_t(i) | \mathcal{F}_t^c] \end{bmatrix} \mid \mathcal{F}_t^c \right]' \times \\
&\quad Var \left[ \begin{bmatrix} s_t(i) - E[s_t(i) | \mathcal{F}_t^c] \\ \xi_t(i) + E[\xi_t(i) | \mathcal{F}_t^c] \end{bmatrix} \begin{bmatrix} s_t(i) - E[s_t(i) | \mathcal{F}_t^c] \\ \xi_t(i) + E[\xi_t(i) | \mathcal{F}_t^c] \end{bmatrix}' \right]^{-1} \begin{bmatrix} s_t(i) - E[s_t(i) | \mathcal{F}_t^c] \\ \xi_t(i) + E[\xi_t(i) | \mathcal{F}_t^c] \end{bmatrix} \\
&= \hat{\theta}_t^c + \begin{bmatrix} \Sigma_t \\ -R_\theta(I_t, \Sigma_t) \Sigma_t \end{bmatrix}' \begin{bmatrix} \Sigma_t + \sigma_s^2 & -R_\theta(I_t, \Sigma_t) \Sigma_t \\ -R_\theta(I_t, \Sigma_t) \Sigma_t & R_\theta(I_t, \Sigma_t)^2 \Sigma_t + \sigma_e^2 \end{bmatrix}^{-1} \begin{bmatrix} s_t(i) - E[s_t(i) | \mathcal{F}_t^c] \\ \xi_t(i) + E[\xi_t(i) | \mathcal{F}_t^c] \end{bmatrix} \\
&= \hat{\theta}_t^c + \frac{\Sigma_t \sigma_e^2}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I_t, \Sigma_t)^2 \Sigma_t} (s_t(i) - E[s_t(i) | \mathcal{F}_t^c]) \\
&\quad - \frac{\sigma_s^2 R_\theta(I_t, \Sigma_t) \Sigma_t}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I_t, \Sigma_t)^2 \Sigma_t} (\xi_t(i) - E[\xi_t(i) | \mathcal{F}_t^c]).
\end{aligned}$$

Similarly, the conditional variance of household  $i$ 's estimate of  $\theta$  is

$$\begin{aligned}
\Sigma_t(i) &= \Sigma_t - Cov \left[ \theta_t, \begin{bmatrix} s_t(i) - E[s_t(i) | \mathcal{F}_t^c] \\ \xi_t(i) + E[\xi_t(i) | \mathcal{F}_t^c] \end{bmatrix} \mid \mathcal{F}_t^c \right]' \times \\
&\quad Var \left[ \begin{bmatrix} s_t(i) - E[s_t(i) | \mathcal{F}_t^c] \\ \xi_t(i) + E[\xi_t(i) | \mathcal{F}_t^c] \end{bmatrix} \begin{bmatrix} s_t(i) - E[s_t(i) | \mathcal{F}_t^c] \\ \xi_t(i) + E[\xi_t(i) | \mathcal{F}_t^c] \end{bmatrix}' \right]^{-1} \times \\
&\quad Cov \left[ \theta_t, \begin{bmatrix} s_t(i) - E[s_t(i) | \mathcal{F}_t^c] \\ \xi_t(i) + E[\xi_t(i) | \mathcal{F}_t^c] \end{bmatrix} \mid \mathcal{F}_t^c \right] \\
&= \Sigma_t - \frac{\Sigma_t^2 \sigma_e^2}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I_t, \Sigma_t)^2 \Sigma_t} - \frac{\sigma_s^2 R_\theta(I_t, \Sigma_t)^2 \Sigma_t^2}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I_t, \Sigma_t)^2 \Sigma_t} \\
&= \frac{\sigma_s^2 \sigma_e^2}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I_t, \Sigma_t)^2 \Sigma_t} \Sigma_t.
\end{aligned}$$

*Proof of Proposition 4:*

To find the optimal level of investment  $I$ , let me conjecture that  $E = E(t, K, I)$ . Then, by the Feynman-Kac Theorem and  $\frac{\Lambda_t}{\Lambda_0}, E_t > 0$ , the function  $E$  that solves each firm's problem (5) must solve the necessary condition

$$0 \geq \sup_{g_t} \frac{\left( a - I_t - \frac{1}{\rho} g_t I_t + \tau_t \right) K_t}{E_t} dt + E \left[ \frac{d(\Lambda_t E_t)}{E[\Lambda_t | \mathcal{F}_t^c] E_t} \mid \mathcal{F}_t^c \right],$$

which can be rewritten as

$$0 \geq \sup_{g_t} \frac{\left(a - I_t - \frac{1}{\rho} g_t I_t + \tau_t\right) K_t}{E_t} dt + E \left[ \frac{dE_t}{E_t} \mid \mathcal{F}_t^c \right] + \frac{E[d\Lambda_t \mid \mathcal{F}_t^c]}{E[\Lambda_t \mid \mathcal{F}_t^c]} + \frac{d\langle \Lambda_t, E_t \mid \mathcal{F}_t^c \rangle}{E[\Lambda_t \mid \mathcal{F}_t^c] E_t}. \quad (\text{B.5})$$

By Proposition 2, the pricing kernel of investor  $j$   $\Lambda_t(j)$  satisfies  $\frac{1}{dt} E \left[ \frac{d\Lambda_t(j)}{\Lambda_t(j)} \mid \mathcal{F}^j \right] = -r_t$ . Thus, by the Law of Iterated Expectations,  $E \left[ \frac{d\Lambda_t}{\Lambda_t} \mid \mathcal{F}^c \right] = -r_t$ , regardless of the distribution of ownership among households. Then, applying Itô's Lemma to  $E$ , equation (B.5) becomes

$$0 \geq \sup_{g_t} \frac{a - I_t}{E_t} K_t - \frac{I_t}{\rho} g_t K_t + \tau_t K_t + \frac{\partial_K E_t}{E_t} \left( I_t \hat{\theta}_t^c - \delta \right) K_t + \frac{1}{2} \frac{\partial_{KK} E_t}{E_t} \sigma_k^2 K_t^2 + \frac{\partial_t E_t}{E_t} - r_t + \frac{1}{dt} \frac{d\langle \Lambda_t, E_t \mid \mathcal{F}_t^c \rangle}{E[\Lambda_t \mid \mathcal{F}_t^c] E_t}, \quad (\text{B.6})$$

where  $\frac{1}{dt} \frac{d\langle \Lambda_t, E_t \mid \mathcal{F}_t^c \rangle}{\Lambda_t E_t}$  is the risk premium on firm equity. Since firms are perfectly competitive, they do not recognize, in equilibrium, that their actions affect the riskless rate  $r_t$  or the pricing kernel of shareholders  $\Lambda_t$ .

Firm effort  $g_t$  is chosen by the firm to achieve its optimal level of investment. Since equation (B.6) is (locally) riskless and linear in investment  $I_t$ , firms are effective risk-neutral and it follows that it must be the case that  $g_t$  satisfies

$$-1 + \partial_K E_t \hat{\theta}_t^c - \frac{1}{\rho} g_t = 0, \quad (\text{B.7})$$

or else there is a riskless gain to changing  $g$  if the marginal return to investment for firm value is positive or negative. By market clearing, the value of firm equity must be such that  $E_t = q_t K_t$ , where  $q_t = \frac{a - I_t}{\rho}$ . To see that  $E_t = q_t K_t$  satisfies the maximized form of equation (B.5), recall from Proposition 2 that  $E_t = q_t K_t$  satisfies at the optimal  $I_t$

$$\Lambda_t(i) \frac{a - I_t}{E_t} K_t dt + E \left[ \frac{d(\Lambda_t(i) E_t)}{E_t} \mid \mathcal{F}_t^i \right] = 0.$$

Let  $u_t(i) = \frac{x_t^k(i) w_t(i)}{q_t K_t}$  be the share of the firm owned by household  $i$ , such that  $\Lambda_t = \int u_t(i) \Lambda_t(i) di = \int e^{-\rho t} u_t(i) \frac{1}{w_t(i)} di$ . It then follows, by linearity and the finiteness of  $\Lambda_t$ , that

$$\frac{1}{dt} \frac{d\langle \Lambda_t, E_t \mid \mathcal{F}_t^c \rangle}{E[\Lambda_t \mid \mathcal{F}_t^c] E_t} = \int u_t(i) \frac{1}{dt} \frac{d\left\langle \frac{1}{w_t(i)}, K_t \mid \mathcal{F}_t^c \right\rangle}{K_t E \left[ \int u_t(j) \frac{1}{w_t(j)} dj \mid \mathcal{F}_t^c \right]} di = -\sigma_k^2 \frac{\int u_t(i) E \left[ \frac{x_t(i)}{w_t(i)} \mid \mathcal{F}_t^c \right] di}{E \left[ \int u_t(j) \frac{1}{w_t(j)} dj \mid \mathcal{F}_t^c \right]}.$$

