Family Knows Best:
Fund Advisors as Talent-Rating Agencies*

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Abstract

Investment companies charge investors a fee for managing their assets and contract with money managers to run the funds. We show that such intermediation is profitable to both the fund family and the manager if there is uncertainty about the manager's skill and the family is better informed than outside investors. Given its informational advantage, family profits stem from expropriating managerial talent. As investors learn about the manager's skill, from observing fund returns, the informational advantage dissipates and profits initially decline. When investors' uncertainty about the manager's skill becomes small, the family intentionally speeds up the rate at which it loses its informational advantage, by letting go of lower skilled managers with good past performance. The accelerated reduction in uncertainty about the manager can be profitable to the family if it stems from such turnover. It facilitates faster building of reputation for retained managers, providing an advantage to higher skilled managers who accept below-market compensation in the short run. The benefit to managers from building reputation faster more than offsets the expropriation costs. We further show that to extract more value from managers, the family optimally sets a lower fund fee for investors to further increase turnover of lower skilled managers.

Keywords: fund management, fund families, managerial compensation, dynamic adverse selection, dynamic reputation.

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1 Introduction

The bulk of assets held and traded are controlled by delegated portfolio managers, with mutual funds playing a central role, holding close to 40% of assets. While research on mutual funds is expansive, a centerpiece that has been mostly ignored is the multi-layered structure of the associated agency relation. Mutual fund contracts are signed between investors and fund families, not between investors and the fund manager employed by the family to manage the assets. Prior work analyzing this agency relation treats the fund family and the portfolio manager as a single entity, ignoring conflicts of interest between them. In this paper, we focus on the contracting implications for this important and under-researched second layer.

We consider a dynamic setting where investors contract with a fund family to invest their wealth into one of its funds. The fund family, in turn, contracts with a fund manager to run this fund. While mutual fund advisory fees are regulated and are typically a fraction of assets under management, contracts between the family and a manager are unregulated, and very little is known about how they are designed. Our analysis provides novel insights as to how these dynamic contracts are likely structured and the underlying economic mechanism driving fund family profits.

Our focus is understanding implications stemming from the fact that the fund family knows more about the manager's ability than do fund investors.[1] The family acts as an intermediary between fund investors and the manager it employs to run the fund. Since the manager's outside employment opportunities are driven by investors' perceptions that depend on fund performance, while the family has better information about the manager's ability than investors, we show that the family is, effectively, a monopsonist for the manager's labor when information asymmetry is high. The family's monopsonist power, enables it to extract economic rents by underpaying higher quality managers.[2]

Over time, as the uncertainty about the manager's skill declines, due to a longer track record, the family looses its informational advantage, reducing its ability to extract rents from higher quality managers. However, the family can affect the rate of decline in its informational advantage by strategically setting its retention policy. While one might conjecture that the family would attempt to slow down the rate of decline, we show that it in fact it increases it. Consequently, some managers will be let go even after good performance.

[1] This natural assumption is supported by empirical evidence in Berk, Van Binsbergen, and Liu (2017).

[2] We abstract away from other roles the family might have, focusing exclusively on profits stemming from asymmetric information about its manager, by precluding any additional family skill as a source of added value.

Dropping lower quality managers improves the pool of retained manager, leading to an increase in assets under management and consequently revenues. Furthermore, a higher speed of separation of low quality managers enhances the value to managers from staying longer in the fund to build their reputation and increase their outside option.This in turn improves the family's rent extraction bargaining position.

We show that the family serves as a reputation building conduit for managers. High quality managers can enhance their reputation faster by working for the fund then building their reputation on their own. In equilibrium, better managers are retained for longer, as they build their reputation, and are relatively more underpaid while working for the family.

The family profits from employing good managers only when it has a sufficiently large informational advantage relative to outside investors. The fund complex lets the manager go when it has little informational advantage about his skill and hires a new manager who is worse in expectation but is more uncertain in the eyes of investors. When public and private perceptions about the manager are close, the benefit to the manager of staying on to further improve his reputation are limited, reducing the family's rent extraction capabilities. Furthermore, since sufficiently low information advantage is associated with good past performance it is also associated with the fund becoming large. Due to decreasing returns to scale, the scope of increasing revenue thus shrinks as well.

We consider a sequence of managers operating one fund for the family to highlight the economic forces that do not require the family to oversee multiple funds.[3] Each manager begins his career working for the family, but can, at any point, leave and open an independent fund.[4] He is privately informed about his skill relative to investors. We assume that when the manager is hired, the fund family also acquires information about his ability. This information is imperfect, and both parties incrementally learn about the manager's skill from his track record of returns.[5] Investors do not observe this private signal about the manager. In every period the family either lets go of the manager or sets the necessary compensation to retain him, which is not observable by outside investors. In equilibrium, compensation is sensitive to the private information of the family-manager pair and, thus, the observable time when he leaves the family signals his private information to outside investors. Investors update their beliefs about the manager's skill based on observable information,

[3] Most prior literature has focused on the spillover effects across multiple funds.

[4] We think of the manager staying in the money-management industry after leaving the fund family, but it applies to a broad specification of alternative careers.

[5] In reality, the manager is likely better informed about his skill, relative to both the fund family and investors. For tractability, we assume that the fund family and the manager share the same private information.

such as returns and when the manager leaves the family. They dynamically allocate wealth to the fund based on their perception of the fund's expected performance subject to decreasing returns to scale, similar to Berk and Green (2004). We characterize the dynamic separating equilibrium, where the private information about the manager is revealed by the time when he leaves the fund family. Unlike classic models of reputation building such as Holmström (1999) and Board and Meyer-ter Vehn (2013), the manager signals his skill by being employed longer by the fund complex, similar to Spence (1973) and Daley and Green (2014).[6]

The family profits from a highly skilled manager only if there is enough public uncertainty about his ability. This stems from two economic forces. First, a high skilled manager generates good returns and increases the future expected fees collected by the family by increasing assets under management. Second, a high skilled manager is more sensitive to the beliefs of investors about his ability when he opens his own fund relative to a low skilled manager since he expects to stay in the money-management industry for longer and suffers a longer-term impact of investor's initial beliefs. As a result, the threat of termination by the family is more effective when negotiating the compensation with a higher skilled manager. In equilibrium, the family lets go of lower skilled managers first, and when the manager leaves to open an independent fund, investors rationally infer that it must be the worst remaining manager still employed given the history of the game. Since the low skill managers leave the family first, the manager's reputation increases with each period that he continues to work for the family above and beyond what can be explained by his track record of returns. This implies that the manager's outside option in each period is to leave the fund and be perceived as the worst remaining manager. At the same time, the fees collected by the family are determined by the investors' beliefs about the average manager still employed, i.e., managers are pooled while working for the fund family. The difference between the average manager working for the family and the worst manager that could still be employed in equilibrium after that history creates the profit wedge for the fund family.

When the public uncertainty about the manager is high, the fund family captures significant profits. Even though the fund's assets under management are not very big since the manager has not yet proved himself, the compensation needed to retain the manager is even smaller. Over time, however, the fund family's profit wedge declines as the manager produces a track record of returns and the

[6] In the model we assume that the fund family has all the bargaining power and, thus, the fund manager does not benefit from this reputation building along the path. If the manager, however, had any positive amount of bargaining power, such reputation building would make his equilibrium payoff strictly exceed his outside option.

public perception about his skill gradually becomes more accurate. This reduces the gap between the average and the lowest quality manager, increasing the bargaining power of the managers still employed and reducing the rents that can be captured by the family. As it finds the higher skilled managers more profitable, it lets go of the worst managers first. As a result, staying to work for the family serves as an endogenous positive signal about the manager's skill. Since this signal is more valuable to better managers, the fund family can strategically exploit highly skilled managers by underpaying while still retaining them. In equilibrium, a higher skilled manager works for below market compensation for the family to signal his superior skill and capture higher fees when he leaves later, while a lower skilled manager is concerned about being discovered and opens his independent fund earlier to collect fund fees. Nevertheless, high skilled managers do become too expensive in time and, eventually, the family prefers to let them go in favor of hiring a lower skilled, but also more uncertain, new manager. The finding that the fund family's profits are decreasing in the uncertainty about the manager's ability is consistent with Gaspar, Massa, and Matos (2006) who find that the fund family attempts to increase the profitability of the younger managers in its funds.

In the separating equilibrium, managers are strictly worse off, relative to a pooling equilibrium where they would all open their own funds from the start. Even though good managers benefit from improved reputation by working for the family, the it can extract much of this value reducing the managers' compensation.[7] Incremental information, however, is socially efficient as it channels more investment towards higher skilled managers. The profit of the family and the speed of information revelation are also increasing in the expected value of the complex from hiring a new manager, as it serves as a commitment device to underpay existing managers, increasing the signaling efficiency of working for the fund family. This is a self-reinforcing mechanism which increases the ex-ante profits of the fund family. The ex-ante payoff to the family may also be decreasing in the total fees charged from investors. Both equilibrium properties occur due to the same economic mechanism: lower profitability of the family, relative to its option value of hiring the new manager, leads to faster and more efficient learning from managerial turnover, increasing the rent the family can extract from the manager.

While the family determines manager's dynamic compensation, it also sets the fund fee rate that investors are charged, as a fraction of assets under management. Varying the fee rate impacts both

[7]This can be relaxed by introducing a bargaining split between the fund manager and the family. In this case, the manager would capture some of the surplus beyond just his outside option.

the dynamic retention strategy and the quality of managers it initially decides to employ. Both channels influence the degree of adverse selection investors face. We show that it is optimal for the family to charge investors a lower fee rate in order to extract more rents from the manager by way of speeding up the endogenous reputation-building mechanism. This result highlights a disconnect between maximizing present value of revenues and present value of profits.

We analyze several extensions of the model. First, , we show that the equilibrium outcome does not change if we allow the fund family to sign private long-term contracts with the manager. Finally, we discuss how the model can be extended to account for other institutional details of the mutual fund industry, such as management in teams and competition for talent by other fund families.

Our findings apply beyond the setting of money management. In any industry where labor markets suffer from adverse selection, and long-term screening contracts are not feasible, an informed intermediary can act as a certification device by letting its worker produce an observable track record as well as credibly reveal private information via retention decisions. The prospect of generating a track record attracts the worker to the firm, while private information allows the firm to profitably bargain with the worker, creating high firm value that we see in talent-driven professions such as economic consulting, legal and financial services, and academia.

1.1 Related Literature

There is considerable literature exploring whether fund managers possess skill. In one of the first papers on the subject Sharpe (1966) finds that skill is persistent across funds. Jensen (1968) shows that the average mutual fund manager does not outperform the market questioning the existence of managerial talent. Carhart (1997) shows that the perceived persistence of fund skill can be explained by mutual fund managers being exposed to stock market momentum in the aggregate.

A number of studies including Hendricks, Patel, and Zeckhauser (1993) and Ippolito (1992) show that fund flows, and correspondingly management fees, respond positively to excess returns generated by the manager. Chevalier and Ellison (1997) document that this relationship is even convex. Berk and Green (2004) show that the positive flow to performance relationship is consistent with lack of aggregate out-performance when investors dynamically allocate wealth and portfolio managers exhibit decreasing returns to scale. We adopt this framework to model the revenues collected by the fund family and explore the implications on the contracts within the fund family.

Several papers explore the role of fund families empirically. Berk, Van Binsbergen, and Liu (2017) document that fund families reallocate assets within the family towards higher skilled managers. This supports our assumption that the fund family is privately informed about the manager's skill. Gaspar, Massa, and Matos (2006) show that fund families engage in favoritism boosting the performance of certain funds over others. In this paper, we show that the family has an incentive to support skilled managers as long as there is sufficient public uncertainty about their skill. Nanda, Wang, and Zheng (2004) document a positive spillover of having a star fund manager on other funds in the family. While we focus on a single fund operated by the family, our findings could extend to capture this empirical pattern. Massa (2003) studies the spillover of strategies across funds belonging to the same family. Kempf and Ruenzi (2007) show that fund managers within the family engage in implicit competition with each other. All of these studies show that the fund family is a first-order economic agent whose strategic incentives must be considered when evaluating mutual fund performance.

Little is known about the compensation contracts of portfolio managers employed by fund families as they are not required to be disclosed. Ma, Tang, and Gómez (2017) is one of a few papers analyzing the qualitative properties of such contracts. They find no clear link between compensation and performance. In this paper, we show that if compensation is endogenous, then the better manager may be under-compensated as they have more to lose in the event of not reaching an agreement with the fund family. Ibert, Kaniel, Van Nieuwerburgh, and Vestman (2017) show that managerial compensation in the fund family does not track the the total fees collected by the fund family, highlighting the necessity to analyze the agency frictions between the fund family and the manager. This paper aims to provide such theoretical foundations.

Admati and Pfleiderer (1994) show how contracting through an informed intermediary may be profitable for outside investors and reduce agency frictions. We find that this result holds even if the intermediary does not own a significant fraction of the project and that intermediation by the family increases information efficiency in this economy as long as there is residual uncertainty. Das and Sundaram (2002) show that in the presence of asymmetric information about the manager's skill, he can signal his ability by choosing an appropriate fee schedule, thus, credibly conveying to the market his type. In this paper, we do not allow for signaling with fees since we make weak assumptions about long-term commitment and, instead, the manager signals his skill by working for the family. Cuoco and Kaniel (2011) and Buffa, Vayanos, and Woolley (2018) study the general equilibrium implications of benchmarking in asset management contracts.

The role of mutual fund families has been under-explored in the theoretical context. Gervais, Lynch, and Musto (2005) show that if the fund family can commit to firing a fixed percentage of managers, then it fires bad ones in equilibrium and channel investment to better managers. They find that this efficiency converges to first-best as the number of funds managed by the fund family increases to infinity. We show that in dynamic labor markets plagued by adverse selection the fund family does not need the commitment to fire bad managers as they become too expensive to retain anyway. This analysis does not rely on the family operating many funds but can be extended. Even if there were many funds offered by the family, it is easy to show that it still captures a significant rent from both investors and managers by exploiting its informational advantage.