Given the optimal position of households in firm equity from Proposition 2, and that  $w_t(i)$  is independent of  $\hat{\theta}_t(i)$  and  $Z^e(i)$  because of the generational structure of the economy, it follows that

$$-\frac{1}{dt} \frac{d \langle \Lambda_t, E_t | \mathcal{F}_t^c \rangle}{E[\Lambda_t | \mathcal{F}_t^c] E_t} = \frac{a - I_t}{q_t} + I_t \hat{\theta}_t^c - r_t - \delta.$$

Thus by direct integration, the linearity of the expectation and covariance operators, and the Law of Iterated Expectations, it follows that

$$\frac{a - I_t}{E_t} K_t + \frac{1}{dt} E \left[ \frac{dE_t}{E_t} | \mathcal{F}_t^c \right] + \frac{1}{dt} \frac{E[d\Lambda_t | \mathcal{F}_t^c]}{E[\Lambda_t | \mathcal{F}_t^c]} + \frac{1}{dt} \frac{d \langle \Lambda_t, E_t | \mathcal{F}_t^c \rangle}{E[\Lambda_t | \mathcal{F}_t^c] E_t} = 0.$$

Therefore, if  $E_t$  satisfies each household's Euler equation, then  $E_t = q_t K_t$  solves each firm's problem. To see that the transversality condition is satisfied, one has from the optimal investment position  $x^k(i)$  from Proposition 2 and the riskless rate from Proposition 1 that

$$\begin{aligned} x^k(i) &= \frac{\frac{\sigma_k^2}{\sigma_A^2} R_\theta(I, \Sigma) (\theta - \hat{\theta}^c) - \xi + I \hat{\theta}^c - \frac{I}{a-I} g - \delta + \sigma_A^2}{\sigma_A^2 + \sigma_k^2} + \frac{1}{\sigma_k^2} \frac{I \Sigma \sigma_e^2 \sigma_s}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} Z^s(i) \\ &+ \frac{1}{\sigma_k^2} \frac{I R_\theta(I, \Sigma) \Sigma \sigma_s^2 \sigma_e}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} Z^e(i), \end{aligned}$$

from which follows that  $E \left[ \int x_T^k(i) di | \mathcal{F}_0^c \right] = \frac{E \left[ I_T \hat{\theta}_T^c - \frac{I_T}{a-I_T} g_T - \xi_T | \mathcal{F}_0^c \right]}{\sigma_A^2 + \sigma_k^2} + \frac{\sigma_A^2 - \delta}{\sigma_A^2 + \sigma_k^2}$ . Assuming that investment  $g$  is unconstrained *a.s.*, it follows that

$$E \left[ \int x_T^k(i) di | \mathcal{F}_0^c \right] = \frac{\rho E \left[ \frac{I_T}{a-I_T} | \mathcal{F}_0^c \right] - e^{-\lambda \xi T} \hat{\xi}_0^c - (1 - e^{-\lambda \xi T}) \bar{\xi} + \sigma_A^2 - \delta}{\sigma_A^2 + \sigma_k^2},$$

since  $\xi_t$  follows an OU process. Consequently, since  $E_t = q_t K_t$ ,  $I_t < a$  *a.s.*, and  $\hat{\xi}_0^c = \bar{\xi}$ , it follows that

$$\lim_{T \nearrow \infty} E[\Lambda_T E_T | \mathcal{F}_0^c] = \lim_{T \nearrow \infty} e^{-\rho T} E \left[ \int x_T^k(i) di | \mathcal{F}_0^c \right] = \lim_{T \nearrow \infty} e^{-\rho T} \frac{\rho E \left[ \frac{I_T}{a-I_T} | \mathcal{F}_0^c \right] - \bar{\xi} + \sigma_A^2 - \delta}{\sigma_A^2 + \sigma_k^2} = 0,$$

and  $E_T$  satisfies the transversality condition. Thus it is the solution to each firm's problem and, from equation (B.7), it follows that

$$g = \rho \left( q \hat{\theta}^c - 1 \right) \mathbf{1} \left\{ I > \underline{I} \cup \hat{\theta}^c \geq \frac{\rho}{a - \underline{I}} \right\}.$$

*Proof of Proposition 5:*

By the second part of Proposition 3

$$\begin{aligned}
& \int \frac{w_t(i)}{W_t} \left( I_t \left( \hat{\theta}_t(i) - \hat{\theta}_t^c \right) - \xi_t(i) \right) di \\
&= I_t \frac{\Sigma_t \sigma_e^2 + \sigma_s^2 R_\theta (I_t, \Sigma_t)^2 \Sigma_t}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I_t, \Sigma_t)^2 \Sigma_t} \left( \theta_t - \hat{\theta}_t^c \right) \int \frac{w_t(i)}{W_t} di - \int \frac{w_t(i)}{W_t} \xi_t(i) di \\
&+ \frac{I_t \Sigma_t \sigma_e^2}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I_t, \Sigma_t)^2 \Sigma_t} \int \frac{w_t(i)}{W_t} (\sigma_s Z_t^s(i) + \sigma_e Z_t^e(i)) di. \tag{B.8}
\end{aligned}$$

Let me define the integral  $X_t$

$$X_t = \int_0^1 \psi_t(i) \nu_t(i) di.$$

where  $\nu_t(i) = \sigma_s Z_t^s(i) + \sigma_e Z_t^e(i)$  and  $\psi_t(i) = \frac{w_t(i)}{W_t} > 0$  is now a weight function, with  $\psi_t(i) \in (0, 1)$  on a set of full measure, whose integral is bounded on any set of positive measure and is 1 over the set  $i \in [0, 1]$ .

Importantly, since the law of motion of the price of firm equity  $q$  and the riskless rate  $r$  by conjecture do not depend on the wealth share or signal noise of any one household, the only difference in the wealth shares of households at time  $t$  are the histories of the fraction of wealth invested in firm equity  $\{x_u(i)\}_{u \leq t}$ , which differ across households only because of differences in signal noise. Therefore, conditional on the initial wealth share of households and the history of the fundamentals  $\mathcal{G}_t = \sigma(\{\theta_u, K_u, \xi_u\}_{u \leq t} \vee w_0)$ , the weights  $\psi_i(t)$  are independent across households.

First, I establish that  $X_t$  converges to its cross-sectional expectation  $E[X_t | \mathcal{G}_t]$  in the  $\mathcal{L}^2$  - norm. As an aside, I do not require convergence *a.s.* and rely on a weaker notion of convergence because of the issues discussed in Judd (1985).

Similar to Uhlig (1996), one can discretize the integral across  $i$  into a Riemann sum  $\Sigma(t, \varphi)$  with a partition  $\varphi$  with  $0 = i_0 < \dots i_j < \dots i_m = 1$  and midpoints  $\phi_j \in [i_{j-1}, i_j]$ ,  $j \in \{1, \dots, m\}$

$$\Sigma(t, \varphi) = \sum_{j=1}^m \psi_t(\phi_j) \nu_t(\phi_j) (i_j - i_{j-1}).$$

Conditional on  $\mathcal{G}_t$ ,  $E[X_t | \mathcal{G}_t]$  is a constant, and one recognizes by Chebychev's Inequality

that

$$\begin{aligned}
E \left[ (\Sigma(t, \varphi) - E[X_t | \mathcal{G}_t])^2 \mid \mathcal{G}_t \right] &= E \left[ \left( \sum_{j=1}^m (\psi_t(\phi_j) \nu_t(\phi_j) - E[X_t | \mathcal{G}_t]) (i_j - i_{j-1}) \right)^2 \mid \mathcal{G}_t \right] \\
&= E \left\{ \sum_{j=1}^m E \left[ (\psi_t(\phi_j) \nu_t(\phi_j) - E[X_t | \mathcal{G}_t])^2 \mid \mathcal{G}_t \right] (i_j - i_{j-1})^2 \right\} \\
&\leq \sum_{j=1}^m (i_j - i_{j-1})^2 \\
&\leq \varepsilon(\varphi),
\end{aligned}$$

where  $\varepsilon(\varphi) = \max_j (i_j - i_{j-1})$ . As  $\varepsilon(\varphi) \searrow 0$ , the above integral converges to the  $\mathcal{L}^2$  distance between  $\Sigma(t, \varphi)$  and  $E[X_t | \mathcal{G}_t]$  on the LHS and 0 on the RHS.