Our model is related to the literature on adverse selection in labor markets pioneered by Greenwald (1986). He shows that asymmetries of information can be a significant barrier for employees switching jobs making the firm currently employing the worker a de-facto monopsonist in the labor market. Acemoglu and Pischke (1998) show that this may give an incentive to the firm to provide general training to its worker since the latter is unable to bargain the newly created value from the firm. In the context of money management Chevalier and Ellison (1999) show that young mutual fund managers may be terminated in response to bad performance. They also show that investors react positively to changing a manager who has been performing poorly. Both findings are consistent with our model and results. Deuskar, Pollet, Wang, and Zheng (2011) characterize the career paths of mutual fund managers who go on to become hedge fund managers. Consistent with the model in this paper, they show that if the manager leaves after a poor performance, they continue to under-perform the market. On the contrary, after a period of good performance, mutual fund managers are allowed to open a hedge fund the side. Given the star-phenomenon in the mutual fund industry, the fund family can benefit both from employing a star manager as well as letting him contract directly with investors.[8]

Spence (1973) is the seminal paper showing how costly signaling may credibly reveal private information. In this paper we explore the implications of this mechanism on optimal compensation within the firm. In the mutual fund industry, the performance of the manager is well observed. As such, the outside opportunities of a single manager constitute this individual's dynamic labor market. We carefully model this labor market showing that better quality managers are more sensitive to their reputation. We derive an equilibrium similar to the one derived in Fuchs and Skrzypacz (2010), where the lowest type gets revealed first. This work is also related to Daley

[8]This behavior is consistent with a minor modification of this model discussed in Section 3.7.

and Green (2012) and Daley and Green (2014) who analyze the interaction of dynamic signaling with the exogenous news. Zryumov (2018) shows that a similar equilibrium structure holds in the context of exogenous changes to the economic environment. We contribute to this literature by micro-founding the gains from trade to reflect the specificity of the labor market for money managers. We set our model in continuous time for tractability. Since the family and the fund manager sign observable contracts in each period, we have to be careful with the equilibrium notion as subgame perfection is not well-defined in continuous time.[9] We also analyze the case when the manager exerts private effort to influence the return process. The implications of agent's effort to reputation building in Board and Meyer-ter Vehn (2013) and career concerns in Bonatti and Hörner (2017). We extend this analysis by allowing the firm to possess private information about its quality along the equilibrium path.

The rest of the paper is structured as follows. Section 2 introduces the model and the equilibrium notion. In Section 3 we derive the equilibrium and characterize the equilibrium rents obtained by the fund family. Section 4 analyzes the variant of the model in which the manager can exert private effort to improve returns and the fund family manipulates his incentives by offering performance-sensitive contracts. Section 5 concludes.

2 Setup

The fund family employs a portfolio manager to operate one of its funds. To highlight the role of the multi-layer contracting, we assume that the family only has one fund and it is operated by a sequence of fund managers indexed by $n \in \mathbb{N}$. Each manager begins his career working for the family but may leave at any time to open his own fund or leave the industry altogether.[10] The fund family and the manager observe an informative signal about his ability. Investors do not observe this information and learn about manager's skill from fund returns and the manager's turnover. The fund is open-ended, and investors allocate wealth into it based on their perception of the manager's skill, similar to Berk and Green (2004). Investors pay a percentage fee f of assets under management for investing into the family's fund, or a percentage fee h of assets under management for investing into the manager's independent fund.[11] The family pays part of this fee

[9] See Simon and Stinchcombe (1989) for details.

[10] We assume the manager cannot run both funds simultaneously. We also assume that once the manager leaves the family, he cannot be re-employed by it.

[11] We assume this fee is exogenous in the baseline model. In Section 3.5 we allow the family to choose it once prior to observing the quality of the manager in its employ.

to the manager in order to retain him in any given period. If the manager leaves the family, he can open his independent fund. Upon the manager leaving the fund, the family expends a labor market search cost I and hires a new manager for its fund which continues to operate.

Time is continuous with $t \in [0, \infty)$. Each period a stage game proceeds as follows. If the manager starts period t working for the family, then he, first, generates a publicly observable return. Both the family and the manager update their private beliefs based on this information. Then the family chooses whether to retain the manager for another period. Whether or not the manager is retained by the family is instantaneously and publicly observed by investors, who update their beliefs about the manager's skill based on these two signals, i.e., observable returns and tenure with the family. Investors then allocate funds competitively. The simplified timing of the game is depicted in Figure 1.

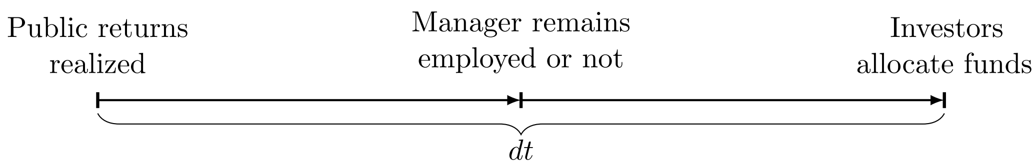


Figure 1: Heuristic timing within a short period of time when the manager works for the family

When the manager and the family separate, he may attempt to open his independent fund. We assume there is no fixed cost of opening the fund, however, the manager has a reservation utility L which he receives if he leaves the money management industry, creating an opportunity cost for staying in the industry.[12] The timing of the subgame in which the manager has left the family is similar to Figure 1, with the difference that the manager may himself choose to leave the money management industry following poor returns.

Now we proceed to formally define the players and payoffs. This is a game of asymmetric information and to highlight the distinction we put a tilde over variables that are privately observable by the family-manager pair but are not observed by investors.[13]

Managers. There is a sequence $n \in \mathbb{N}$ of ex-ante identical risk-neutral managers who are sequentially hired by the fund family. For every manager n we denote the cumulative profit generated by

[12]The cost of opening the independent fund creates a barrier to entry for the fund manager lowering his bargaining power over compensation with the fund family. If this cost is not very large, the analysis remains qualitatively unchanged.

[13]In this notation random variable ξ is observed by all the players in the game, while random variable $\tilde{\eta}$ is observed by the manager and the family.

the manager who is working in the industry to time t by R_t. Process $R = (R_t)_{t\geq 0}$ evolves according to

$$dR_t = A_t \cdot \Big[(\mu - g(A_t))\ dt - dN_t^{\theta} \Big].$$

In the above expression $A_t \geq 0$ denotes the assets under management of the fund at time t. Process $N = (N_t)_{t\geq 0}$ is Poisson with a constant arrival intensity $\lambda(1-\theta_n)$. Random variable $\theta_n \in \{0, 1\}$ denotes the underlying ability of the manager n and is not directly observed by any of the players. If $\theta_n = 0$, then the manager is of low ability and may suffer a loss to his portfolio. If $\theta_n = 1$, then he is of high ability and does not suffer this loss. We assume that $\{\theta_n\}_{n\in\mathbb{N}}$ are independent random variables, i.e., represent manager-specific skill. A low skill manager is less profitable than the high skill manager since the prospect of the loss reduces expected returns.

In what follows we drop subscript n refering to each manager relative to the beginning of his employment by the fund family. The manager does not know θ but is endowed with a private signal about it. We identify this private signal with his private posterior $\tilde{p}_0 \sim F(\cdot)$ which he holds at $t = 0$. Our modeling choice for the return process as a negative Poisson technology stems from its tractability benefits in experimentation settings as highlighted in Hörner and Skrzypacz (2018).[14] Parameter μ denotes the maximum return that the manager can ever generate on investments. The increasing function $g(\cdot)$ captures decreasing returns to scale, in the spirit of Berk and Green (2004), as more assets under management dampen the value the manager can deliver to investors. We make the following assumption about $F(\cdot)$ which simplifies exposition going forward.

Assumption 1. *Distribution $F(\cdot)$ has convex support on $[\underline{p}, \overline{p}]$.*

The convexity of support implies that all intermediate beliefs between $\underline{p}$ and $\overline{p}$ occur with positive probability. The manager's knowledge about his type at $t = 0$ is imperfect, and he learns about it from the returns he generates. Denote by $\tilde{p}_t$ the private posterior belief of the manager and the fund family given the returns up to time t. It is given by

$$\tilde{p}_t \overset{def}{=} \mathrm{P}\left(\theta = 1 \,\middle|\, (R_u)_{u\leq t},\, \tilde{p}_0\right). \tag{1}$$

Throughout the paper we refer to $\tilde{p}_t$ as the (expected) skill of the manager at time t. If the manager has generated a negative shock prior to time t, i.e., $N_t > 0$, then the manager learns that he is unskilled, i.e., $\tilde{p}_t = \theta = 0$. On the other hand, if a loss had not occurred, the manager updates positively about his skill. As a result, the manager with a prior $\tilde{p}_0$ holds a private belief $\tilde{p}_t$ at time

[14] The findings are qualitatively similar in a Brownian model, but the formal analysis carries technical difficulties.

t given by Bayes rule

$$\tilde{p}_t = \frac{\tilde{p}_0}{\tilde{p}_0 + (1-\tilde{p}_0)\cdot e^{-\lambda t}} \cdot \mathbb{I}\left\{N_t^\theta = 0\right\}, \tag{2}$$

which can be equivalently rewritten in differential form as

$$d\tilde{p}_t = \lambda\tilde{p}_t(1-\tilde{p}_t) - \tilde{p}_t \cdot dN_t^\theta.$$

The fund family and the manager negotiate his compensation in every period. We denote the agreed upon wage in period t by $\tilde{w}_t$. It is not observed by outside investors and may reflect the private posterior of the manager given by $\tilde{p}_t$. We denote by τ the time when the manager departs. His exit from the family is observed by investors, and they update their beliefs at time τ based on the equilibrium distribution of $\tilde{p}_\tau$. At this point the manager may be able to remain in the money management industry and, for expositional clarity, we refer to this part of his career as him opening a hedge fund. While operating the hedge fund, he collects a fee h on assets under management, which we assume is exogenous.[15] We denote the hedge fund's assets at time t by Y_t. The total fees collected by the manager if he runs his own fund at time t is $h \cdot Y_t$. The manager may choose to subsequently exit the hedge fund industry altogether and receive his reservation utility L. We denote this time by η. The manager is risk-neutral and discounts future consumption at a rate of r. The realized $t = 0$ discounted utility of manager n can be written as

$$\int_{\tau_{n-1}}^{\tau_n} e^{-rt}\tilde{w}_t\,dt + \int_{\tau_n}^{\eta_n} e^{-rt}hY_t\,dt + e^{-r\eta_n}\cdot L.$$

Fund Family. The fund family is long-lived, risk-neutral, and discounts the future at rate r. At the beginning of each period the family collects a percentage fee f, which we assume is exogenous,[16] of assets under management. The fund is open-ended, and the fund size A_t at time t is determined by the beliefs of investors about the manager's skill. Total revenue of the family in period t is given by fA_t. The flow profit of the family is the difference between its revenue and the compensation cost $fA_t - \tilde{w}_t$. We refer to this term as the profit wedge of the family and show that it crucially depends on the informational advantage the family has about the manager relative to the outside investors. We elaborate further on how the wage process $\tilde{w} = (\tilde{w}_t)_{t\geq 0}$ is determined for each manager once we formalize the behavior of investors and the implied outside option for the manager.

The fund family chooses a stopping time τ_n when to let go of manager n. A natural interpretation

[15]The analysis is qualitatively unchanged if the manager is compensated by a performance sensitive component when he opens his fund.

[16]We relax this assumption in Section 3.5.

is that the family has a real option to let the manager go given the flow wage cost determined by the manager's outside option of opening his own fund. The realized discounted profit of the family is given by[17]

$$\sum_{n=1}^{\infty}\left(\int_{\tau_{n-1}}^{\tau_n} e^{-rt}\left(fA_t-\tilde{w}_t\right)\,dt-I\cdot e^{-r\tau_{n-1}}\right).$$

Investors. There is a set of competitive risk-neutral investors who discount cash flows at rate $\rho > 0$. They observe all past returns generated by the fund as well as whether or not the manager is currently employed by the family. Importantly, they do not observe the manager's private signal $\tilde{p}_0$ and must rely on public information to learn about it. We denote by k_t the belief, held by investors in equilibrium, about the manager who has not generated a negative return and who leaves the family and opens his own fund at time t. Similarly, we denote by q_t the belief, held by investors, about the manager who is still working for the complex at time t. Process q_t denotes the public posterior at time t about the managers who still work for the family. It is computed by Bayes rule for any $t \in [\tau_{n-1}, \tau_n]$ given the equilibrium stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ which may be informative of the manager's private information

$$q_t = \mathrm{P}_t\left(\theta = 1 \;\middle|\; N_t^\theta - N_{\tau_{n-1}}^\theta = 0\right) = \mathrm{E}_t\left[\tilde{p}_t \;\middle|\; N_t^\theta - N_{\tau_{n-1}}^\theta = 0\right]. \tag{3}$$

If the manager is currently working for the family, investors allocate wealth, i.e., choose A_t, until the expected return generated by the fund is equal to their opportunity cost of capital

$$\mu - g(A_t) - f - \lambda(1-q_t) = \rho \qquad \Rightarrow \qquad A_t = g^{-1}\big(\lambda q_t + \mu - \rho - f - \lambda\big).$$

A higher belief q_t implies a lower risk of investors suffering a loss and thus increases the assets they are willing to invest. With a slight abuse of notation, we can write $A_t = A(\lambda q_t - f)$. It is increasing in the public perception of the manager's quality and decreasing in the fee charged by the family. We impose the following tractability assumption on the assets under management function $A(\cdot)$.

Assumption 2. *Decreasing returns to scale satisfy*

$$fA'(\lambda q - f)q(1-q) \leq hA'(\lambda k - h)k(1-k)$$

for any $q \geq \mathrm{E}\left[\tilde{p}_0 \;\middle|\; \tilde{p}_0 \geq k\right]$ *and* $k \geq \underline{k}$.

In equilibrium, the time when the manager leaves the family is informative about his type since his

[17] Wage $\tilde{w}_t$ paid to the manager at time t is unobserved by outside investors and differs across managers of different skill.

compensation is linked to his private information and so does the incentive of the family to retain him. Investors assign belief k_t to the manager who leaves the family at time t after a history of good returns to open his own fund. Once the manager has left the family, subsequent updating about his quality is driven solely by publicly generated returns.[18] Time s public posterior about the manager who leaves the family at time $t < s$ and belief k_t can be computed by applying Bayes rule between s and t

$$\pi(s;t,k_t) \overset{def}{=} \mathrm{P}\left(\theta = 1 \;\middle|\; (R_u)_{u\in[t,s]},\, k_t\right) = \frac{k_t}{k_t + (1-k_t)\cdot e^{-\lambda(s-t)}} \cdot \mathbb{I}\left\{N_s^\theta = 0\right\}. \tag{4}$$

In a separating equilibrium it is the case that $\tilde{p}_\tau = k_\tau$ when the manager leaves the family and, thus, the public posterior at time t given by $\tilde{p}_t = \pi(k_\tau, t-\tau)$ for $t > \tau$ coincides with the manager's private posterior post-exit. While this is true on the equilibrium path, it is important to evaluate the value the manager can obtain by leaving the family if $k_s \neq \tilde{p}_s$. Note that k_t itself need not follow Bayes rule as it is given by the type of the manager who investors expect to separate at time t. If the manager quits at time t and opens his independent fund, his assets under management at time s, denoted by $Y_{t,s}$, must satisfy

$$\mu - g(Y_{t,s}) - h - \lambda + \lambda \cdot \pi(s;t,k_t) = \rho, \qquad Y_{t,s} = g^{-1}\big(\lambda\pi(s;t,k_t) + \mu - \rho - h - \lambda\big).$$

With a slight abuse of notation, we write $Y_{t,s} = Y(\lambda\pi(s;t,k_t) - h)$. Process $k = (k_t)_{t\geq 0}$ determines the outside option of the manager in every instance as it pins down the starting size of the fund that he can open if he leaves the family. Once the manager leaves the family, the subgame dynamics are driven by Bayesian updating (4) and the manager has no way to influence investor beliefs beyond generating good returns. We make the following assumption on the outside option of the manager ensuring that the manager leaves the money management industry if he generates a loss, but would otherwise prefer to stay and manage his own fund.

Assumption 3. *Manager's outside option L satisfies $L \in \left[\frac{hA(-h)}{r}, \frac{hA(\lambda\underline{p}-h)}{r}\right]$.*

We make four assumptions simplifying the solution of the model. Assumption 1 states that $F(\cdot)$ has full support on $[\underline{p}, \overline{p}]$ allowing us to characterize the equilibrium using first order conditions. Assumption 2 implies that local optimality implies global optimality. The assumption that returns follow a negative Poisson process imply that this first order approach translates into dynamics driven by nonlinear differential equations with initial conditions. This ensures the existence and

[18] We can show that if the manager would have chosen to leave the money management industry, he would optimally do so right after leaving the family, rather than after having generated good returns over any time period.

uniqueness of the smooth equilibrium. Finally, Assumption 3 is a sufficient, but not a necessary, condition that ensures the existence of a separating equilibrium introduced below.

2.1 Retention Constraint and Equilibrium Definition

Investors assign a belief k_t to whichever manager happens to leave at time t, regardless of the true underlying skill $\tilde{p}_t$. Thus, opening the fund at a starting belief level k_t is the outside option available to every manager at time t. Even when the family has full bargaining power the lifetime utility of the manager must be weakly greater than the expected value he would get by leaving immediately and opening his own fund. It is incentive compatible for the manager of type $\tilde{p}_t$ to stay with the fund from time 0 until time τ if and only if it is better than leaving the fund immediately given investor beliefs at any time $t < \tau$. Thus, along the path of good returns the wage process $\tilde{w} = (\tilde{w}_t)_{t\geq 0}$ for manager of type $\tilde{p} = (\tilde{p}_t)_{t\geq 0}$ must satisfy[19]

$$\mathrm{E}_{\tilde{p}_t}\left[\int_t^{\tau_n} e^{-r(s-t)}\tilde{w}_s ds + \int_{\tau_n}^{\eta_n} e^{-r(s-t)} hY_{\tau_n,s} ds + e^{-r(\eta_n - t)} L\right] \geq \mathrm{E}_{\tilde{p}_t}\left[\int_t^{\eta_n} e^{-r(s-t)} hY_{t,s} ds + e^{-r(\eta_n - t)} L\right] \quad (5)$$

for all $t \in [\tau_{n-1}, \tau_n]$ and the expectations are taken with respect to future returns, conditional on the belief about θ at time t equal to $\tilde{p}_t$.

We focus on pure strategy separating equilibria in which investors infer the manager's private information based on his returns and when he exits the fund. We focus on subgame perfect equilibria in which parties can only commit to short-term contracts. The difficulty of introducing subgame perfection into a dynamic game with frequent actions is well known[20] and we must take special care in defining it to avoid such complications.

Definition 1. *A pure strategy separating equilibrium is a sequence of stopping times* $(\tau_n)_{n\in\mathbb{N}}$*, investor belief processes* $(q_t)_{t\geq 0}$ *and* $(k_t)_{t\geq 0}$*, and wage process* $(\tilde{w}_t)_{t\geq 0}$ *such that the following three properties are satisfied.*

(i) Fund family optimality: stopping times $(\tau_n)_{n\in\mathbb{N}}$ *and wage* $\tilde{w}$ *solve the optimal stopping problem of the family*

$$\left\{(\tau_n)_{n\in\mathbb{N}},\, \tilde{w}\right\} \in \underset{\{(\hat{\tau}_n)_{n\in\mathbb{N}},\, \hat{w}\}}{\arg\max}\ \mathrm{E}_{\tilde{p}_0}\left[\sum_{n=1}^{\infty}\left(\int_{\hat{\tau}_{n-1}}^{\hat{\tau}_n} e^{-rt}\left(fA_t - \hat{w}_t\right)\, dt - I\cdot e^{-r\hat{\tau}_{n-1}}\right)\right], \quad (6)$$

[19]With a slight abuse of notation we write that the wage is paid as a flow $\tilde{w}_t\, dt$. To account for a possibility of a discontinuous process $(k_t)_{t\geq 0}$ it would be sufficient to work with process $(\tilde{W}_t)_{t\geq 0}$ referring to the cumulative wage paid to the manager of type $\tilde{p}_t$ up to time t. If $\tilde{W}_t$ were continuous, then $d\tilde{W}_t = \tilde{w}_t\, dt$.

[20]See Simon and Stinchcombe (1989) for details. In this model, the wage offered by the family in every period is the frequent observable action.

subject to (5) *being satisfied with equality for every manager* $n \in \mathbb{N}$ *and every* $t \in (\tau_{n-1}, \tau_n]$*.*

(ii) Investor rationality: q_t *given by* (3)*, the equilibrium is separating* $k_{\tau_n} = \tilde{p}_{\tau_n}$ *for all* $n \in \mathbb{N}$*, and belief process* $(k_t)_{t\geq 0}$ *is smooth for any* $t \in (\tau_{n-1}, \tau_n)$*.*

(iii) Stationarity: for any $t \in (\tau_{n-1}, \tau_n]$ *and* $t' \in (\tau_{n'-1}, \tau_{n'}]$ *such that* $t - \tau_{n-1} = t' - \tau_{n'-1}$ *it follows that* $q_t = q_{t'}$ *and* $k_t = k_{t'}$*.*

Two features are important in the above equilibrium definition. First, the family chooses when to let the manager go, while at the same time satisfying the manager's retention constraint (5). In a subgame perfect equilibrium, however, the family cannot commit to paying the manager more than necessary for him to stay an additional period. This is why a strict inequality (5) cannot be sustained in equilibrium. To keep matters technically simple, we impose this intuitive subgame perfection requirement directly in the equilibrium definition. Second, the family is solving the optimal retention problem for every manager type given the equilibrium belief process $(k_t)_{t\geq 0}$. This is why stopping time τ, and wage $(\tilde{w}_t)_{t\geq 0}$ depend on the private information of the family and the manager.

Definition 2. *An equilibrium is monotone if, after every history, the set of managerial types still employed by the family, i.e., the support of* $\tilde{p}_t$ *conditional on* $t < \tau$*, is convex.*

We show that there exists a unique separating equilibrium which is monotone. Definition 2 captures the idea that at any point in time only the very best or the very worst managers are let by the family. Definition 2 is a selection tool that helps us choose an equilibrium in a large set of possible equilibria of the dynamic signaling environment.

3 Equilibrium Analysis

Now we proceed to derive the unique monotone separating equilibrium. First, we characterize the manager's outside option if he leaves the family given a belief process k investors assign to him upon exit after a history of good returns. This pins down the dynamic outside option of the manager which, in turn, lets us derive the equilibrium compensation when he works for the family. We then proceed to derive the equilibrium learning dynamics from the incentive compatibility condition of the family to let go of the manager when investors' belief k_t about the exiting type coincides with the private belief $\tilde{p}_t$. We then proceed to analyze the properties of this separating equilibrium.

3.1 Managerial Subgame

Denote by $U(\tilde{p}, k)$ to be the expected value to the manager who has a private belief $\tilde{p}$ about his skill, while the public belief is equal to k. This corresponds to the manager's continuation utility from leaving the fund family after a history of good returns and being perceived as having expected skill k. If the manager leaves the family after a loss, he is publicly revealed to be unskilled, i.e., $\theta = 0$. Public beliefs impact assets under management and total fees that the manager can collect, while the private belief $\tilde{p}$ is the likelihood that he is a skilled type $\theta = 1$. If the manager runs his fund up until the time when he suffers a loss, his continuation utility can be written as

$$U(\tilde{p}, k) \stackrel{def}{=} \mathrm{E}_{\tilde{p}} \left[\int_0^{\eta} e^{-rt} hA\left(\lambda\pi(t; 0, k) - h\right)\, dt + e^{-r\eta} \cdot L \right]. \tag{7}$$

where $\pi(t; 0, k)$ is the public posterior of a manager who is perceived as type k when opening his own fund after he generates good returns between 0 and t and η is the first time when $dN_t^\theta = 0$ after the manager opens his fund.[21] To keep exposition simple we assume the manager prefers to stay in the money-management industry up until he has generated a loss, i.e., $hA(-h) \leq rL \leq hA(\lambda \underline{p} - h)$. Then, if the manager leaves the family at time t, his expected payoff can be naturally written as $U(\tilde{p}_t, k_t)$. Using law of iterated expectations we can decompose (7) as

$$U(\tilde{p}, k) = \tilde{p} \cdot U(1, k) + (1 - \tilde{p}) \cdot U(0, k). \tag{8}$$

If the manager knows his type perfectly, then computation of $U(\theta, k)$ is quite easy as the high type earns fees in perpetuity, while the low type earns them until the exponentially distributed random time when he suffers a loss

$$\begin{cases} U(1, k) = \displaystyle\int_0^{\infty} e^{-rt} \cdot hA\left(\lambda\pi(t; 0, k) - h\right)\, dt, \\ U(0, k) = \displaystyle\int_0^{\infty} e^{-(r+\lambda)t} \cdot hA\left(\lambda\pi(t; 0, k) - h\right)\, dt + \dfrac{\lambda}{r + \lambda} \cdot L. \end{cases} \tag{9}$$

Lemma 1. *The expected payoff of the manager of type $\tilde{p}$ is given by (8). Functions $U(\theta, k)$ are given by (9) and can be obtained as the unique solutions to differential equations*[22]

$$rU(\theta, k) = hA(\lambda k - h) + \lambda k(1 - k) \cdot \frac{\partial}{\partial k} U(\theta, k) + \lambda(1 - \theta)(L - U(\theta, k)), \qquad \theta \in \{0, 1\} \tag{10}$$

[21] By this we mean that the manager does not incur a loss between 0 and t.

[22] If k is very low, then it may be optimal for the manager to exit the fund management industry at the very beginning. We assume that all the managers start with sufficiently optimistic beliefs that they choose to exit the industry only after generating a loss $dN_t^\theta = 1$.

subject to the boundary condition

$$U(\theta,1)=\frac{hY(\lambda-h)}{r+\lambda(1-\theta)}+\frac{\lambda(1-\theta)L}{r+\lambda(1-\theta)},\qquad \theta\in\{0,1\}.$$

Sketch of Proof. Equations (10) is obtained by applying Ito's lemma to the realized utility of the manager. The general solutions can be obtained in semi-closed form via hypergeometric functions. It implies that there exist a unique solutions satisfying the boundary condition at $k=1$, where both ordinary differential equations exhibit regular singularities. □

It may be the case that k is so low that it is unprofitable for the manager of quality $\tilde{p}$ to manage the fund. Since beliefs only increase as the manager is generating good returns, it implies that the optimal time to leave the money management industry is either immediately, or upon obtaining a negative return. In this case the manager's outside option takes a simple form $\max\left[U(\tilde{p},k),L\right]$. In what follows we assume that L is sufficiently small so that the manager only leaves the mutual fund industry once he is revealed to be a low type, i.e., $U(\underline{p},\underline{p})\geq L$.

Lemma (1) provides a tractable solution for the outside option of the manager $U(\tilde{p},k)$ which depends on both $\tilde{p}$ and k and which is essential for pinning down wages in equilibrium. It also allows us to derive an important comparative static result that higher skilled managers value reputation more.

Proposition 1. *A manager with a higher private belief about his skill is more sensitive to changes in investors' beliefs:* $\frac{\partial}{\partial k}U(\tilde{p}',k)>\frac{\partial}{\partial k}U(\tilde{p}'',k)$ *for any* $\tilde{p}'>\tilde{p}''$.

A manager who holds a higher belief in his ability expects to work in the money management industry for a longer time. A higher skilled manager is, effectively, more patient and, hence, is more sensitive to assets under management in the long run than a less skilled manager. Thus, he is more sensitive to changes in his reputation since it has a longer-term impact on his assets under management. Proposition 1 can be seen from the following derivation

$$\frac{\partial}{\partial k}U(\tilde{p},k)=\frac{\partial}{\partial k}U(1,k)-\frac{\partial}{\partial k}U(0,k)=\int_0^\infty\underbrace{\left[e^{-rt}-e^{-(r+\lambda)t}\right]}_{>0}\cdot\underbrace{\left[h\cdot\frac{\partial}{\partial k}A\big(\lambda\pi(t;0,k)-h\big)\right]}_{>0}dt>0,$$

as long as there is any residual public uncertainty about the manager's type, i.e., $k\in(\underline{k},1)$. Proposition 1 is economically important as it implies that better managers are more willing to sacrifice short term gains in favor of public learning about their quality. This is an important incentive that partly determines the equilibrium turnover and wage distribution within the fund family.

3.2 Fund Family Equilibrium

First, we formulate our first result that there exists a separating equilibrium.

Proposition 2. *There exists a monotone separating equilibrium. It is characterized by a threshold p^*. The fund family retains all well-performing managers as long as investor beliefs about the separating manager $k_t < p^*$. For $k_t \geq p^*$ the fund family gradually lets go of the worst managers in favor of hiring new managers.*

We outline the key steps in the equilibrium construction here and relegate some of the technical details to the general proof in the Appendix. To keep the exposition simple and capture the key economic forces, suppose that distribution $F(\cdot)$ is continuous on $[\underline{p}, \overline{p}]$. Denote by M the ex-ante expected value of the family. It is given by

$$M = \mathrm{E}\left[\sum_{n=1}^{\infty}\left(\int_{\tau_{n-1}}^{\tau_n} e^{-rt}\left(fA(\lambda q_t - f) - \tilde{w}_t\right)\, dt - I \cdot e^{-r\tau_{n-1}}\right)\right],$$

where the expectation operator is taken both across all potential realized return paths and ex-ante fund manager types $\tilde{p}_0$. In a stationary equilibrium M is the ex-ante continuation value collected by the family when it lets go of an existing manager and hires a new one from the same distribution of types $F(\cdot)$. The expected value to the fund family conditional on fund manager of type $\tilde{p}_0$ can then be written as

$$\mathrm{E}_{\tilde{p}_0}\left[\int_0^{\tau_1} e^{-rt}(fA(\lambda q_t - f) - \tilde{w}_t)\, dt + e^{-r\tau_1} \cdot M\right] - I.$$

In a monotone separating equilibrium investors believe that the types of the managers working for the fund family at time t are always in some interval $[a_t, b_t]$. It must be the case that $b_t = \bar{p}_t = \pi(\bar{p}, t)$, i.e., it includes the highest ex-ante type, since, otherwise, the low type managers would have an incentive to leave the fund family earlier and be assigned the belief of being the highest type. This implies investors perceive the manager leaving the fund family after any history to be the worst manager that could still be employed by the family up to that time given its equilibrium retention policy. We refer to this as the cutoff manager type and it corresponds to k_t. At $t = 0$ the lowest managerial type is $\underline{p}$ and hence $k_0 = \underline{p}$.