Therefore

$$\lim_{\varepsilon(\varphi) \searrow 0} E \left[ (\Sigma(t, \varphi) - E[X_t | \mathcal{G}_t])^2 \mid \mathcal{G}_t \right] = 0.$$

By Dominated Convergence and Slutsky's Theorem

$$\lim_{\varepsilon(\varphi) \searrow 0} E \left[ (\Sigma(t, \varphi) - E[X_t | \mathcal{G}_t])^2 \mid \mathcal{G}_t \right] = E \left[ (X_t - E[X_t | \mathcal{G}_t])^2 \mid \mathcal{G}_t \right].$$

Therefore

$$E \left[ (X_t - E[X_t | \mathcal{G}_t])^2 \mid \mathcal{G}_t \right] = 0,$$

which does not depend on the wealth share or signal noise of any individual household because  $E[X_t | \mathcal{G}_t] = g(\tilde{\omega}_t)$  for some  $\tilde{\omega}_t \in \mathcal{G}_t$ .

Since the choice of partition  $\varphi$  was arbitrary, the convergence result did not depend on my choice of partition, and therefore  $X_t$  and its convergence to  $g(\tilde{\omega}_t)$  in  $\mathcal{L}^2$  are well-defined. Furthermore, since convergence is in  $\mathcal{L}^2$ , the integral is  $g(\tilde{\omega}_t)$  *a.s.* and I can choose a modification of the process, if need be, under which it is always 0.<sup>43</sup> Given that this convergence is ex-post the realized sample path of the aggregate state variables  $\mathcal{G}_t$ , this convergence also holds unconditionally.

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<sup>43</sup>Though the convergence implies that the variance of  $X_t$  is zero over time,  $X_t$  can deviate from its expected value on a negligible subset of times.



Recognizing that  $E[\nu_t(i) \mid \mathcal{G}_t] = 0$ , it follows that

$$\begin{aligned} g(\tilde{\omega}_t) &= \frac{\Sigma_t \sigma_e^2}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I_t, \Sigma_t)^2 \Sigma_t} E[\psi_t(i) \nu_t(i) \mid \mathcal{G}_t] \\ &= \frac{\Sigma_t \sigma_e^2}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I_t, \Sigma_t)^2 \Sigma_t} E[\psi_t(i) \mid \mathcal{G}_t] E[\nu_t(i) \mid \mathcal{G}_t] = 0, \end{aligned}$$

since  $\psi_t(i)$  is independent of  $\nu_t(i) \forall i$  and  $E[\nu_t(i) \mid \mathcal{G}_t] = 0$ . Similarly, I can apply a weak LLN to  $\int_0^1 \frac{w_t(i)}{W_t} di$ , which holds on subintervals of  $[0, 1]$  *a.s.*, to arrive at

$$\begin{aligned} W_t &= E[w_t(i) \mid \mathcal{G}_t], \\ \int \xi(i) di &= \xi, . \end{aligned}$$

Thus equation (B.8) becomes

$$\int \frac{w_t(i)}{W_t} \left( I_t \left( \hat{\theta}_t(i) - \hat{\theta}_t^c \right) - \xi_t(i) \right) di = I_t \frac{\Sigma_t \sigma_e^2 + \sigma_s^2 \Sigma_t R_\theta (I_t, \Sigma_t)^2}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I_t, \Sigma_t)^2 \Sigma_t} \left( \theta_t - \hat{\theta}_t^c \right) - \xi_t,$$

*Proof of Proposition 1:*

Substituting  $q = \frac{a-I}{\rho}$  and  $P(i) = \frac{D(i)}{\rho}$ , optimal household demand for firm equity  $x(i)$  from Proposition 2, and optimal firm investment  $g$  from Proposition 4 into the market clearing condition for the market for riskless debt (9), and imposing  $W > 0$  and Proposition (5), one has that

$$r = \frac{\sigma_A^2}{\sigma_A^2 + \sigma_k^2} I \frac{\Sigma \sigma_e^2 + \sigma_s^2 R_\theta (I, \Sigma)^2 \Sigma}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I, \Sigma)^2 \Sigma} \left( \theta - \hat{\theta}^c \right) + \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi + \rho + \frac{\sigma_A^2 \left( I \hat{\theta}^c - \frac{I}{a-I} g - \delta \right) - \sigma_k^2 \sigma_A^2}{\sigma_A^2 + \sigma_k^2}.$$

and therefore, matching this with the conjectured representation equation (11), it follows

that

$$\begin{aligned}
r_0 &= \frac{\sigma_A^2 \left( \rho + I\hat{\theta}^c - \frac{I}{a-I}g - \delta \right) + \sigma_k^2 \rho - \sigma_k^2 \sigma_A^2}{\sigma_A^2 + \sigma_k^2}, \\
r_\theta &= \frac{\sigma_A^2}{\sigma_A^2 + \sigma_k^2} I \frac{\Sigma \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma}, \\
r_\xi &= \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2},
\end{aligned}$$

where  $R_\theta$  satisfies

$$R_\theta(I, \Sigma) = \frac{\sigma_A^2}{\sigma_k^2} I \frac{\Sigma \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma},$$

which expands to equation (10) and confirms the conjecture. Given optimal firm equity demand  $x(i)$  from Proposition 2, it follows that  $X(i)$  is given by

$$X(i) = \left[ \begin{array}{c} \frac{\rho - \frac{I}{a-I}g + I\hat{\theta}(i) - r - \delta}{\frac{\sigma_k^2}{\sigma_A^2}} \\ \frac{\rho + \xi(i) - r}{\sigma_A^2} \end{array} \right].$$

*Proof of Proposition 6:*

The discriminant  $\Delta$  of the cubic equation (10)

$$\begin{aligned}
\Delta &= 18 \frac{\sigma_A^4}{\sigma_k^4} \frac{\sigma_e^4}{\sigma_s^2} I^2 \left( \frac{1}{\sigma_s^2} + \frac{1}{\Sigma} \right) - 4 \frac{\sigma_A^8}{\sigma_k^8} I^4 \frac{\sigma_e^2}{\sigma_s^2} + \frac{\sigma_A^4}{\sigma_k^4} \sigma_e^4 I^2 \left( \frac{1}{\sigma_s^2} + \frac{1}{\Sigma} \right)^2 \\
&\quad - 4 \left( \frac{1}{\sigma_s^2} + \frac{1}{\Sigma} \right)^3 \sigma_e^6 - 27 \frac{\sigma_A^4}{\sigma_k^4} \frac{\sigma_e^4}{\sigma_s^4} I^2
\end{aligned}$$

Since  $I \leq a$ ,  $\Sigma \geq 0$ , and  $\sigma_s < 1$ , so that  $\frac{1}{\sigma_s^2} + \frac{1}{\Sigma} > 1$ , it follows that

$$\Delta < \left( a^2 \frac{\sigma_A^4}{\sigma_k^4} (18 + \sigma_s^2) - 4\sigma_e^2 \right) \frac{\sigma_e^4}{\sigma_s^2} \left( \frac{1}{\sigma_s^2} + \frac{1}{\Sigma} \right)^2 - 4 \frac{\sigma_A^8}{\sigma_k^8} I^4 \frac{\sigma_e^2}{\sigma_s^2} - 27 \frac{\sigma_A^4}{\sigma_k^4} \frac{\sigma_e^4}{\sigma_s^4}.$$

If  $\Delta < 0$ , then equation (10) has only one real, nonnegative root. It is thus sufficient that

$$\sigma_e > \frac{a}{2} \frac{\sigma_A^2}{\sigma_k^2} \sqrt{18 + \sigma_s^2},$$

for  $R_\theta$  to be unique. Rewriting equation (10) as

$$\left( R_\theta - \frac{\sigma_A^2}{\sigma_k^2} I \right) \left( \frac{1}{\sigma_e^2} R_\theta^2 + \frac{1}{\sigma_s^2} \right) + \frac{1}{\Sigma} R_\theta = 0,$$

it is clear, since  $R_\theta \geq 0$ , that  $R_\theta \leq \frac{\sigma_A^2}{\sigma_k^2} I$ ,  $\partial_I R_\theta \geq 0$ , and  $\partial_\Sigma R_\theta \geq 0$ . Furthermore, since  $R_\theta \leq I$ , it follows that  $\frac{\partial R_\theta}{\partial \sigma_e} \leq 0$  and  $\frac{\partial R_\theta}{\partial \sigma_s} \leq 0$ .