The equilibrium career dynamics of a given manager is split into two regions. In the first region, the difference between the average investor beliefs q_t and the worst manager k_t is sufficiently high so that the fund family retains all managers given their good performance. In the second region, the difference between q_t and k_t is small and the fund family gradually lets go of some of the well-performing managers based on its private information about them. Assumption 2 implies that the equilibrium dynamics either start in the second region, or transition once from the first region to the second region.

Fund family's assets under management are governed by the expected ability q_t of the manager still employed at time t. For a given M define by $Q(k)$ the lowest public belief level such that the total fees collected by the fund family in that instant make it worthwhile to match the fees the manager would collect in that instant were he to leave and be perceived by investors as type k. It is the unique solution to the equation[23]

$$fA(\lambda Q(k) - f) = hA(\lambda k - h) + rM. \tag{11}$$

Since M is an equilibrium object, function $Q(\cdot)$ is also determined in equilibrium. As long as $q_t > Q(k_t)$ the family prefers not to let go of any of its managers who have not generated the observable loss to investors, i.e., $N_t^\theta = 0$. If $\mathrm{E}[\tilde{p}_0] > Q(\underline{p})$, then the fund family does not fire managers based on private information $\tilde{p}_0$ for a positive length of time and investors only learn about the manager's type from public returns. For any t such that $q_t > Q(k_t)$ investors learn about the manager according to the simple Bayes rule for all t s.t. $q_t > Q(k_t)$ we have

$$q_t = \frac{q_0}{q_0 + (1 - q_0) \cdot e^{-\lambda t}} \cdot \mathbb{I}\left\{N_t^\theta = 0\right\}, \qquad k_t = \frac{k_0}{k_0 + (1 - k_0) \cdot e^{-\lambda t}} \cdot \mathbb{I}\left\{N_t^\theta = 0\right\},$$

where $q_0 = \mathrm{E}[\tilde{p}_0]$ and $k_0 = \underline{p}$. Define t^* to be the first time when the information wedge of the fund family shrinks enough so that the flow profit of the fund family is exactly equal to its flow opportunity cost. Formally

$$t^* = \inf\{t : \, q_t = Q(k_t)\}. \tag{12}$$

If the fund family does not do anything, then incremental learning would imply that the fund family becomes unprofitable next period.[24] This implies that the fund family begins to let go of the lowest quality managers, resulting in the cutoff process k_t to be increasing for $t > t^*$ for two reasons. First, good returns increase the belief about *all* types and hence k_t increases. Second, being retained by the family is a good signal about the manager since he is not the worst type that would have been

[23]It may be the case that $Q(k) > 1$, in which case it is not, technically, a belief level.

[24]Assumption 2 implies that beliefs converge as a result of Bayesian learning at that point.

let go otherwise. Define process $\Gamma = (\Gamma)_{t\geq 0}$ resulting from learning based on retention via

$$dk_t = \lambda k_t(1-k_t)\,dt + d\Gamma_t, \tag{13}$$

subject to $\Gamma_0 = 0$. The first term, $\lambda k_t(1-k_t)$, represents positive Bayesian updating about type k_t over a length of time dt if there are no losses in that period. The second term, $d\Gamma_t$, represents the incremental information conveyed by the retention decision of the family over length of time dt. In a separating equilibrium where retention is a positive signal it must be the case that Γ_t is weakly increasing. The derivative of the monotone function is defined almost everywhere and we denote by γ_t its derivative, i.e., $\gamma_t = \dot{\Gamma}_t$. According to this definition, for $q_t \geq Q(k_t)$ we have $\gamma_t = 0$.[25]

The manager's outside option if he leaves the fund family at time t is exactly $U(\tilde{p}_t, k_t)$. Since the fund family has all of the bargaining power in negotiating compensation, this is also the expected value of working for the fund family in any equilibrium. If working for the fund family leads to improved reputation, as captured by a faster increase in k_t, then the fund manager would be willing to forgo some of the compensation and still remain with the fund family. Considering (7), the necessary wage to retain the manager in period t is given by

$$\begin{aligned}\tilde{w}_t = w_t(\tilde{p}_t) &= hA(\lambda k_t - h) - \frac{\partial}{\partial k}U(\tilde{p}_t, k_t)\cdot\gamma_t \\ &= hA(\lambda k_t - h) - \left(\tilde{p}_t \cdot \frac{\partial}{\partial k}U(1,k_t) + (1-\tilde{p}_t)\cdot\frac{\partial}{\partial k}U(0,k_t)\right)\cdot\gamma_t.\end{aligned} \tag{14}$$

Equation (14) is a result obtained by analyzing (8) and (10), where we can see that the manager's marginal value of improved reputation, i.e., increase in k_t, is equal to $\frac{\partial}{\partial k}U(\tilde{p}_t, k_t)$. His required required compensation $\tilde{w}_t$ are the fees he would have collected that period if he were to leave and open his fund minus the present value of improvement in reputation γ_t assigned by investors if he were to delay his exit from the family for another instance multiplied by the marginal expected value of this reputation given by $\frac{\partial}{\partial k}U(\tilde{p}_t, k_t)$. Higher reputation improves his outside value to investors and, consequently, increases his bargaining power with the fund family going forward.

Facing such compensation costs, the fund family lets go of the manager when its flow profit falls below its opportunity cost. This implies that at each time τ when the manager is let go it must be that

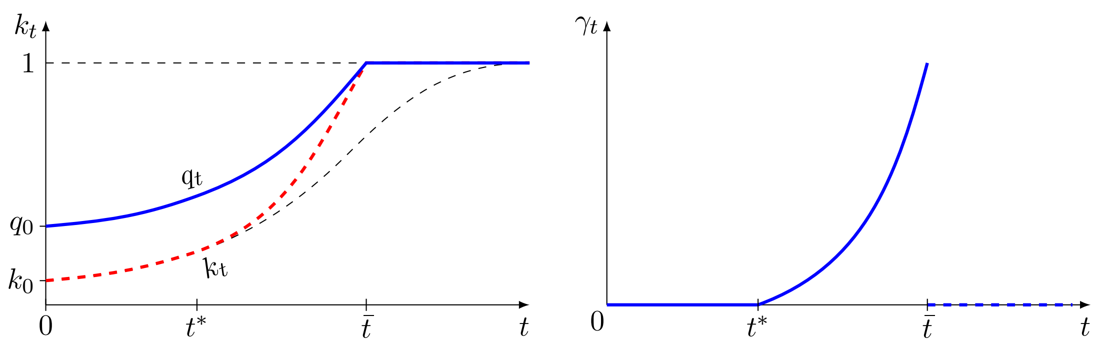
$$\underbrace{rM}_{\text{opportunity cost}} = \underbrace{fA(\lambda q_\tau - f) - w_\tau(\tilde{p}_\tau)}_{\text{flow profit}}. \tag{15}$$

[25] We conjecture and verify an equilibrium in which Γ_t is continuously differentiable and γ_t finite everywhere.

On the one hand, the family benefits from a higher average belief q_t about its manager as the total fees are increasing with respect to public perception of the manager's skill. On the other hand, a higher perception about the manager who is leaving the fund, i.e., the level of k_t at time t, increases his bargaining power and lowers profits, even for a high belief q_t. The family lets go of the manager when its flow profit falls below its equilibrium opportunity cost. Formally, the stopping time τ is optimal if the family prefers to let go of the manager at time τ immediately, rather than employing him for another period and letting him go then.

In a separating equilibrium it must be the case that $k_\tau = \tilde{p}_\tau$ meaning that it is incentive compatible for the family to let go of the cutoff type in a way that is consistent with investor expectations.[26] Examining (15) and (14) we can see that for $t > t^*$ it must be the case that the managers are leaving continuously. Wage Equation (14) implies that for every $t > t^*$ in which separation occurs on equilibrium path, i.e., if $\gamma_t > 0$, it must be the case that

$$\gamma_t = \frac{rM + hA(\lambda k_t - h) - fA(\lambda q_t - f)}{k_t \cdot \frac{\partial}{\partial k} U(1, k_t) + (1 - k_t) \cdot \frac{\partial}{\partial k} U(0, k_t)}. \tag{16}$$



(a) Equilibrium dynamics for k_t, q_t, and standard Bayesian updating.

(b) Rate of learning through retention γ_t.

Figure 2: Equilibrium learning about the cutoff type as a combination of Bayesian learning and learning through retention.

Figure 2 highlights the equilibrium learning dynamics when the information asymmetry is initially small in the sense that $q_0 > Q(\underline{p})$. Then for $t < t^*$ the difference between the average manager type q_t and the cutoff type k_t is sufficiently high so that both evolve solely based on Bayesian learning. As we see in 2a, this informational asymmetry decreases due to departures of managers who generated a loss and revealed themselves to be unskilled. Such departures increase both the

[26] Note that while $\tilde{p}_t$ is unknown by investors, in a separating equilibrium observing the separation time τ allows investors to correctly infer the level of $\tilde{p}_\tau$.

average type q_t and the cutoff type k_t, with the gap eventually decreasing over time. At $t = t^*$ the profit from retaining the lowest skilled manager is exactly equal to the family's opportunity cost, i.e., $q_{t^*} = Q(k_{t^*})$, at which point the family begins to let go of the lower expected skill managers, even if they have been generating good returns up to that point. For $t \in [t^*, \bar{t}]$ the family gradually lets go of the lowest skilled managers, at an increasing rate depicted in 2b. The informational asymmetry between the family-manager pair and investors decreases at an accelerated pace, yet the fund family is able to profit from this as better managers are are willing to accept lower pay in order to build reputation, as the belief about the cutoff type increases, thus increasing their expected value of opening the fund in the future.

Ex-ante Value to the Family. This completes the equilibrium construction during the employment spell of a single manager. What is left to do is characterize the ex-ante value of the fund family M which impacts these equilibrium dynamics. Intuitively, the discounted value of the fund family's rents in excess of the opportunity cost has to be equal to its search cost. Formally, this implies that

$$\mathrm{E}\left[\int_0^{\tau^M} e^{-rt} \cdot \left(fA_t\left(\lambda q_t^M - f\right) - \tilde{w}_t^M(\tilde{p}_t) - rM\right)\, dt\right] = I. \tag{17}$$

where q^M and $\tilde{w}^M$ carry superscript M since they implicitly depend on it. The solution to the above equation always exists. If search cost I is low then it implies that the value to the fund family M is high, and t^* may be 0 in equilibrium. If search cost I is high, however, then M is low, t^* is positive and there are both stages of the dynamic separation in equilibrium. The following Lemma establishes that if search cost I is sufficiently high, then there is a unique solution to (17).

Lemma 2. *Suppose that λ is sufficiently small and I is sufficiently large. Then the separating equilibrium is unique.*

Lemma 2 provides a sufficient condition under which M is uniquely pinned down in equilibrium and other stationary equilibria cannot arise. Intuitively, if I is large, then it must be the case that t^* is sufficiently high in equilibrium. This implies that the majority of the fund family's rents arise in the first stage of the employment spell of the manager, i.e., $t < t^*$, when the fund family can underpay the managers due to relatively large adverse selection in the markets. If search cost I was low, on the other hand, the value of re-sampling to the fund family could be so large that it would lead to t^* being low and the majority of the family's profits would arise from the reputation building phase of the manager's career, i.e., $t \in [t^*, \bar{t}]$, when the manager is willing to work for a very low wage as he expects to increase his outside opportunities at high rate. The possibility for equilibrium multiplicity arising in our model is a novel feature in the employment literature. It arises from the combination of the limited commitment of the fund family to long-term contracts and its preference to keep the contract private as to not reveal the manager's skill to outside investors.

3.3 Equilibrium Properties of Managerial Compensation

Careers and compensation of the manager exhibit four properties.

Proposition 3. *In equilibrium careers of the managers exhibit four properties*

(i) all managers get paid the same compensation for $t \leq t^*$;

(ii) better fund managers have longer employment spells with the fund family;

(iii) for $t \in (t^*, \bar{t}]$ better managers are underpaid relative to their instantaneous outside option if and only if $q_t < Q(k_t)$;

(iv) if manager leaves after t^* and a history of positive returns, he obtains discontinuous positive jump in compensation when he opens his own fund;

(v) if $f = h$, his assets under managements decline when he leaves the family.

The wage of the manager after being employed for time t by the family can be written as the fees he would have collected were he to run his own fund that instant less the rent he is willing to give up to the fund family. Denote $\phi(k_t) = \frac{\partial}{\partial k}U(0, k_t)/\frac{\partial}{\partial k}U(1, k_t)$. We can combine (14) and (16) to obtain

$$w_t(\tilde{p}_t) = hA(\lambda k_t - h) - \left[rM + hA(\lambda k_t - h) - fA(\lambda q_t - f)\right]^+ \cdot \frac{\tilde{p}_t + (1 - \tilde{p}_t) \cdot \phi(k_t)}{k_t + (1 - k_t) \cdot \phi(k_t)}. \tag{18}$$

Using the definition of t^* it follows that $w_t(\tilde{p}_t) = hA(\lambda k_t - h)$ for any type of the manager employed by the family for a time $t \leq t^*$ since it makes a positive profit on all types of managers as the informational asymmetry is quite big. For $t > t^*$ there is turnover driven by private information with lower skilled managers leaving the fund, despite having generated high returns. This implies that investors incrementally learn about managers who remain with the fund family, beyond return performance based Bayesian updating. Since higher skilled managers are more sensitive to public beliefs about their ability captured by the fact that $\phi(k) < 1$ for any $k \in (0, 1)$, they put a higher value on the turnover source of reputation building. The fund family understands this and underpays better managers more, while still retaining them. When such managers do eventually leave the family, they obtain a discontinuous increase in compensation. We summarize these findings in Proposition 3.