To find  $\partial_I R_\theta$  and  $\partial_\Sigma R_\theta$  explicitly, I apply the Implicit Function Theorem to equation (10) to find that

$$\begin{aligned} \partial_I R_\theta &= \frac{\sigma_A^2}{\sigma_k^2} \frac{R_\theta^2 + \frac{\sigma_e^2}{\sigma_s^2}}{3R_\theta^2 - 2\frac{\sigma_A^2}{\sigma_k^2} I R_\theta + \left( \frac{1}{\sigma_s^2} + \frac{1}{\Sigma} \right) \sigma_e^2}, \\ \partial_\Sigma R_\theta &= \frac{\sigma_e^2}{\Sigma^2} \frac{R_\theta}{3R_\theta^2 - 2\frac{\sigma_A^2}{\sigma_k^2} I R_\theta + \left( \frac{1}{\sigma_s^2} + \frac{1}{\Sigma} \right) \sigma_e^2}, \end{aligned} \tag{B.9}$$

and, substituting with equation (10), I arrive at

$$\begin{aligned} \partial_I R_\theta &= \frac{\sigma_A^2}{\sigma_k^2} \frac{\left( R_\theta^2 + \frac{\sigma_e^2}{\sigma_s^2} \right) R_\theta}{\left( 2R_\theta - \frac{\sigma_A^2}{\sigma_k^2} I \right) R_\theta^2 + \frac{\sigma_A^2}{\sigma_k^2} \frac{\sigma_e^2}{\sigma_s^2} I}, \\ \partial_\Sigma R_\theta &= \frac{\left( \frac{\sigma_e}{\Sigma} R_\theta \right)^2}{\left( 2R_\theta - \frac{\sigma_A^2}{\sigma_k^2} I \right) R_\theta^2 + \frac{\sigma_A^2}{\sigma_k^2} \frac{\sigma_e^2}{\sigma_s^2} I}. \end{aligned}$$

As  $\sigma_e \nearrow \infty$ , the unique root of equation (10) tends to

$$R_\theta(I, \Sigma) \searrow \frac{\sigma_A^2}{\sigma_k^2} I \frac{\Sigma}{\Sigma + \sigma_s^2}.$$

As  $\sigma_s \nearrow \infty$ , equation (10) reduces to

$$\left( R_\theta(I, \Sigma)^2 - \frac{\sigma_A^2}{\sigma_k^2} I R_\theta(I, \Sigma) + \frac{\sigma_e^2}{\Sigma} \right) R_\theta(I, \Sigma) = 0,$$

from which follows that  $R_\theta(I, \Sigma) = 0$  is a root. To see that the other two potential roots cannot be an admissible solution, I recognize that when  $\sigma_s \nearrow \infty$ , households have no private information about  $\theta_t$  to aggregate in financial markets, and therefore financial prices cannot reveal any information about  $\theta_t$ . Thus  $R_\theta(I, \Sigma) \searrow 0$  as  $\sigma_s \nearrow \infty$ .

Similarly, applying the Implicit Function Theorem to equation (10), and substituting

with equation (10), one has that

$$\partial_{\sigma_e} R_\theta = \frac{2}{\sigma_e} \frac{R_\theta^2 \left( R_\theta - \frac{\sigma_A^2}{\sigma_k^2} I \right)}{\left( 3R_\theta - 2\frac{\sigma_A^2}{\sigma_k^2} I \right) R_\theta + \left( \frac{1}{\sigma_s^2} + \frac{1}{\Sigma} \right) \sigma_e^2} = - \frac{2}{\sigma_e} \frac{R_\theta^3 \left( \frac{\sigma_A^2}{\sigma_k^2} I - R_\theta \right)}{\left( 2R_\theta - \frac{\sigma_A^2}{\sigma_k^2} I \right) R_\theta^2 + \frac{\sigma_A^2}{\sigma_k^2} \frac{\sigma_e^2}{\sigma_s^2} I} \leq 0, \quad (\text{B.10})$$

from which follows that as  $\sigma_e \searrow 0$ , one has from equation (10) that

$$R_\theta(I, \Sigma) \nearrow \frac{\sigma_A^2}{\sigma_k^2} I,$$

which must be the limiting point since the only other admissible root is  $R_\theta = 0$ , but that violates  $\partial_{\sigma_e} R_\theta \leq 0$ .

Furthermore, as  $\sigma_s \searrow 0$ , one equation (10) also has that

$$R_\theta(I, \Sigma) \nearrow \frac{\sigma_A^2}{\sigma_k^2} I.$$

Thus  $R_\theta(I, \Sigma) \in \frac{\sigma_A^2}{\sigma_k^2} I \left[ \frac{\Sigma}{\Sigma + \sigma_s^2}, 1 \right]$ .

Finally, differentiating equation (B.10) by  $I$ , and substituting with equation (B.9), it follows that

$$\partial_{I\sigma_e} R_\theta = - \frac{2IR_\theta^3 \left( \frac{\sigma_A^2}{\sigma_k^2} \right)^2 \left( R_\theta^2 + \frac{\sigma_e^2}{\sigma_s^2} \right) \left( \frac{\sigma_A^2}{\sigma_k^2} I - R_\theta \right)}{\sigma_e \left( \left( 2R_\theta - \frac{\sigma_A^2}{\sigma_k^2} I \right) R_\theta^2 + \frac{\sigma_A^2}{\sigma_k^2} \frac{\sigma_e^2}{\sigma_s^2} I \right)^3} \left( 3\frac{\sigma_e^2}{\sigma_s^2} - R_\theta^2 \right).$$

Since  $\frac{\sigma_A^2}{\sigma_k^2} I \frac{\Sigma}{\Sigma + \sigma_s^2} \leq R_\theta \leq \frac{\sigma_A^2}{\sigma_k^2} I$ , which are the limit points as  $\sigma_e \nearrow \infty$  and  $\sigma_e \searrow 0$ , it follows that  $\partial_I R_\theta$  is hump-shaped in  $\sigma_e$  between  $\frac{\sigma_A^2}{\sigma_k^2}$  and  $\frac{\sigma_A^2}{\sigma_k^2} \frac{\Sigma}{\Sigma + \sigma_s^2}$ . From equations (10) and (B.9),  $\partial_I R_\theta = \frac{\sigma_A^2}{\sigma_k^2}$  when  $\sigma_e = 0$  and when

$$\sigma_e^2 = \left( 2(1 - I^2)\Sigma + \sigma_s^2 \right) \left( I \frac{\sigma_A^2}{\sigma_k^2} \frac{2\Sigma}{2\Sigma + \sigma_s^2} \right)^2 \leq \left( \frac{\sigma_\theta^2}{\lambda_\theta} + \sigma_s^2 \right) \left( a \frac{\sigma_A^2}{\sigma_k^2} \right)^2.$$

Thus it follows that  $\partial_I R_\theta \leq \frac{\sigma_A^2}{\sigma_k^2}$  if  $\sigma_e \geq a \frac{\sigma_A^2}{\sigma_k^2} \sqrt{\frac{\sigma_\theta^2}{\lambda_\theta} + \sigma_s^2}$ .

*Proof of Proposition 7:*

When  $\theta$ , then optimal investment  $I$  and the firm equity price  $q$  are given by equations

(16) and (18)

$$q = \frac{a - I}{\rho}$$

and

$$g = \rho(q\theta - 1) \mathbf{1} \left\{ I > \underline{I} \cup \theta \geq \frac{\rho}{a - \underline{I}} \right\}.$$

Since all households are now perfectly informed, it follows that the only heterogeneity among them is their return to trees. Following the arguments of Proposition 2, their optimal policies are

$$\begin{aligned} c(i) &= \rho w(i), \\ X(i) &= \begin{bmatrix} \frac{\rho - \frac{I}{a-I}g + I\theta - r - \delta}{\sigma_k^2} \\ \frac{\rho + \xi(i) - r}{\sigma_A^2} \end{bmatrix}. \end{aligned}$$

By the market clearing condition for riskless debt (9), it follows that

$$r = \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi + \rho + \frac{\sigma_A^2 (I\theta - \frac{I}{a-I}g - \delta) - \sigma_k^2 \sigma_A^2}{\sigma_A^2 + \sigma_k^2}.$$

*Proof of Proposition 8:*

When households are perfectly informed about  $\theta$ , they consume a fixed fraction of their wealth and follow identical investment strategies

$$\begin{aligned} c(i) &= \rho w(i), \\ X(i) &= \begin{bmatrix} \frac{\rho - \frac{I}{a-I}g + I\theta - r - \delta}{\sigma_k^2} \\ \frac{\rho + \xi(i) - r}{\sigma_A^2} \end{bmatrix}, \end{aligned}$$

when not hit by the preference shock. Since firms still learn from prices, it follows that the optimal  $g$  still satisfies

$$g = \rho \left( q\hat{\theta}^c - 1 \right) \mathbf{1} \left\{ I > \underline{I} \cup \hat{\theta}^c \geq \frac{\rho}{a - \underline{I}} \right\}.$$