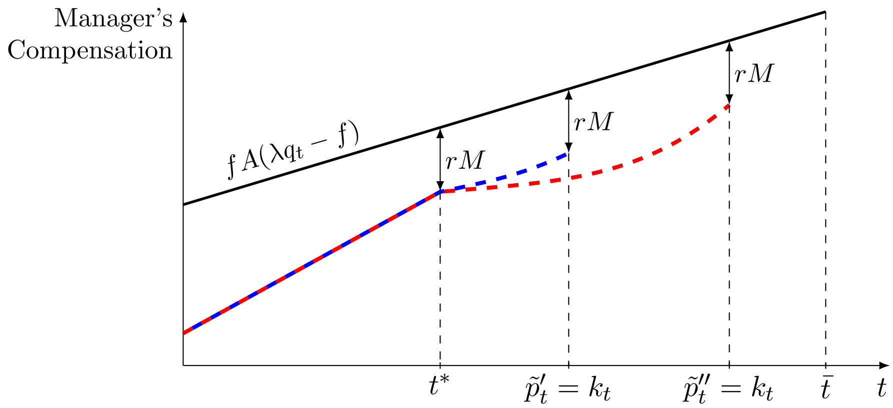


Figure 3: Wage dynamics $w_t(\tilde{p}'_t)$ (in blue) and $w_t(\tilde{p}''_t)$ (in red) for two types $\tilde{p}'_t < \tilde{p}''_t$. Lower skill manager is paid more but is let go earlier.

3.4 Fund Family Profits

The family collects a flow profit in every period which is the difference between the total fees collected from investors and the wage paid to the manager. Based on the characterization of the wage in (18), the flow profit of the family employing the manager of type $\tilde{p}_t$ is given by the difference between collected fees and wages paid out $fA(\lambda q_t - f) - w_t(\tilde{p}_t)$. The behavior of this profit wedge is different depending on the equilibrium value to the fund family of sampling a new manager.

High Information Asymmetry Profits. When the fund family first hires the fund manager, it has the greatest amount of private information about him. In this case, the family effectively retains the manager regardless of his skill, unless he has generated a loss. If the manager is to leave the family at time t, he is perceived as having skill k_t and his fees in that instant would be $hA(\lambda k_t - h)$. If the family can match this in the form of a wage, then the manager might as well stay with the family for at least another instant. The instantaneous profit obtained by the family for $t \in [0, t^*)$ in excess of its opportunity cost is given by

$$fA(\lambda q_t - f) - hA(\lambda k_t - h) - rM > 0. \tag{19}$$

This profit wedge is strictly positive up until $t = t^*$.[27] At t^* it is exactly equal to 0 indicating that the informational advantage over outside investors has decreased so much that the fund family cannot profitably continue while retaining all of its well-performing managers. Assumption 2 guarantees that assets under management $A(\cdot)$ are well behaved and the profit wedge (19) does not increase if updating solely on returns.

Low Information Asymmetry Profits. For $t \geq t^*$ the revenue of the family is insufficient to retain the manager regardless of his skill. After some algebraic manipulations, the profit flow of the family over its opportunity cost can be written as

$$\underbrace{\left(fA\left(\lambda q_t - f\right) - hA(\lambda k_t - h) - rM\right)}_{\leq 0} \cdot \underbrace{\left(k_t - \tilde{p}_t\right)}_{\leq 0} \cdot \underbrace{\frac{1 - \phi(k_t)}{k_t + (1 - k_t) \cdot \phi(k_t)}}_{>0} \geq 0. \tag{20}$$

When the first term becomes negative, learning through turnover implies that the better managers are willing to be underpaid, implying that total fund family rent in excess of its opportunity cost is weakly positive. The negative information rent is then also multiplied by the negative term $k_t - \tilde{p}_t$ which scales by how much the skilled managers benefit from accelerated learning about their ability. It is interesting that in equilibrium the profit flow of the family is always a multiple of the "unconditional" profit wedge (19). For $t < t^*$ it is identical for all managers, while for $t > t^*$ it depends on the private information about each manager. As the information rent declines, the family can charge the best managers for the public certification opportunity it provides them.

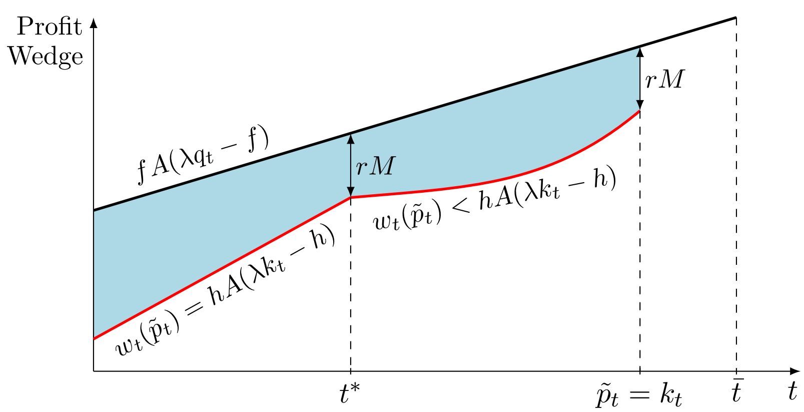


Figure 4: Period t profit of the family is the difference between the black and red lines, captured in blue.

The dynamics of the principal's profit wedge are illustrated in Figure 4. The profit flow between

[27] Depending on parameters of the model t^* may be equal to 0.

$t = 0$ and $t = t^*$ is independent of the manager's skill as long as he has not generated a loss, and assets under management, as well as managerial compensation, follow Bayesian learning dynamics based on observable returns. The profit wedge of the family monotonically decreases from $fA(\lambda q_0 - f) - hA(\lambda k_0 - h)$ to rM over the length of time $[0, t^*]$. After t^* the type of the worst manager is improving over time. This implies that the profit wedge of the fund family increases as the wage paid to the manager decreases, as the latter is willing to be under-compensated in exchange for faster growth of his reputation based on turnover. This cannot go on forever and the fund family eventually lets go of the manager. For $t \geq t^*$ the profit of the fund family given any manager is inverse U-shaped starting and ending with the fund family being locally indifferent between keeping the manager on and its opportunity cost of hiring a new manager.

Valuable Managers. It is useful to point out that the residual value of the manager to the family, as measured by the residual profits is given by

$$V_t(\tilde{p}_t, k_t) = \mathrm{E}_{\tilde{p}_t}\left[\int_t^\tau e^{-r(s-t)}(fA(\lambda q_s - f) - w_s(\tilde{p}_s))\, ds + e^{-r(\tau - t)} \cdot M\right].$$

It depends on both the current belief about the manager $\tilde{p}_t$ as well as the residual uncertainty about the manager, as captured by k_t. As such, managers who are valuable for the fund family are those, for whom there is a large difference between $\tilde{p}_t$ and k_t, i.e., they are not likely to leave the family soon. Young managers satisfy this since, for them, the adverse selection in the labor markets is greatest. This is consistent with the empirical finding of Gaspar, Massa, and Matos (2006) who show that a fund family is more likely to assist a young manager generate a positive α, rather than a well-established one.

3.5 Optimal Fund Family Fee

In this section we show that the family does not collect the maximum possible fees in each period as it balances out rent extraction from investors and rent extraction from managers. The objective of the family is to maximize its expected revenues less the compensation paid to the manager. To maximize this difference, it may not be optimal to maximize total fees, i.e., revenues, collected from investors, but instead choose it to be able to extract more value from the manager.

Suppose the family chooses fee f before hiring the manager. This implies that the family does not know the skill of the manager working for it, and the fee is not used to signal the quality of the manager.[28] Denote by $V(\tilde{p}, k; f)$ who hires a manager of type $\tilde{p}$ and the lowest type of manager is given by k which is exogenous. The family then solves

$$\max_f \mathrm{E}\left[V(\tilde{p}, k; f)\right].$$

Importantly, the family does not observe $\tilde{p}$ when choosing the fee. This captures the idea that the contract between investors and the fund family is quite sticky. For simplicity, we focus on a flat fee structure, but the analysis can be extended to performance contingent fees.[29]

To illustrate the point, consider the decreasing returns to scale function to be logarithmic, i.e., $g(x) = \alpha \log(x)$. Such specification implies that the total fees collected by the family at time t are equal to

$$f \cdot A\left(\lambda q_t - f\right) = f \cdot e^{\frac{\lambda q_t - f}{\alpha}}.$$

If the objective was to maximize total revenues, the family can simply maximize fees collected in every period. The optimal fee is then a constant given by $f^{fb} = \alpha$ as can be seen from the first order condition

$$e^{\frac{\lambda q_t - f}{\alpha}} - \frac{f}{\alpha} \cdot e^{\frac{\lambda q_t - f}{\alpha}} = 0.$$

We show that if the family is strategic about the wages it pays to its managers, it should set the fee f^* strictly below f^{fb}.

Proposition 4. *Suppose $g(x) = \alpha \log(x)$. The optimal fee f^* chosen by the family before it hires the manager collects strictly less fees from investors than the fee that maximizes revenues, i.e., $f^* \neq f^{fb}$.*

The intuition for maximum fees in Proposition 4 being sub-optimal stems from the fact that signaling is most efficient when the delay is expensive. Skilled managers are willing to sacrifice more wages to be revealed sooner. It implies that instead of charging investors a higher fee, the family can offer a lower fee on its fund and underpay the manager more. The manager accepts this as it implies a bigger process of cutoff beliefs $(k_t)_{t \geq 0}$ speeding up the time when he can open an independent fund. Proposition 4 can be interpreted as the family being willing to pass the gains from revealing talent from the managers to investors.

[28] For a model of signaling with fees, see Das and Sundaram (2002).

[29] See Das and Sundaram (2002) for the analysis of the effects of fulcrum and incentive fees on investor welfare.

3.6 Equilibrium of the Limit to the Binary Distribution

In this section we consider the limiting equilibrium that arises as we take the limit of investor beliefs (q,k) of equilibria derived in Section 3.2 as the distribution of managerial types converges continuously to a binary support. This construction is useful to analytically highlight additional properties of the family profits and managers' compensation. We further use this limiting equilibrium in Section 4 to analyze the extension of the model in which the manager exerts unobservable effort, thus endogenizing the arrival rate λ of the return process.

Denote by F^b a binary distribution on $\{\underline{p},\overline{p}\}$ with probabilities $\{\alpha, 1-\alpha\}$. For ease of notation, define by $\underline{p}_t$ and $\overline{p}_t$ as the private posterior of the manager-family pair about the former's skill given that the manager has generated good returns for t periods, namely $\underline{p}_t = \pi(\underline{p};0,t)$ and $\overline{p}_t = \pi(\overline{p};0,t)$. Denote by M^b the ex-ante value to the fund family from paying the search cost and hiring a new manager. As in the previous section, define by t^{b*} the first time when the fund family's profits absent turnover are equal to the fund family's opportunity cost

$$t^{b*} = \inf\Big\{t:\ fA\left(\lambda\pi\left(\alpha\underline{p} + (1-\alpha)\overline{p};0,t\right) - f\right) = hA\big(\lambda\underline{p}_t - h\big) + rM^b\Big\}.$$

Assumption 2 implies that once the fund family has begun letting go of the less skilled manager, then it will continue to gradually let go of them in order to mitigate the decrease in the informational advantage. Similarly, define by $\hat{t}^{b*}$ first time when, in order for the fund family to remain profitable, it cannot retain any of its low skill managers

$$\hat{t}^{b*} = \inf\Big\{t:\ fA\big(\lambda\overline{p}_t - f\big) = hA\big(\lambda\underline{p}_t - h\big) + rM^b\Big\}.$$

In the case of the binary distribution, investor belief process q^b about the fund managers retained by the fund family at time t can be characterized in closed form as

$$q^b_t = \begin{cases} \pi\left(\alpha\underline{p} + (1-\alpha)\overline{p};0,t\right) & \text{if} \quad t < t^{b*}, \\ Q^b\left(\underline{p}_t\right) & \text{if} \quad t \in \left[t^{b*}, \hat{t}^{b*}\right], \\ \overline{p}_t & \text{if} \quad t > \hat{t}^{b*}. \end{cases} \tag{21}$$

where $Q^b(\cdot)$ is the solution to $fA(\lambda Q^b(k) - f) = hA(\lambda k - h) + rM^b$, similar to (11). For $t \in [0, t^{b*}]$ the informational advantage of the fund family relative to investors is high and, thus, the fund family does not let go of the manager unless he generated a loss. Investor beliefs are simply equal to the conditional average about the initial type. For $t \in \left[t^{b*}, \hat{t}^{b*}\right]$ the fund family gradually lets go of the low skill manager in order to keep the family's revenues above its opportunity costs. Investors update about the average type based on both returns and turnover. For $t > \hat{t}^{b*}$ only the low type managers remain employed until they become too expensive. Even though investors know that in equilibrium it is the high type employed by the fund family, these managers cannot capture this value by leaving immediately as it would not be incentive compatible for the low skill managers.

Given process q^b, define the process for the cutoff type k^b as the solution to the differential equation

$$\dot{k}_t^b = \begin{cases} \lambda k_t^b (1 - k_t^b) & \text{if} \quad t \leq \hat{t}^{b*}, \\ \dfrac{rM^b + hA(\lambda k_t^b - h) - fA(\lambda q_t^b - f)}{k_t^b \cdot \dfrac{\partial}{\partial k} U(1, k_t^b) + (1 - k_t^b) \cdot \dfrac{\partial}{\partial k} U(0, k_t^b)} + \lambda k_t^b (1 - k_t^b) & \text{if} \quad t > \hat{t}^{b*}. \end{cases} \tag{22}$$

subject to an initial condition $k_0^b = \underline{p}$. The above expression states that as long as q_t^b is sufficiently large, then the fund family retains a positive fraction of the low type managers, implying that $k_t^b = \underline{p}_t$ for $t \leq \hat{t}^{b*}$. In (22) we write this in differential form, but $k_0^b = \underline{p}$, and the two have identical laws of motion in $\left[0, \hat{t}^{b*}\right]$. For $t > \hat{t}^{b*}$ the total fees collected by the family are small relative to the required compensation by the manager. As the beliefs of investors about the separating managers improve, the better managers are willing to work for below market wage in order to build reputation. Even though there are no cutoff types in the binary distribution, we are considering the limit of distributions satisfying convex support Assumption 1, which implies that one can think of an infinitesimal sliver of intermediate types covering the interval $(\underline{p}, \overline{p})$, depicted in grey in Figure 5, in addition to the two atoms at $\{\underline{p}, \overline{p}\}$. We formalize this intuition below.