It follows by market clearing condition for riskless debt (9) that the riskless rate satisfies

$$r = \frac{\sigma_A^2}{\sigma_A^2 + \sigma_k^2} I\theta + \frac{\sigma_k^2}{\sigma_A^2 + \sigma_k^2} \xi + \rho - \frac{\sigma_A^2 \left( \frac{I}{a-I}g + \delta + \sigma_k^2 \right)}{\sigma_A^2 + \sigma_k^2}.$$

As  $\sigma_s \searrow 0$ , from the law of motion of  $\hat{\theta}^c$  and  $\hat{\theta}(i)$  from Proposition 3, it follows that  $\Sigma(i) \searrow 0$  and  $R_\theta$ , the loading of the riskless rate  $r$  on the firm expectational error  $\theta - \hat{\theta}^c$ , converges to

$$R_\theta \rightarrow \frac{\sigma_A^2}{\sigma_k^2} I.$$

Thus the change in the conditional variance  $\Sigma_t$  converges to

$$\frac{d\Sigma}{dt} \rightarrow \sigma_\theta^2 - 2\lambda_\theta \Sigma - I^2 \frac{\Sigma^2}{\sigma_k^2} - \frac{(I\sigma_\theta^2 + I(g - \lambda_\theta)\Sigma)^2}{(I\sigma_\theta)^2 + \sigma_\xi^2}$$

Thus it follows that  $\Sigma$  does not converge to 0 as  $\sigma_s \searrow 0$ , reflecting the uncertainty that firms still face about  $\theta$  from observing only  $\log K$  and  $r$ , and  $g$  does not converge to its perfect-information benchmark value.

Since  $\hat{\theta}(i) \rightarrow \theta$ , it follows that the investment strategy of households  $X(i)$  converges to

$$X(i) = \left[ \begin{array}{c} \frac{\rho - \frac{I}{a-I}g + I\theta - r - \delta}{\sigma_k^2} \\ \frac{\rho + \xi(i) - r}{\sigma_A^2} \end{array} \right],$$

from which it follows that the riskless rate  $r$  approaches its representative agent benchmark value. Thus beliefs, prices, and optimal policies in the economy with informational frictions approach their representative agent benchmark values as  $\sigma_s \searrow 0$ .

*Proof of Proposition 9:*

Define the vector-valued markov process  $V_t = \left[ \begin{array}{cccc} \theta_t & \hat{\theta}_t^c & I_t & \Sigma_t \end{array} \right]$  with law of motion

$$dV_t = b(V_t) dt + \sigma(V_t) d\vec{Z}_t,$$

where

$$b(V_t) = \left[ \begin{array}{c} \lambda_\theta (\bar{\theta} - \hat{\theta}^c) + \left( \sigma_{\hat{\theta}k}(X_t) \frac{I}{\sigma_k} + \sigma_{\hat{\theta}r}(X_t) \frac{\partial_{\Sigma} R_\theta(X_t) \frac{d\Sigma}{dt} + \partial_I R_\theta(X_t) I g(X_t) + R_\theta(X_t) (\lambda_\xi - \lambda_\theta)}{\sqrt{(R_\theta(X_t) \sigma_\theta)^2 + \sigma_\xi^2}} \right) (\theta - \hat{\theta}^c) \\ I_t g_t \\ \frac{d\Sigma}{dt} \end{array} \right],$$

and

$$\sigma(V_t) = \begin{bmatrix} \sigma_\theta & 0 & 0 \\ \sigma_{\hat{\theta}r}(V_t) \frac{R_\theta(X_t)\sigma_\theta}{\sqrt{(R_\theta(X_t)\sigma_\theta)^2 + \sigma_\xi^2}} & \sigma_{\hat{\theta}k}(V_t) & \sigma_{\hat{\theta}r}(V_t) \frac{\sigma_\xi}{\sqrt{(R_\theta(X_t)\sigma_\theta)^2 + \sigma_\xi^2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and  $\vec{Z} = \begin{bmatrix} Z^\theta & Z^k & Z^\xi \end{bmatrix}$ . I derived the law of motion for  $\hat{\theta}_t^c$  under the physical measure by substituting for  $\tilde{Z}^k$  and  $\tilde{Z}^r$  in Proposition 3.

Consider a function  $F(V_T)$  that is  $\mathcal{F}_T$ -adapted. Then by the Clark-Ocone-Haussman version of the (local) Martingale Representation Theorem,  $F(V_T)$  can be written as

$$F(V_T) = E[F(V_0) \mid \mathcal{F}_0] + \int_0^T E[D_t F(V_T) \mid \mathcal{F}_t] d\vec{Z}_t,$$

where  $D_t F(V_T)$  is the Malliavin derivative of  $F$  and  $E[D_t F(V_T) \mid \mathcal{F}_t]$  is the non-anticipating coefficient on  $d\vec{Z}_t$ . Following Borovicka, Hansen, and Scheinkman (2014), I link the Malliavin Derivative of  $F(V_T)$  at time  $t$ ,  $D_t F(V_T)$ , to an object that I can interpret as an impulse response to a unit shock  $\vec{Z}_t$ ,  $\nabla_{v_0} F$ , for initial conditions  $V_t = v_0$  as

$$\nabla_{v_0} F = E[D_t F(V_T) \mid V_t = v_0].$$

Since the process  $V_t$  is Markovian,  $V_t = v_0$  is a sufficient statistic for the filtration  $\mathcal{F}_t$ . From Fournié et al (1999) Property P2,  $D_t F(V_T)$  can be expressed as

$$D_t F(V_T) = H_T H_t^{-1} \sigma(V_t) \mathbf{1}_{\{t \leq T\}},$$

where  $H_t$  is the first-variation process defined by  $H_t = \frac{\partial V_t}{\partial v}$  with  $H_0 = Id_4$ , where  $Id_4$  is the  $4 \times 4$  identity matrix, and law of motion

$$dH_t = b'(V_t) H(t) dt + \sigma_{H\theta}(V_t) H_t dZ_t^\theta + \sigma_{Hk}(V_t) H_t dZ_t^k + \sigma_{H\xi}(V_t) H_t dZ_t^\xi,$$

with the prime denoting the derivative of  $b(V_t)$ ,  $\sigma_{H\theta}(V_t)$  the derivative of the column vector corresponding to  $Z^\theta$ , and similarly for  $\sigma_{Hk}(V_t)$  and  $\sigma_{H\xi}(V_t)$ .

Define  $\phi_v(\vec{\eta}(v_0), V_T)$  to be the impulse response function of  $F(V_T)$  with initial conditions

$V_0 = v_0$  and  $H_0 = Id_4$  in the direction  $\vec{\eta}(v_0)$ . Thus it follows that<sup>44</sup>

$$\phi_v(\vec{\eta}(v_0), V_T) = \vec{\eta}'(v_0) \nabla_x F = \vec{\eta}'(v_0) E[H_T | V_0 = v_0] \sigma(v_0).$$

*Proof of Proposition 10:*

From Proposition 1, it follows that each household's demand for the risky assets can be written as  $X(i) = [x^k(i) \ x^A(i)]'$ , where

$$\begin{aligned} x^k(i) &= \frac{\frac{\sigma_k^2}{\sigma_A^2} R_\theta(I, \Sigma) (\theta - \hat{\theta}^c) - \xi + I\hat{\theta}^c - \frac{I}{a-I}g - \delta + \sigma_A^2}{\sigma_A^2 + \sigma_k^2} + \frac{1}{\sigma_k^2} \frac{I\Sigma\sigma_e^2\sigma_s}{(\Sigma + \sigma_s^2)\sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} Z^s(i) \\ &\quad - \frac{1}{\sigma_k^2} \frac{IR_\theta(I, \Sigma)\Sigma\sigma_s^2\sigma_e}{(\Sigma + \sigma_s^2)\sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} Z^e(i), \\ x^A(i) &= -\frac{\frac{\sigma_k^2}{\sigma_A^2} R_\theta(I, \Sigma) (\theta - \hat{\theta}^c) - \xi + I\hat{\theta}^c - \frac{I}{a-I}g - \delta - \sigma_k^2}{\sigma_A^2 + \sigma_k^2} + \frac{\sigma_e}{\sigma_A^2} Z^e(i). \end{aligned} \quad (\text{B.11})$$

Substituting my expression for  $q$  into the law of motion of household wealth  $w_t(i)$  equation (12), it follows by Itô's Lemma that

$$\begin{aligned} d \log w(i) &= (1 - X(i)' \mathbf{1}_{2 \times 1}) (r - \rho) dt + X(i)' \begin{bmatrix} (I\theta - \frac{I}{a-I}g - \delta) dt + \sigma_k dZ^k \\ \xi(i) dt + \sigma_A dZ^A \end{bmatrix} \\ &\quad - \frac{1}{2} X(i)' \begin{bmatrix} \sigma_k^2 & 0 \\ 0 & \sigma_A^2 \end{bmatrix} X(i) dt, \end{aligned}$$