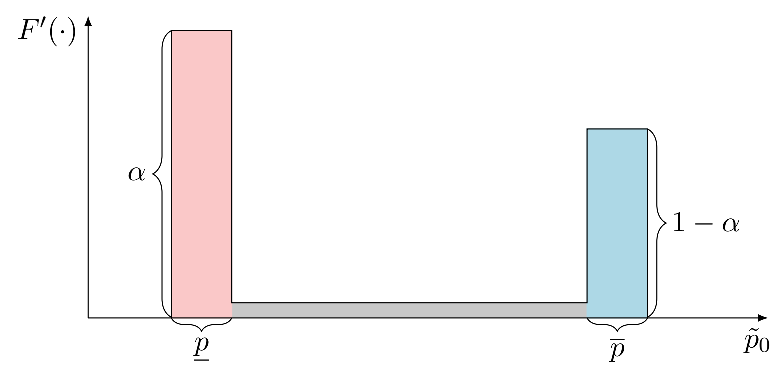


Figure 5: Continuous Limit to the Binary Distribution of Initial Beliefs.

Proposition 5. *Let $\{F^n(\cdot)\}_{n \in \mathbb{N}}$ be a sequence of distributions with continuous supports over $[\underline{p}, 1]$ converging to $F^b(\cdot)$. There exists a sequence of separating equilibria characterized by investor belief* *processes $(q_t^n, k_t^n)_{t\geq 0}$ and fund family expected value M^n under distribution $F^n(\cdot)$ such that*

$$\lim_{n\to\infty} q_t^n = q_t^b, \qquad \lim_{n\to\infty} k_t^n = k_t^b, \qquad \lim_{n\to\infty} M^n = M^b.$$

Sketch of Proof. The equilibrium constructed in Section 3.2 is continuous in the distribution of F, as are the differential equations for both belief processes. If F_n converge uniformly to F^b, then the differential equations also converge to the limiting distribution, i.e., solutions corresponding to distribution F^b. □

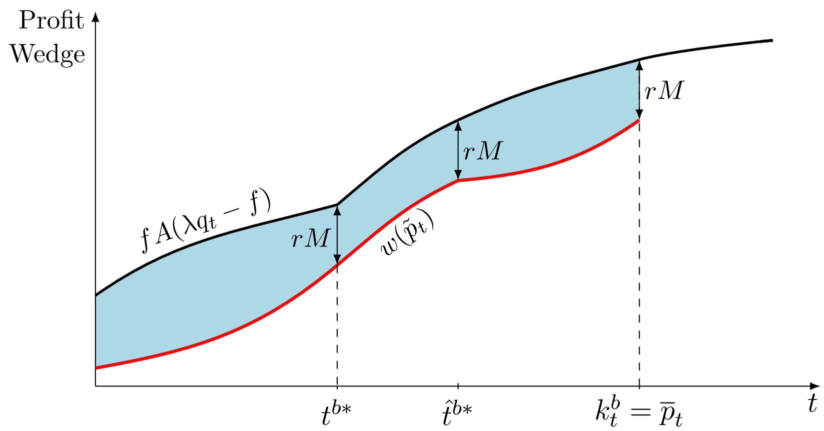


Figure 6: Each period t profit of the family is the difference between the black and red lines, captured in blue. Initial distribution of initial priors $F(\cdot)$ is continuously distributed in the interval $(\underline{p}, \overline{p})$.

The structure of the limiting equilibrium is rather parsimonious. First, the fund family profits from the difference between the assets under management and the outside option of the managers. When this profit wedge declines due to public learning from returns, the fund family can maintain profitability by letting go of low skilled managers. Subsequently, when all low type managers leave the fund, the family profits by holding up the high skilled manager and extracting the value of improved reputation from him by competitively setting wages.

Binary Limit Equilibrium Properties. We use the analytic structure of the equilibrium in order to show that the properties of the fund family are decreasing in the informational asymmetry about the manager.

Lemma 3. *Suppose search cost I is sufficiently large and λ sufficiently small. The expected value to the fund family is decreasing in $\underline{p}$.*

Sketch of Proof. Under these parametric conditions the separating equilibrium is unique and M^b is small. This implies that t^{b^*} is sufficiently large relative to the final separating time $\bar{t}^b$. As the initial belief increases, the wages of the manager increase more than the increased profits by the fund family. □

Equilibrium Selection Mechanism. In a model with two types, the slow separation of low skilled managers is a result of the fund family profiting from the assets under management being driven by the perceived quality of its talent pool. The separation of the high skilled manager in this model happens at a different time than under the Riley signaling outcome . This stems from the fact that we require that all types of managers between $\underline{p}$ and $\overline{p}$ choose to separate at their optimal time, even if the likelihood of these managers being present is quite small. As such, we use the continuous type separation equilibrium as a refinement tool to be able to later focus on the binary type model.

3.7 Other Possible Extensions

Now we discuss what results would change and what results would remain qualitatively the same if we were to better tailor the model to known empirical patterns.

Long-term Commitment. Suppose the family could sign long-term private contracts with the managers. I.e., at $t = 0$ the family can commit to a long-term wage profile $\{\tilde{w}_t^c\}$ satisfying (5). This potentially allows the family to compensate the manager cheaper by deferring compensation until a later date. The equilibrium is, essentially, unchanged if the family can commit to long-term contracts.[30] The intuition for this result is that the promise-keeping constraint of the family is never binding since the agent's outside option is gradually increasing.

Superstar managers. A number of papers have documented that having a very successful manager working for the family increases flows into other funds within the family (see Khorana and Servaes (1999) and Nanda, Wang, and Zheng (2004) for details). Such a phenomenon can be captured in the present model by changing the decreasing returns to scale function when the manager works for himself versus if he works for the family. If the assets under management for the mutual fund family and the managers' hedge funds are not governed by the same decreasing returns to scale

[30]By essentially unique we mean that the beliefs of investors $(q_t, k_t)_{t\geq 0}$ and the expected $t = 0$ payoffs to all parties are the same.

function $g(\cdot)$, or the opportunity cost of investors who would be open to investing in hedge funds is higher than that of mutual fund investors, then it may be optimal for the family to retain the best managers forever, even though for intermediate managers the mechanism described in this section applies without change. It would result in partial separation where managers would separate up to an endogenous level $\bar{k}$.

General equilibrium effects. We have assumed that the family is a monopolist in providing the fund to investors. This assumption can be relaxed and we can allow for the family to compete with the fund manager after he leaves and opens his own fund. We can accommodate this by allowing the outside option of the family to depend on the manager's type when he leaves as well as the perception about his ability by outside investors, i.e., M becomes a function of $\tilde{p}_t$ and k_t written as $M(\tilde{p}_t, k_t)$. The same arguments as before hold and, as long as $M(\tilde{p}, k)$ is bounded away from 0 the qualitative equilibrium structure survives.

Brownian versus Poisson returns technology. The negative loss technology may seem quite unrealistic in the application to fund management. It is possible to reformulate this model with Brownian returns as much of the mechanism is the same. It is possible to derive the dynamics for the cutoff type. With a Poisson returns process, separation occurs if either the manager is performing well or he generates a negative signal in which case there is no more residual uncertainty. With a Brownian returns process, negative news are not perfect and it will imply similar separation dynamics as in Daley and Green (2012), where the worst managers leave the family either after very bad returns or after very good returns.

Competing fund families. We assumed the fund family is a monopsonist in the labor market for the manager's labor. Even if there was another family operating in the market, the derived equilibrium survives. Given the positive search cost s, if a manager started his career by being employed by one fund family, the other fund family is not willing to interview managers who were let go by other families since, in equilibrium, it infers that his skill must be equal to k_t, just like the outside investors. As such, along the equilibrium path there is no informational asymmetry between it and outside investors, and even if cost $s > 0$ were to be paid, it would not result in future profits. This implies that the manager is effectively locked out working for other fund families in equilibrium.

Team management. Recent practice has been for funds within the family to be overseen by teams of two to five money managers. Naturally, this introduces a problem for the markets to infer which of the team members is truly skilled. The analysis in this paper can be applied to the team as a whole where the entire team can leave the family and open start a fund on their own. It also, potentially, introduces the possibility of a staggered exit in which the first party that leaves the team is less skilled. As such, the longer the team member stays with the fund, the higher his perceived level of skill is. While this paper focuses on the case when a single manager governs the fund, the same mechanism applies to team production.

3.8 Summary of Results

We have characterized the rents obtained by the fund family when in the presence of adverse selection in the labor market for money managers. We show that the family may serve as a credible money-burning device for fund managers who are willing to sacrifice short-term compensation in favor of reputation. We derive implications for the manager's compensation and profits obtained by the family in equilibrium.

We show that skilled managers are employed longer by the family, as they are more sensitive to changes in their reputation and, thus, are more willing to sacrifice short-term compensation for it. Proposition 3 characterizes the properties of the manager's compensation. It does not discriminate across managers in initial employment period $[0, t^*]$ when the adverse selection is sufficiently severe, that the family can extract rents from all types of managers. Only the lowest type gets his fair market value in this case. For $t > t^*$ the family lets go of the worst managers gradually. Since skilled managers are the most worried about their reputation, they are much more willing to be underpaid in exchange for faster information revelation about their skill. The fund family profits from the inability of money managers to credibly convey their type to the market. The family profits most when the difference between $\tilde{p}_t$ and k_t is high. It implies that managers most profitable for the fund family are both skilled and yet undervalued by investors.

4 Private Effort and Incentives

[This section contains preliminary results.]

We extend the main model by allowing the manager to affect the generated returns by exerting costly private effort. We view this as the manager deviating from his benchmark portfolio in an attempt to find more profitable opportunities. Once the manager is able to exert effort, fund family prefers to introduce performance sensitivity into the compensation structure in order to manipulate the manager's effort to maximize its profits. The time series of this performance sensitivity is the focus of our analysis.

4.1 Extended Setup

We assume that the manager improves the expected return by reducing the likelihood of a loss in a given period. In this sense, effort and ability serve as substitutes in this model. In our Negative Poisson model of returns, we assume that for a given effort level $\tilde{e}_t$ the manager of type θ generates a period t net return of

$$A_t \left(\mu \, dt - g(A_t) \, dt - dN_t^{\theta,\tilde{e}} \right)$$

where $N^{\theta,\tilde{e}}$ is a Poisson process with time-varying and state dependent intensity $\lambda(1-\theta)(1-\tilde{e}_t)$. Higher effort e_t implies a lower likelihood of generating a loss and can, thus, be interpreted as a risk-management decision by the fund manager. In this setting effort is not valuable for type $\theta = 1$ managers, while it lowers the likelihood of a loss for a type $\theta = 0$ managers. The manager's private cost of effort $\tilde{e}$ is given by $i(\tilde{e})$.

In addition to the manager's effort affecting instantaneous expected returns, it also affects the rate of learning about ability from returns. This implies that the manager's beliefs evolve according to

$$d\tilde{p}_t = \lambda(1-\tilde{e}_t)\tilde{p}_t(1-\tilde{p}_t) \, dt - \tilde{p}_t \, dN_t^{\theta,\tilde{e}}.$$

Investors do not observe the on path equilibrium effort choice of the manager $\tilde{e}$. Instead, they conjecture an effort level exerted by the manager in equilibrium and allocate wealth based on the expected value of this effort and the expected skill of the manager. Denoting by $\tilde{e}_t^I$ the expected effort exerted by the manager at time t, investors allocate assets under management A_t up to the point when their marginal return is equal to their opportunity cost

$$\mu - g(A_t) - \lambda \mathrm{E}\left[(1-\tilde{p}_t)(1-\tilde{e}_t)\right] - f = \rho \quad \Rightarrow \quad A_t = g^{-1}\left(\mu - \rho - f - \lambda \mathrm{E}\left[(1-\tilde{p}_t)(1-\tilde{e}_t)\right]\right).$$

Note that different types of managers exert different levels of effort. Investors care about the product of the two and, as a result, the relevant belief of investors along the equilibrium path can then be defined as

$$m_t = 1 - \mathrm{E}\left[(1-\tilde{p}_t)(1-\tilde{e}_t)\right] = \mathrm{E}\left[\tilde{p}_t + \tilde{e}_t - \tilde{p}_t\tilde{e}_t\right].$$

Under this belief process, augmented for agent's private effort, assets under management at time t can be written as $A_t = A(\lambda m_t - f)$, consistent with our prior notation, but where m_t captures the effects of agent's effort.

Once the fund manager has the scope to affect returns, the fund family would like to manipulate the incentives of the manager to its advantage. As such, flat wage contracts seize to be optimal, and we extend the contracting space to allow the family to offer short-term performance-based compensation to the fund manager. Manager's contract in period t is a pair of a wage $\tilde{w}_t$ and pay for performance sensitivity $\tilde{\beta}_t$ on the net return of the manager. The cumulative compensation for the manager is given by

$$d\tilde{W}_t = \tilde{w}_t\, dt + \tilde{\beta}_t \left(\lambda(1 - m_t)\, dt - dN_t^{\theta,\tilde{e}}\right).$$

The above contract violates limited liability of the manager in a given instant if $\tilde{\beta}_t > 0$ and the manager experiences a loss.[31] We assume that the agent's effort is observable by the fund family, but not to outside investors whether or not the manager is working for the fund family. The equilibrium definition is almost identical to the one in Section 2, but now the equilibrium definition also includes effort process $\{\tilde{e}_t\}_{t\geq 0}$ for the manager when he works for the fund family and not. Effort process is a function of the private information of the family-manager pair and is not observed by outside investors.

4.2 Equilibrium Construction

Similar to the previous analysis, equilibrium characterization relies on first describing the outside option of the manager if he holds a private belief $\tilde{p}$ about his skill, why investors perceive him as having skill k. This analysis is qualitatively similar to one carried out in Section 3.1, but more technically involved since the manager exerts private effort along the equilibrium path.[32] We then formally characterize the equilibrium incentives provided by the fund family to the manager. The manager responds to the combination of contractual incentives offered by the fund and career concerns incentives stemming from his hope to open an independent fund in the future.

Denote by $U^\theta(\tilde{p}, k)$ the equilibrium value function of the manager who has private belief about his skill $\tilde{p}$ and who investors perceive as having expected skill k, conditional on θ. Note that this is

[31] We focus on short-term contracts to avoid the necessity for additional state variables.

[32] For this reason we only state the existence of the equilibrium here, but relegate the technical details to the Appendix.

the value to the manager in equilibrium, conditional on the actual realization of θ. The manager's expected value can be written as

$$U(\tilde{p}, k) = \tilde{p} \cdot U^1(\tilde{p}, k) + (1 - \tilde{p}) \cdot U^0(\tilde{p}, k).$$

Similarly, define $V_t^{\theta}(\tilde{p})$ as the expected value to the fund family who perceives its manager as having skill $\tilde{p}$ conditional on θ at time t.