Substituting for  $X(i)$  with equations (B.11) and aggregating across households, one then

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<sup>44</sup>While  $\phi_v(\vec{\eta}(v_0), V_T)$  characterizes impulses to a linear,  $\mathcal{F}_0$ -measurable combination of the state vector  $V_T$ , the approach can be generalized to differentiable functions of  $V_T$ ,  $f(V_T; v_0)$ , and to non-differentiable functions  $h(V_T; v_0)$  using the appropriate integration-by-parts formula.



has that

$$\begin{aligned}
& \int_0^1 d \log w(i) di \\
= & \frac{\frac{\sigma_k^2}{\sigma_A^2} R_\theta(I, \Sigma) \left( \theta - \hat{\theta}^c \right) - \xi + I \hat{\theta}^c - \frac{I}{a-I} g - \delta + \sigma_A^2}{\sigma_A^2 + \sigma_k^2} \left( \left( I \theta - \frac{I}{a-I} g - \delta \right) dt + \sigma_k dZ^k - \sigma_A dZ^A \right) + \frac{\sigma_A^2}{\sigma_A^2 + \sigma_k^2} \\
& + \frac{1}{2} \left( 2 \frac{\sigma_e^2}{\sigma_A^2} - \frac{\sigma_A^2 \sigma_k^2}{\sigma_A^2 + \sigma_k^2} - \frac{\sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2}{\sigma_k^2} \left( \frac{I \Sigma \sigma_s \sigma_e}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} \right)^2 \right) dt \\
& + \frac{1}{2} \frac{\xi^2 + 2 \sigma_k^2 \xi - \left( \frac{\sigma_k^2}{\sigma_A^2} R_\theta(I, \Sigma) \left( \theta - \hat{\theta}^c \right) + I \hat{\theta}^c - \frac{I}{a-I} g - \delta \right)^2}{\sigma_A^2 + \sigma_k^2} dt.
\end{aligned}$$

With equation (B.12), I can express aggregate flow utility as  $\int_0^1 \log c_t(i) di = \log \rho + \log w_0 + \int_0^1 \int_0^t d \log w_u(i) dudi$ .

Assuming  $E \left[ \int_0^T \left( \frac{\sigma_k^2}{\sigma_A^2} R_\theta(I_u, \Sigma_u) \left( \theta_u - \hat{\theta}_u^c \right) + I_u \hat{\theta}_u^c - \frac{I_u}{a-I_u} g_u - \delta + \sigma_A^2 - \xi_u \right)^2 du \mid \mathcal{F}_0 \right] < \infty \forall T$ , utilitarian welfare at time 0 in the economy  $U = E \left[ \int_0^\infty e^{-\rho t} \int_0^1 \log \frac{c_t(i)}{c_0(i)} didt \mid \mathcal{F}_0 \right]$  under the physical measure  $\mathcal{P}$  defined on  $\mathcal{F}_0$  is then given by

$$\begin{aligned}
U = & \frac{1}{\rho} E \left[ \int_0^\infty e^{-\rho t} \frac{\frac{\sigma_k^2}{\sigma_A^2} R_\theta(I_t, \Sigma_t) \left( \theta_t - \hat{\theta}_t^c \right) - \xi_t + I_t \hat{\theta}_t^c - \frac{I_t}{a-I_t} g_t - \delta + \sigma_A^2}{\sigma_A^2 + \sigma_k^2} \left( I_t \theta_t - \frac{I_t}{a-I_t} g_t - \delta \right) dt \mid \mathcal{F}_0 \right] \\
& + \frac{1}{2\rho} E \left[ \int_0^\infty e^{-\rho t} \frac{\xi_t^2 + 2 \sigma_k^2 \xi_t - \left( \frac{\sigma_k^2}{\sigma_A^2} R_\theta(I_t, \Sigma_t) \left( \theta_t - \hat{\theta}_t^c \right) + I_t \hat{\theta}_t^c - \frac{I_t}{a-I_t} g_t - \delta \right)^2}{\sigma_A^2 + \sigma_k^2} dt \mid \mathcal{F}_0 \right] \\
& + \frac{1}{2\rho^2} \left( \frac{\sigma_e^2}{\sigma_A^2} - \frac{\sigma_A^2 \sigma_k^2}{\sigma_A^2 + \sigma_k^2} \right) - \frac{1}{2\rho \sigma_k^2} E \left[ \int_0^\infty e^{-\rho t} \left( \frac{I_t \Sigma_t \sigma_s \sigma_e \sqrt{\sigma_e^2 + \sigma_s^2 R_\theta(I_t, \Sigma_t)^2}}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I_t, \Sigma_t)^2 \Sigma_t} \right)^2 dt \mid \mathcal{F}_0 \right]. \quad (\text{B.13})
\end{aligned}$$

where I have applied integration by parts to each integral to express the double integrals as

single integrals.<sup>45</sup> Recognizing that

$$\begin{aligned} \frac{\frac{\sigma_k^2}{\sigma_A^2} R_\theta(I, \Sigma) \left( \theta - \hat{\theta}^c \right) - \xi + I\hat{\theta}^c - \frac{I}{a-I}g - \delta + \sigma_A^2}{\sigma_A^2 + \sigma_k^2} &= \frac{qK}{qK + P}, \\ -\frac{\frac{\sigma_k^2}{\sigma_A^2} R_\theta(I, \Sigma) \left( \theta - \hat{\theta}^c \right) - \xi + I\hat{\theta}^c - \frac{I}{a-I}g - \delta - \sigma_k^2}{\sigma_A^2 + \sigma_k^2} &= \frac{P}{qK + P}, \end{aligned}$$

and defining  $w$  such that  $d \log w = \int_0^1 d \log w(i) di$ , it follows from equation (B.12) and Itô's Lemma that  $w$  has the law of motion

$$\begin{aligned} \frac{dw}{w} &= \frac{qK}{qK + P} \left( \left( I\theta - \frac{I}{a-I}g - \delta \right) dt + \sigma_k dZ^k \right) + \frac{P}{qK + P} \left( \xi + \sigma_A dZ^A \right) dt \\ &\quad - \frac{1}{2} \left( \left( \frac{\sigma_A P}{qK + P} \right)^2 + 2 \frac{\sigma_e^2}{\sigma_A^2} + \frac{1}{\sigma_k^2} \left( \frac{I \Sigma \sigma_s \sigma_e \sqrt{\sigma_e^2 + \sigma_s^2} R_\theta(I, \Sigma)^2}{(\Sigma + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I, \Sigma)^2 \Sigma} \right)^2 \right) dt. \end{aligned} \quad (\text{B.14})$$

Thus I can think of the economy as having a representative household who owns all firm equity and trees in the economy, and whose wealth evolves according to (B.14). Substituting for  $\frac{qK}{qK+P}$  and  $\frac{P}{qK+P}$  in equation (B.13), I arrive at the expression in the proposition.

*Proof of Proposition 11:*

To find the law of motion of the probability law of the economy  $p_t \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right)$ , I find the probability law implied by households and firms whose optimization is consistent with their HJB equations. This is commonly referred to as the Kolmogorov Forward Equation. To find this, I recognize that, under the optimal control for the change in investment

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<sup>45</sup>For instance, an integral of the form  $E \left[ \int_0^\infty e^{-\rho t} \int_0^t F(I_u, \Sigma_u) du dt \mid \mathcal{F}_0 \right]$  can be rewritten as

$$\begin{aligned} &E \left[ \int_0^\infty e^{-\rho t} \int_0^t F(I_u, \Sigma_u) du dt \mid \mathcal{F}_0 \right] \\ &= -E \left[ \int_0^\infty \left( \int_0^t F(I_u, \Sigma_u) du \right) d \left( \int_t^\infty e^{-\rho s} ds \right) \mid \mathcal{F}_0 \right] \\ &= -E \left[ \int_0^t F(I_u, \Sigma_u) du \int_t^\infty e^{-\rho s} ds \mid \mathcal{F}_0 \right]_{t=0}^{t=\infty} + E \left[ \int_0^\infty \left( \int_t^\infty e^{-\rho s} ds \right) F(I_u, \Sigma_u) dt \mid \mathcal{F}_0 \right] \\ &= \frac{1}{\rho} E \left[ \int_0^\infty e^{-\rho t} F(I_u, \Sigma_u) dt \mid \mathcal{F}_0 \right] \end{aligned}$$

This approach is equivalent to stacking the double integrals and redefining them.