Proposition 6. *Suppose the effort cost $i(\cdot)$ is sufficiently convex. There exists a separating equilibrium characterized conditional value functions $\{U_\theta(\tilde{p}, k)\}$ and $\{V_\theta(\tilde{p}, m, k)\}$ of the manager and the principal respectively given posterior $\tilde{p}$, cutoff type k_t, investor beliefs m_t, and underlying type θ. The fund family offers manager pay for performance sensitivity $\tilde{\beta}_t$ at time t given by*

$$\tilde{\beta}_t = V_t^0 (\tilde{p}_t) - M. \tag{23}$$

Manager's equilibrium effort is determined by the composition of the pay for performance sensitivity of the contract and his career concerns

$$c'(\tilde{a}_t) = \lambda (1 - \tilde{p}_t) \left(\tilde{\beta}_t + U^0(\tilde{p}_t, k_t) - L \right). \tag{24}$$

The fund family does not provide incentives to the manager when he is let go.

We proceed to construct this separating equilibrium. In any equilibrium, the fund family observes the returns of the manager and, thus, his expected value is going to track his outside option outside the fund family given by $U(\tilde{p}_t, k_t)$. Because effort does not affect the type $\theta = 1$ manager, it implies that by exerting effort, the manager can marginally delay the arrival of the negative shock conditional on $\theta = 0$. Denote the effort exerted by the manager absent incentives from the fund family by $\tilde{e}_t$, where the superscript captures the fact that this would be the effort exerted by the manager outside of the fund family. It solves

$$c'(\tilde{e}_t) = \underbrace{\lambda(1 - \tilde{p}_t)}_{\substack{\text{Marginal} \\ \text{Probability} \\ \text{of Loss}}} \cdot \underbrace{(U_0(\tilde{p}_t, k_t) - L)}_{\substack{\text{Lifetime} \\ \text{Utility Lost} \\ \text{Conditional on Loss}}}$$

If the fund family offers the manager additional pay-for-performance sensitivity $\tilde{\beta}_t$, then it only affects the manager if he is a low type. Thus, it serves as a conditional incentive and we obtain (24).

The fund family also internalizes the convex effort cost of the manager since it increases the compensation necessary to keep him from leaving. The fund family's value at stake is the difference between its value of retaining a low skill manager, $\theta = 0$, and letting him go while hiring a new manager. As such, managers effort is always efficient along the equilibrium path given investor beliefs, which results in (23), which is the unique implementation of this welfare maximizing effort level.

Denote by $\underline{e}_t$ the effort exerted along the equilibrium path by the lowest manager still employed by the fund family. The resulting dynamics for the belief process can be written as

$$\dot{k}_t = \lambda(1 - e(k_t)) \cdot k_t(1 - k_t) + \gamma_t,$$

where the first term captures learning from returns and the second term captures learning from turnover. Given process $k = (k_t)_{t \geq 0}$ the compensation necessary to retain the manager adjusts for the incremental costs/benefits of effort. Since the fund family can lower the manager's wage when it requires him to exert more effort, i.e., setting a higher β_t, the manager does not collect information rents.[33] The resulting wage making the manager indifferent between leaving the fund family in that instant, and staying while exerting effort $\tilde{e}_t$ versus $\tilde{e}_t^o$ given by

$$\tilde{w}_t = hA(\lambda k_t - h) + c(\tilde{a}_t) - c(\tilde{e}_t) + \lambda(1 - \tilde{p}_t) \cdot (\tilde{a}_t - \tilde{e}_t) \cdot (L - U^0(\tilde{p}_t, k_t)) - \frac{\partial}{\partial k} U(\tilde{p}_t, k_t) \cdot \gamma_t. \quad (25)$$

At termination time τ it must be the case that $\underline{e}_\tau = \underline{e}_\tau^o$ for the cutoff type. This implies that along the equilibrium path γ_τ is given by

$$\gamma_\tau = \frac{rM + hA(\lambda k_\tau - h) - fA(\lambda m_\tau - f)}{k \cdot \frac{\partial}{\partial k} U(1, k_\tau) + (1 - k_\tau) \cdot \frac{\partial}{\partial k} U(0, k_\tau)},$$

similar to (16) with the exception that m_τ depends on the effort exerted by the manager at termination given by $\underline{e}_\tau$, and that the equilibrium dynamics for processes k and m follow endogenous learning paths.

The choice of incentives $\tilde{\beta}_t$ is driven by the interaction of the rents obtained by the fund family from the informational asymmetry as well as the rents it can extract from its managers. In any equilibrium, the expected value to the fund family from a high expected skill manager is higher than from a low expected skill manager. Even if the manager is $\theta = 0$, a more confident manager is willing to be underpaid, resulting in higher profits for the fund family.

[33] We assume that the fund family observes the effort chosen by the manager implying that this break-even constraint holds off-equilibrium path as well.

Lemma 4. *In equilibrium higher skilled managers are given more performance sensitive contracts* $\tilde{\beta}'_t < \tilde{\beta}''_t$ *if* $\tilde{p}'_t < \tilde{p}''_t$ *after any history in which the manager is still employed. The incentives of the low type manager are declining over time, i.e.,* $\dot{\beta}^L_t < 0$.

When type $\underline{p}$ managers are hired, they generate the highest profit for the fund family. Over time, this profit is decreasing and so are the residual rents relative to re-sampling and hiring a new manager. At $t = t^{b*}$, however, the fund family is indifferent between keeping and letting go of the low type manager, the fund family provides 0 incentives, i.e., $\tilde{\beta}_t = 0$ for $t \in [t^{b*}, \hat{t}^{b*}]$ for the lowest type of the manager. Eventually, the low type manager is let go. The high type managers are profitable at this point, and the fund family continues to provide incentives since it is concerned about losing these rents if the manager generates a loss.

Remark. The challenge in analyzing reputation building under moral hazard lies in the identification of the fund family's belief process m_t about the employed manager which integrates the efforts of the managers across types. When the manager is exerting effort, this distribution is endogenous and needs to be tracked as a state variable. We solve this problem by focusing on the limiting equilibrium when the distribution of types converges to binary. This pins down the smooth outside option of the fund family while, at the same time, allows to characterize investor beliefs along the equilibrium path. While the equilibrium can be characterized through a set of first order differential equations, the existence of the solution of such a system is not evident. When there are only two types of managers, the dimension of this system can be significantly reduced, and we can prove the existence of a solution that satisfies all first order conditions.

5 Conclusion

In this paper, we show that fund families can profitably serve as intermediaries in the labor market for skilled money managers. We show that the fund family profits from skilled managers only if there is significant public uncertainty about their skill, that it underpays better managers, and that it is selective at the initial hiring stage. The fund family profits from adverse selection in this labor market in the beginning. However, as asymmetry of information shrinks as a result of observed returns, the fund family speeds up reputation formation for skilled managers by credibly letting go of the worst manager sfirst. Finally, we show that the family optimally undercharges investors as it increases the rents it can extract from its money managers. This line of research is important for understanding the role of fund families intermediating 40% of financial assets.

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Appendix

Proof of Lemma 1

If the manager leaves with reputation k, his continuation utility is given by

$$U(\theta, k) = \mathrm{E}_\theta \left[\int_0^\eta e^{-rs} hY(\lambda\pi(k;0,s) - h)\, ds + e^{-r\eta} L \right]$$

where $\eta = \inf \left\{ t : N_t^\theta - N_{\tau_n}^\theta > 0 \right\}$. The realized value to the manager of type θ at time t can be expressed as

$$U_t(\theta, k) = \int_0^{\eta \wedge t} e^{-rs} hY(\lambda\pi(k;0,s) - h)\, ds + e^{-rt} U(\theta, \pi(k;0,t)) \cdot \mathbb{1}\{t < \eta\} + e^{-r\eta} L \cdot \mathbb{1}\{\eta \le t\}.$$

Process $U_t(\theta, k)$ is a Levy martingale with respect to the filtration of type θ manager. By Ito's lemma for Poisson processes the differential form for $U_t(\theta, k)$ is given by

$$\begin{aligned} dU_t(\theta, k) =& e^{-rt} hY(\lambda\pi(k;0,t) - h) - rU(\theta, \pi(k;0,t)) \\ &+ e^{-rt} \lambda\pi(k;0,t)(1 - \pi(k;0,t)) \cdot \partial_2 U(\theta, \pi(k;0,t)) \\ &+ e^{-rt}(L - U(\theta, \pi(k;0,t)) \cdot dN_t^\theta. \end{aligned}$$

The martingale condition requires that this drift is 0 on average. Multiplying both sides by e^{rt} then

$$\begin{aligned} rU(\theta, \pi(k;0,t)) = hY(\lambda\pi(k;0,t) - h) + \lambda\pi(k;0,t)(1 - \pi(k;0,t)) \cdot \partial_2 U(\theta, \pi(k;0,t)) \\ + (L - U(\theta, \pi(k;0,t)) \cdot \lambda(1 - \theta) \end{aligned}$$

A change of variables $\pi(k;0,t) = k$ implies

$$rU(\theta, k) = hA(\lambda k - h) + \lambda k(1 - k) \cdot U(\theta, k) + (L - U(\theta, k) \cdot \lambda(1 - \theta). \tag{A.1}$$

The above is a linear differential equation. The general solution at $t = 1$ solves

$$(r + \lambda(1 - \theta)) \cdot U^G(\theta, k) = \lambda k(1 - k) \cdot \partial_2 U^G(\theta, k). \tag{A.2}$$

which is bounded at $k = 1$ since

$$\frac{r + \lambda(1 - \theta)}{\lambda} \cdot \frac{1}{k(1 - k)} = \partial_2 \left(\ln \left(U^G(\theta, k) \right) \right).$$

This implies that $U^G(\theta, k)$ is bounded at $k = 0$ and $k = 1$. This implies that there exists a unique solution such that

$$U(\theta, 1) = \frac{hA(\lambda - h)}{r + \lambda(1 - \theta)} + \frac{\lambda(1 - \theta)L}{r + \lambda(1 - \theta)}.$$

Proof of Proposition 1

Self-contained in the main text.

Proof of Proposition 2.

Equilibrium construction given M. The belief process $(q_t)_{t\geq 0}$ about the average manager working for the family is also shaped by learning from returns and learning about the cutoff type. We show that, given process $(k_t)_{t\geq 0}$ we can construct the corresponding process of beliefs $(q_t)_{t\geq 0}$. The average quality of the manager working for the family at $t = 0$ is simply $q_0 = \mathrm{E}[\tilde{p}_0]$. For a given t and q define

$$K(t, q) = \sup \left\{ k : \frac{\mathrm{E}\left[\tilde{p}_0 \,\middle|\, \tilde{p}_0 > \frac{k}{k + (1-k)e^{\lambda t}}\right]}{e^{-\lambda t} + \mathrm{E}\left[\tilde{p}_0 \,\middle|\, \tilde{p}_0 > \frac{k}{k+(1-k)e^{\lambda t}}\right] \cdot (1 - e^{-\lambda t})} < q \right\}. \tag{A.3}$$

Note that

$$\dot{k}_t = \frac{d}{dt} K(t, q_t) = \lambda K_t(q_t)(1 - K_t(q_t)) + (\dot{q}_t - \lambda q_t(1 - q_t)) \cdot \frac{\partial}{\partial q} K_t(q_t)$$

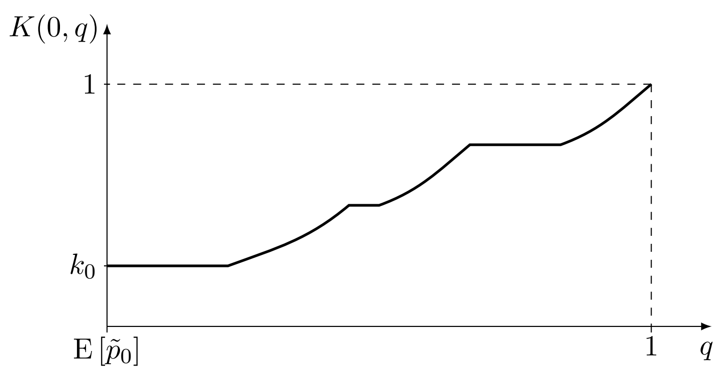

Figure 7: Lowest Type Conditional on the Average Type. Flat regions of $K(0, q)$ arise due to atoms in the distribution of ex-ante types $F(\cdot)$.

Lemma A.1 (Equilibrium ODE). *In the unique monotone separating equilibrium the right deriva-*

tive of thebelief process $(q_t)_{t\geq 0}$ satisfies

$$\dot{q}_{t+} = \begin{cases} \dfrac{\Big[rM + hA(\lambda K(t,q_t) - h) - fA(\lambda q_t - f)\Big]^+}{\frac{\partial}{\partial q}U\big(K(t,q_t),K(t,q_t)\big)\cdot\frac{\partial}{\partial q}K(t,q_t)} + \lambda q_t(1-q_t) & \text{if} \quad \dfrac{\partial}{\partial q}K(t,q) > 0, \\ \lambda K(t,q_t)(1-K(t,q_t))\cdot Q'(K(t,q_t)) & \text{if} \quad \dfrac{\partial}{\partial q}K(t,q) = 0. \end{cases}$$

The initial condition is $q_0 = \max\big[\mathrm{E}\left[\tilde{p}_0\right], Q(k_0)\big]$. At the points of the resulting process q_t we have $q_t = Q(K(t,q_{t-}))$.

Proof. Process k_t captures the dynamics for the lowest type agent. It follows the following process

$$dk_t = \lambda k_t(1-k_t)\,dt + d\Gamma_t.$$

Incremental learning through retention is positive and hence Γ_t is an increasing process. Since we focus on a smooth belief process, there exists $\gamma_t = \dot{\Gamma}_t$. Then

$$dk_t = \lambda k_t(1-k_t)\,dt + \gamma_t\,dt.$$

The minimal necessary compensation required by the manager in any given instance is given by

$$\tilde{w}_t = fA(\lambda q_t - f) - \gamma_t\cdot\frac{\partial}{\partial k}U(\tilde{p}_t, k_t).$$

In a smooth equilibrium agent of type $\tilde{p}_t = k_t$ leaves whenever

$$fA(\lambda q_t - f) - w_t(\tilde{p}_t) \leq rM.$$

This implies two possibilities along the equilibrium path

- Marginal type does not change: $\gamma_t = 0$. This implies that

  $$w_t(\tilde{p}_t) = hA(\lambda k_t - h) = hA(\lambda q_t - f) - M.$$

  In this case $q_t = Q(k_t)$ in order to satisfy the above necessary condition for the fund family to be indifferent between retaining the agent and not.