$g \left( I_s, \Sigma_s, \hat{\theta}_s^c; \tau^I \right)_{s \geq 0}$ ,  $\mathcal{D}^g f = 0$ , where  $\mathcal{D}^g$  is the infinitesimal generator that satisfies

$$\begin{aligned} \mathcal{D}^g f &= \partial_{\hat{\theta}^c} f \lambda_{\theta} \left( \bar{\theta} - \hat{\theta}^c \right) + \partial_{\Sigma} f \frac{d\Sigma}{dt} + \partial_I f I g + \frac{1}{2} \partial_{\hat{\theta}^c \hat{\theta}^c} f \left( \sigma_{\hat{\theta}^c}^2 + \sigma_{\hat{\theta}^r}^2 \right) \\ &+ \partial_{\hat{\xi}^c} f \lambda_{\xi} \left( \bar{\xi} - \hat{\xi}^c \right) + \frac{1}{2} \partial_{\hat{\xi}^c \hat{\xi}^c} f \left( R_{\theta}^2 \sigma_{\hat{\theta}^c}^2 + \frac{\left( \sigma_{\xi}^2 - R_{\theta} \left( \partial_{\Sigma} R_{\theta} \frac{d\Sigma}{dt} + \partial_I R_{\theta} I g + (\lambda_{\xi} - \lambda_{\theta}) R_{\theta} \right) \Sigma \right)^2}{(R_{\theta} \sigma_{\theta})^2 + \sigma_{\xi}^2} \right) \\ &+ \partial_{\hat{\theta}^c \hat{\xi}^c} f \left( R_{\theta} \sigma_{\hat{\theta}^c}^2 + \frac{\sigma_{\xi}^2 - R_{\theta} \left( \partial_{\Sigma} R_{\theta} \frac{d\Sigma}{dt} + \partial_I R_{\theta} I g + (\lambda_{\xi} - \lambda_{\theta}) R_{\theta} \right) \Sigma}{\sqrt{(R_{\theta} \sigma_{\theta})^2 + \sigma_{\xi}^2}} \sigma_{\hat{\theta}^r} \right), \end{aligned}$$

and  $\sigma_{\hat{\theta}^c}$  and  $\sigma_{\hat{\theta}^r}$  are given in Proposition 3 appropriately modified for the position cost  $\tau^r$ .

In the above, I made use of the market signal  $S = R_{\theta} (I, \Sigma) \theta + \xi$  to write  $\hat{\xi}^c$  as

$$\hat{\xi}^c = \xi + R_{\theta} (I, \Sigma) \left( \theta - \hat{\theta}^c \right) = S - R_{\theta} (I, \Sigma) \hat{\theta}^c,$$

to calculate the law of motion of  $\hat{\xi}^c$ .

Standard results establish that the law of motion of the probability law  $p_t \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right)$  is given by

$$\partial_t p_t \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) = \mathcal{D}^{g*} p_t \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right), \quad (\text{B.15})$$

where  $\mathcal{D}^{g*}$  is the adjoint of  $\mathcal{D}^g$  and is the time-homogeneous infinitesimal generator associated with the Koopman operator. Importantly,  $\mathcal{D}^{g*}$  is a (uniformly) elliptic operator that has divergence form. When  $p_t$  has reached its stationary distribution  $p$ , where  $p = \lim_{t \nearrow \infty} p_t$ , it follows that  $\partial_t p_t = 0$ . Thus equation (B.15) is a second-order parabolic equation. Rewriting  $p_t$  when it has reached its stationary distribution, suppressing arguments, I arrive at the expression given in the proposition.

That  $p_t \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right)$  will satisfy the conservation of mass law  $\int p_t \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) dI d\Sigma d\hat{\theta}^c d\hat{\xi}^c = 1$ , where the integral is understood to be taken over the entire space  $[I, a] \times \left[ 0, \frac{\sigma_{\theta}^2}{2\lambda_{\theta}} \right] \times \mathbb{R}^2$ , gives rise to my spatial boundary conditions. Notice that I can rewrite the KFE as

$$\nabla \cdot \phi \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) = 0,$$

where

$$\phi \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) = \left[ \begin{array}{c} \left( \begin{array}{c} -\lambda_\theta \left( \bar{\theta} - \hat{\theta}^c \right) p + \frac{1}{2} \partial_{\hat{\theta}^c} \left\{ \left( \sigma_{\hat{\theta}^k}^2 + \sigma_{\hat{\theta}^r}^2 \right) p \right\} \\ + \frac{1}{2} \partial_{\hat{\xi}^c} p \left( R_\theta \sigma_{\hat{\theta}^k}^2 + \frac{\sigma_\xi^2 - R_\theta \left( \partial_\Sigma R_\theta \frac{d\Sigma}{dt} + \partial_I R_\theta I g + (\lambda_\xi - \lambda_\theta) R_\theta \Sigma \right)}{\sqrt{(R_\theta \sigma_\theta)^2 + \sigma_\xi^2}} \sigma_{\hat{\theta}^r} \right) \end{array} \right) \\ \frac{d\Sigma}{dt} p \\ I g p \\ \left( \begin{array}{c} \frac{1}{2} \partial_{\hat{\xi}^c} p \left( R_\theta^2 \sigma_{\hat{\theta}^k}^2 + \frac{\left( \sigma_\xi^2 - R_\theta \left( \partial_\Sigma R_\theta \frac{d\Sigma}{dt} + \partial_I R_\theta I g + (\lambda_\xi - \lambda_\theta) R_\theta \Sigma \right) \right)^2}{(R_\theta \sigma_\theta)^2 + \sigma_\xi^2} \right) \\ + \frac{1}{2} \partial_{\hat{\theta}^c} \left\{ p \left( R_\theta \sigma_{\hat{\theta}^k}^2 + \frac{\sigma_\xi^2 - R_\theta \left( \partial_\Sigma R_\theta \frac{d\Sigma}{dt} + \partial_I R_\theta I g + (\lambda_\xi - \lambda_\theta) R_\theta \Sigma \right)}{\sqrt{(R_\theta \sigma_\theta)^2 + \sigma_\xi^2}} \sigma_{\hat{\theta}^r} \right) \right\} \\ - \lambda_\xi \left( \bar{\xi} - \hat{\xi}^c \right) p \end{array} \right) \end{array} \right].$$

Here  $\phi \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right)$  is the "probability flux" representing the flow or flux of particles through the point  $\left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right)$ . Consequently, a reflecting boundary condition will ultimately impose that the flux through boundary points must be zero.

For  $\hat{\theta}^c$ , which has unbounded support, one has that for  $\varepsilon > 0$  arbitrary that

$$\lim_{|\hat{\theta}^c| \nearrow \infty} \left( \hat{\theta}^c \right)^{2(1+\varepsilon)} p \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) = 0 \quad \forall I,$$

and similarly for  $\hat{\xi}^c$ , while for  $\Sigma$ , one has that  $\partial_\Sigma p \left( I, \frac{\sigma_\theta^2}{2\lambda_\theta}, \hat{\theta}^c, \hat{\xi}^c \right) = 0$ , since  $\frac{\sigma_\theta^2}{2\lambda_\theta}$  is a reflecting boundary, and  $\lim_{\Sigma \searrow 0} p \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) = 0$ , since arbitrarily small precision becomes arbitrarily unlikely given that new unobservable innovations to  $\theta_t$  occur at each instant.

Integrating this expression over the entire space, imposing that  $\int \partial_t p_t \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) dI d\Sigma d\hat{\theta}^c d\hat{\xi}^c = \partial_t \int p_t \left( I, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) dI d\Sigma d\hat{\theta}^c d\hat{\xi}^c = 0$ , applying the Divergence Theorem, it follows that the appropriate "reflecting" boundary condition for  $I$  is  $\hat{n}_{I=\underline{I}} \cdot \phi \left( \underline{I}, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) = \hat{n}_{I=a} \cdot \phi \left( a, \Sigma, \hat{\theta}^c, \hat{\xi}^c \right) = 0 \quad \forall \left( \Sigma, \hat{\theta}^c, \hat{\xi}^c \right)$ , where  $\hat{n}_{I=i}$  is the unit (outward) normal vector perpendicular to the  $I = i$  boundary. The intuition for these two boundary conditions is that the probability flux, or flow, through the two walls  $I = \underline{I}$  and  $I = a$  must be zero for probability mass not to leak out through them.

*Proof of Corollary 1:*

Since  $\left( r_t, \hat{\theta}_t^c, \hat{\xi}_t^c \right) \in \mathcal{F}_t^c \subseteq \mathcal{F}_t$ , from the expression for the riskless rate  $r$  in Proposition 1,

and it follows

$$\xi_t - \hat{\xi}_t^c = -R_\theta(I_t, \Sigma_t) (\theta_t - \hat{\theta}_t^c),$$

and therefore

$$E \left[ \left( \xi_t - \hat{\xi}_t^c \right)^2 \mid \mathcal{F}_t^c \right] = R_\theta(I_t, \Sigma_t)^2 \Sigma_t.$$

Let  $U^c$  be ex-ante utilitarian welfare under the common knowledge filtration. Then  $U^c$  satisfies

$$U^c = E \left[ \int_0^\infty e^{-\rho t} \int_0^1 \log c_t(i) di dt \mid \mathcal{F}_0^c \right] = E \left[ E \left[ \int_0^\infty e^{-\rho t} \int_0^1 \log c_t(i) di dt \mid \mathcal{F}_0 \right] \mid \mathcal{F}_0^c \right] = E[U \mid \mathcal{F}_0^c].$$

It then follows from the expression for  $U$  from 10, since  $\hat{\theta}_t^c \sim \mathcal{N}(\theta_t, \Sigma_t)$  and  $E[\xi_0 \mid \mathcal{F}_0^c] = \bar{\xi}$ , that the above reduces by the LIE to

$$\begin{aligned} U^c &= \frac{1}{\rho \sigma_A^2} E \left[ \int_0^\infty e^{-\rho t} I_t \Sigma_t R_\theta(I_t, \Sigma_t) dt \mid \mathcal{F}_0^c \right] + \frac{1}{2\rho} \frac{1 - \left( (1 - \tau^r) \frac{\sigma_k^2}{\sigma_A^2} \right)^2}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2} E \left[ \int_0^\infty e^{-\rho t} \Sigma_t R_\theta(I_t, \Sigma_t)^2 dt \mid \mathcal{F}_0^c \right] \\ &+ \frac{1}{2\rho} E \left[ \int_0^\infty e^{-\rho t} \frac{\left( \hat{\xi}_t^c \right)^2 + 2(\delta + \sigma_k^2 - \tau^r) \hat{\xi}_t^c}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2} dt \mid \mathcal{F}_0 \right] - \frac{1}{\rho} E \left[ \int_0^\infty e^{-\rho t} \frac{\left( I_t \hat{\theta}_t^c - \frac{I_t}{a - I_t} g_t \right) \hat{\xi}_t^c}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2} dt \mid \mathcal{F}_0^c \right] \\ &+ \frac{1}{2\rho} E \left[ \int_0^\infty e^{-\rho t} \frac{\left( I_t \hat{\theta}_t^c - \frac{I_t}{a - I_t} g_t \right)^2 + 2(\sigma_A^2 - \delta + \tau^r) \left( I_t \hat{\theta}_t^c - \frac{I_t}{a - I_t} g_t \right)}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2} dt \mid \mathcal{F}_0^c \right] \\ &- \frac{1}{2\rho (1 - \tau^r)^2 \sigma_k^2} E \left[ \int_0^\infty e^{-\rho t} \left( \frac{\sigma_s \sigma_e I_t \Sigma_t \sqrt{\sigma_e^2 + \sigma_s^2 R_\theta(I_t, \Sigma_t)^2}}{(\Sigma_t + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta(I_t, \Sigma_t)^2 \Sigma_t} \right)^2 dt \mid \mathcal{F}_0^c \right] \\ &+ \frac{1}{2\rho^2} \frac{\delta^2 - 2\delta \sigma_A^2}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2} + \frac{1}{2\rho^2} \left( \frac{\sigma_e^2}{\sigma_A^2} - \frac{(1 - \tau^r) \sigma_A^2 \sigma_k^2 + 2\tau^r (\delta + \sigma_A^2)}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2} \right). \end{aligned} \tag{B.16}$$

Assume now that the economy is initialized from the stationary distribution  $p(\hat{\theta}^c, I, \Sigma)$  and that the stationary distribution is bounded  $p(\hat{\theta}^c, I, \Sigma) \in \mathcal{L}^\infty(\mathbb{R}, [0, \frac{\sigma_\theta^2}{2\lambda_\theta}], [\underline{I}, a])$ . Let  $U_p^c$  be the expected welfare in economy under the stationary distribution, and  $E^p[\cdot]$  be the expectation operator w.r.t. the stationary distribution. Then the first expectation, when

taken w.r.t. the stationary distribution, can be rewritten as

$$\begin{aligned}
& E^p \left[ \int_0^\infty e^{-\rho t} I_t \Sigma_t R_\theta (I_t, \Sigma_t) dt \right] \\
&= \int_0^\infty e^{-\rho t} \int P_t I_0 \Sigma_0 R_\theta (I_0, \Sigma_0) p \left( \hat{\theta}_0^c, \Sigma_0, I_0 \right) d\hat{\theta} d\Sigma dI dt \\
&= \int_0^\infty e^{-\rho t} \int I_0 \Sigma_0 R_\theta (I_0, \Sigma_0) P_t^* p \left( \hat{\theta}_0^c, \Sigma_0, I_0 \right) d\hat{\theta} d\Sigma dI dt, \tag{B.17}
\end{aligned}$$

where  $P_t = e^{t\mathcal{D}^g}$  is the Ruelle-Frobenius-Perron operator and  $P_t^* = e^{t\mathcal{D}^{g*}}$  is its adjoint, often called the Koopman operator.  $P_t^*$  is defined such that, for a bounded, Borel measurable function  $f$  and measure  $\nu$   $\langle P_t f, \nu \rangle = \int_{\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+} P_t f d\nu = \langle f, P_t^* \nu \rangle$ . Probabilistically,  $P_t^*$  corresponds to time-reversal and acts on measures, whereas  $P_t$  acts on functions. By construction, since  $\mathcal{D}^{g*} p = 0$ ,

$$e^{t\mathcal{D}^{g*}} p \left( \hat{\theta}_0^c, \Sigma_0, I_0 \right) = p \left( \hat{\theta}_0^c, \Sigma_0, I_0 \right),$$

and therefore equation (B.17) simplifies to

$$E^p \left[ \int_0^\infty e^{-\rho t} I_t \Sigma_t R_\theta (I_t, \Sigma_t) dt \right] = \frac{1}{\rho} E^p [I_0 \Sigma_0 R_\theta (I_0, \Sigma_0)].$$

This result obtains under the assumption that  $\Sigma$  is essentially bounded. Since  $\Sigma \leq \frac{\sigma_\theta^2}{2\lambda_\theta}$  from Proposition 3, this assumption is justified for  $\sigma_\theta$  finite and  $\lambda_\theta > 0$ . It follows from these results and equation (B.16), that  $U_p^c$  takes the form

$$\begin{aligned}
U_p^c &= \frac{1}{\rho^2} E^p \left[ \frac{\frac{1}{2} \left( I_0 \hat{\theta}_0^c - \frac{I_0}{a-I_0} g_0 \right)^2 + (\sigma_A^2 - \delta + \tau^r) \left( I_0 \hat{\theta}_0^c - \frac{I_0}{a-I_0} g_0 \right) - \left( I_0 \hat{\theta}_0^c - \frac{I_0}{a-I_0} g_0 \right) \hat{\xi}_0^c}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2} \right] \\
&+ \frac{1}{\rho^2} E^p \left[ \frac{I_0 \Sigma_0 R_\theta (I_0, \Sigma_0)}{\sigma_A^2} - \frac{(1 - \tau^r)^2 \sigma_k^4 \Sigma_0 R_\theta (I_0, \Sigma_0)^2}{2 \sigma_A^4 \sigma_A^2 + (1 - \tau^r) \sigma_k^2} \right] \\
&- \frac{1}{2\rho^2 (1 - \tau^r)^2 \sigma_k^2} E^p \left[ \left( \frac{\sigma_s \sigma_e I_0 \Sigma_0 \sqrt{\sigma_e^2 + \sigma_s^2 R_\theta (I_0, \Sigma_0)^2}}{(\Sigma_0 + \sigma_s^2) \sigma_e^2 + \sigma_s^2 R_\theta (I_0, \Sigma_0)^2 \Sigma_0} \right)^2 \right] + \frac{1}{2\rho^2} \frac{\sigma_e^2}{\sigma_A^2} \\
&+ \frac{1}{2\rho^2} \frac{\delta^2 + \bar{\xi}^2 + \frac{\sigma_\xi^2}{2\lambda_\xi} + 2(\delta + \sigma_k^2 - \tau^r) \bar{\xi} - 2\delta \sigma_A^2 - (1 - \tau^r) \sigma_A^2 \sigma_k^2 - 2\tau^r (\delta + \sigma_A^2)}{\sigma_A^2 + (1 - \tau^r) \sigma_k^2}.
\end{aligned}$$