- Marginal type increases: $\gamma_t > 0$. The necessary condition for separation is given by

  $$\gamma_t = \frac{rM + hA(\lambda k_t - h) - fA(\lambda q_t - f)}{\partial_2 U(k_t, k_t)}.$$

This implies that

$$w_t(\tilde{p}_t) = hA(\lambda k_t - h) - \Big(rM + hA(\lambda k_t - h) - fA(\lambda q_t - f)\Big) \cdot \frac{\partial_2 U(\tilde{p}_t, k_t)}{\partial_2 U(k_t, k_t)}.$$

In both cases it is weakly optimal for the fund family to let go of the lowest type manager k_t. In both cases the flow rent to the fund complex is exactly equal to its opportunity cost. The continuation rent for managers for $k < k_t$ in the future is then weakly below opportunity cost. The evolution of the profit margin of manager of type $\hat{k}_t$ is given by

$$fA(\lambda q_t - f) - hA(\lambda k_t - h) - \gamma_t \cdot \partial_2 U(\hat{k}_t, k_t) - rM.$$

If $\hat{k}_t < k_t$ then the above term is weakly negative, and equal to 0 at exactly $\hat{k}_t = k_t$. We need to show that it cannot become positive in the future.

- If $\gamma_t > 0$, the above is clearly negative for $\hat{k}_t < k_t$.
- If $\gamma_t = 0$, then it implies that the profit on the manager keeps the fund complex exactly indifferent.

Also, once separation begins, Assumption 2 implies that the difference between the fees collected by the fund complex and the fees that would be collected by investors decreases. □

The time t wage to the manager of type $\tilde{p}_t \geq k_t$ is given by

$$w_t(\tilde{p}_t) = \begin{cases} hA(\lambda k_t - h) & \text{if} \quad t \leq t^*, \\ hA(\lambda k_t - h) - \Big(rM + hA(\lambda k_t - h) - fA(\lambda q_t - f)\Big) \dfrac{\partial_2 U(\tilde{p}_t, k_t)}{\partial_2 U(k_t, k_t)} & \text{if} \quad t > t^*. \end{cases}$$

The flow profit of the family $fA(\lambda q_t - f) - w_t(\tilde{p}_t)$ is given by

$$\begin{cases} fA(\lambda q_t - f) - hA(\lambda k_t - h) & \text{if} \quad t \leq \hat{t}, \\ rM + (\tilde{p}_t - k_t)\Big(rM + hA(\lambda k_t - h) - fA(\lambda q_t - f)\Big) \dfrac{\partial_{12} U(\tilde{p}_t, k_t)}{\partial_2 U(k_t, k_t)} & \text{if} \quad t > \hat{t}. \end{cases}$$

If $t \leq t^*$ the profits of the family are higher than the opportunity cost rM. Thus option exercise is suboptimal. For $t > t^*$ the value obtained by the family is higher than rM if $\tilde{p}_s > k_t$ and less than rM if $\tilde{p}_s \leq k_t$. It implies that it is globally incentive compatible for the family to fire the manager of type $\tilde{p}_t$ at time t. This verifies that, for a given M, the equilibrium dynamics are uniquely pinned down.

Fixed point M. The profit to the fund family as a function of M can be written as

$$M + \int_0^{t^*\wedge\tau^M} e^{-rt}\left(fA(\lambda q_t^M - f) - hA(\lambda k_t^M - h) - rM\right)\,dt$$
$$+ \int_{\hat{t}\wedge\tau^M}^{\tau^M} e^{-rt}\left(rM + hA(\lambda k_t^M - h) - fA(\lambda q_t^M - f)\right)(q_t - k_t)\frac{1-\phi(k_t^M)}{k_t^M + (1-k_t^M)\phi(k_t^M)}\,dt$$

where $\phi(k) = \frac{\partial_2 U(0,k)}{\partial_2 U(1,k)}$. Then to determined an equilibrium level of M it must be that

$$M = M + \mathrm{E}\left[\int_0^{t^*\wedge\tau^M} e^{-rt}\left(fA(\lambda q_t^M - f) - hA(\lambda k_t^M - h) - rM\right)\,dt\right]$$
$$+ \mathrm{E}\left[\int_{\hat{t}\wedge\tau^M}^{\tau^M} e^{-rt}\left(rM + hA(\lambda k_t^M - h) - fA(\lambda q_t^M - f)\right)(q_t - k_t)\frac{1-\phi(k_t^M)}{k_t^M + (1-k_t^M)\phi(k_t^M)}\,dt - I\right].$$

It implies that M solves

$$I = \mathrm{E}\left[\int_0^{t^*\wedge\tau^M} e^{-rt}\left(fA(\lambda q_t^M - f) - hA(\lambda k_t^M - h) - rM\right)\,dt\right]$$
$$+ \mathrm{E}\left[\int_{\hat{t}\wedge\tau^M}^{\tau^M} e^{-rt}\left(rM + hA(\lambda k_t^M - h) - fA(\lambda q_t^M - f)\right)(q_t - k_t)\frac{1-\phi(k_t^M)}{k_t^M + (1-k_t^M)\phi(k_t^M)}\,dt\right],$$

where the left hand side is the endogneous resampling cost I and the right hand side is independent of I. The right hand side is continuous in M and we can show that it is bounded for $M \in [0,\infty]$. Thus for every search cost I in a range $[\underline{I}, \overline{I}]$, which corresponds to the range of values of the right hand side there exists an equilibrium. This proves that the equilibrium exists.

Proof of Lemma 2

To investigate equilibrium uniqueness we must analyze

$$I = \hat{v}(M) = \hat{v}_1(M) + \hat{v}_2(M) \tag{A.4}$$

where

$$\hat{v}_1(M) = \mathrm{E}\left[\int_0^{t^*\wedge\tau^M} e^{-rt}\left(fA(\lambda q_t^M - f) - hA(\lambda k_t^M - h) - rM\right)\,dt\right],$$
$$\hat{v}_2(M) = \mathrm{E}\left[\int_{\hat{t}\wedge\tau^M}^{\tau^M} e^{-rt}\left(rM + hA(\lambda k_t^M - h) - fA(\lambda q_t^M - f)\right)(q_t - k_t)\frac{1-\phi(k_t^M)}{k_t^M + (1-k_t^M)\phi(k_t^M)}\,dt\right].$$

We can decompose the profit of the fund family into two parts. $\hat{v}_1(M)$ captures the fund family's profits from the difference in fees between the family and the standalone fund. $\hat{v}_2(M)$ captures the fund family's profits from underpaying better managers. Noting the definition of γ_t we can write

$$\hat{v}_2(M) = \mathrm{E}\left[\int_{t^*\wedge\tau}^{\tau} e^{-rt} \cdot \gamma_t (q_t - k_t)\left(\frac{\partial}{\partial k}U(1,k_t) - \frac{\partial}{\partial k}U(0,k_t)\right) dt\right]$$

Note that $\dot{k}_t = \gamma_t - \lambda k_t(1-k_t)$ which implies that the reputation building profit for the fund family is equal to

$$\begin{aligned}
\hat{v}_2(M) \leq& \mathrm{E}\left[\int_{t^*\wedge\tau}^{\tau} e^{-rt}(q_t - k_t)\left(\frac{\partial}{\partial k}U(1,k_t) - \frac{\partial}{\partial k}U(0,k_t)\right)(\dot{k}_t - \lambda k_t(1-k_t))\, dt\right] \\
\leq& \mathrm{E}\left[\int_{t^*\wedge\tau}^{\tau} e^{-rt}(1 - k_t)\left(\frac{\partial}{\partial k}U(1,k_t) - \frac{\partial}{\partial k}U(0,k_t)\right)\dot{k}_t\, dt\right] \\
\leq& \mathrm{E}\left[\int_{k^*}^{1} e^{-rt}(1 - k)\left(\frac{\partial}{\partial k}U(1,k) - \frac{\partial}{\partial k}U(0,k)\right) dk\right] \\
\leq& \mathrm{E}\left[\int_{k_0}^{1} e^{-rt(k)} \cdot \frac{rU(1,k) - (r+\lambda)U(0,k)}{k}\, dk\right] \\
=& \mathrm{E}\left[\int_{\underline{p}}^{1} \frac{rU(1,k) - (r+\lambda)U(0,k)}{k}\, dk\right]
\end{aligned}$$

We want to show that $\hat{v}(0) > \hat{v}(M)$ for any $M > 0$ for some parameters. This would be sufficient to claim that for a high enough search cost I the solution to (A.4) is unique. If $M = 0$ then the fund family does not sample a new manager and employes him in perpertuity. In order to make this argument we need

$$\mathrm{E}\left[\int_0^{\tau^\infty} e^{-rt}(fA(\lambda q_t - f) - hA(\lambda k_t - h))\right] \geq \tag{A.5}$$

$$\mathrm{E}\left[\int_0^{t^*\wedge\tau^M} e^{-rt}\left(fA(\lambda q_t^M - f) - hA(\lambda k_t^M - h) - rM\right) dt\right] \tag{A.6}$$

$$+\mathrm{E}\left[\int_{\hat{t}\wedge\tau^M}^{\tau^M} e^{-rt}\left(rM + hA(\lambda k_t^M - h) - fA(\lambda q_t^M - f)\right)(q_t - k_t)\frac{1-\phi(k_t^M)}{k_t^M + (1-k_t^M)\phi(k_t^M)}\, dt\right] \tag{A.7}$$

A sufficient condition for this is

$$\mathrm{E}\left[\int_0^{\tau^\infty} e^{-rt}(fA(\lambda q_t - f) - hA(\lambda k_t - h))\right] \geq \tag{A.8}$$

$$\mathrm{E}\left[\int_0^{t^*\wedge\tau^M} e^{-rt}\left(fA(\lambda q_t^M - f) - hA(\lambda k_t^M - h) - rM\right) dt\right] \tag{A.9}$$

$$+\mathrm{E}\left[\int_{\underline{p}}^{1} \frac{rU(1,k) - (r+\lambda)U(0,k)}{k}\, dk\right] \tag{A.10}$$

Now up to t^* both q_t^M and k_t^M follow bayesian updating. Thus they coincide with the case if $q_t =$

Thus a sufficient condition of uniqueness becomes

$$\begin{aligned}
&\mathrm{E}\left[\int_0^{t^*(M)\wedge\tau^\infty} e^{-rt} rM\,dt\right] + \mathrm{E}\left[\int_{t^*(M)\wedge\tau^\infty}^{\tau^\infty} e^{-rt}(fA(\lambda q_t - f) - hA(\lambda k_t - h))\,dt\right] \\
\geq &\mathrm{E}\left[\int_{k^*(M)}^{1} \frac{rU(1,k)-(r+\lambda)U(0,k)}{k}\,dk\right]
\end{aligned}$$

where $t^*(M)$ are functions of M. If $\lambda = 0$ then for $M > 0$ the above is satisfied. Based on the expression for $v(M)$ we have $\hat{v}_1'(0) < 0$ and $\hat{v}_2'(0) = 0$ since $t^*(0) = \infty$. This implies that there exists a lower bound $\underline{M}$ for which we need to verify the above equation. This implies that there also exists $\underline{\lambda}$ such that the equilibrium exists. Moreover if λ is sufficiently small, then a higher search cost implies that M is lower since $\hat{v}'(0) < 0$.

Proof of Proposition 3

Follows from equilibrium construction.

Proof of Proposition 4

The optimal fee chosen by the family solves

$$\max_f \mathrm{E}\left[\max_\tau \mathrm{E}\left[\int_0^\tau e^{-rt} fA(\lambda q_t - f)\,dt + e^{-r\tau}(U(\tilde{p}_\tau, k_\tau) + M)\right]\right].$$

Denote by $\tau(f)$ the equilibrium stopping time when the manager is fired. Also, note that since f is publicly observable, $(k_t) = (k_t(f))$ depends on f. Since $k_t(f)$ is either constant or satisfies

$$k_t(f) = \lambda k_t(1-k_t) + \frac{rM(f) + hA(\lambda k_t(f) - h) - fA(\lambda q_t(f) - h)}{k_t \frac{\partial}{\partial k} U(1,k) + (1-k_t)\frac{\partial}{\partial k}U(0,k)}$$

then $\frac{d}{df}k_t(f) > 0$. By envelope theorem $M'(f) = 0$ and thus the first order condition with respect to f is given by

$$\mathrm{E}\left[\int_0^{\tau(f)} e^{-rt}\frac{d}{df} fA(\lambda q_t(k_t(f)) - f)\,dt + e^{-r\tau(f)}\frac{d}{df}U(\tilde{p}_{\tau(f)}, k_{\tau(f)})\right] = 0,$$

which can be rewritten as

$$\begin{aligned}
&\mathrm{E}\left[\int_0^{\tau(f)} e^{-rt}(A(\lambda q_t(f) - f) - fX'(\lambda q_t(f) - f))\,dt\right] \\
= -&\mathrm{E}\left[\int_0^{\tau(f)} e^{-rt}\lambda fA'(\lambda q_t(f) - f)\cdot\frac{d}{df}q_t(f)\,dt + e^{-r\tau(f)}\cdot\frac{d}{dk}U(p_{\tau(f)}, k_{\tau(f)})\cdot\frac{d}{df}k_t(f)\right] < 0
\end{aligned}$$

where the last inequality is negative since $\frac{d}{df}q_t(f) > 0$ and $\frac{d}{df}k_t(f) > 0$.

Proof of Proposition 5

The equilibrium for each n can be solved for via an initial value problem characterized in Lemma A.1. This ODE is continuous with respect to the underlying distribution since $K(t,q)$ is continuous in distribution. Thus, as the lowest type converges to $\underline{p}$ and as the distribution converges to binary, the limiting equilibrium is characterized by A.1.

Proof of Lemma 3

Suppose λ is sufficiently small and I is sufficiently large. Then, in equilibrium M is sufficiently small. It is easy to see that for $t \in [0, t^*]$ since the profit, in excess of the opportunity cost, is simply given by

$$fA(\lambda q_t - f) - hA(\lambda k_t - h) - rM$$

which is decreasing in k_t for $t < t^*$ by Assumption 2. If M is sufficiently small, then t^* is high and, thus, the effect of the reputation stage is low. Thus, if M is low, the fund family loses if the information asymmetry is lower.

Dynamic Commitment Does not Add Value

Lemma A.2. *If the fund family has commitment power, the equilibrium derived in Proposition* (2) *is, essentially, unchanged.*

Proof. Since both the family and the manager are risk-neutral and share the same discount rate, every contract could be identified with a bonus B_t paid to the manager if he leaves at time t. This menu of contracts satisfies ex-ante individual rationality of the manager if

$$p_0 U(1, k_0) + (1 - p_0)U(0, k_0) \leq \left(p_0 + (1 - p_0)e^{-\lambda t}\right) \cdot e^{-rt}(B(p_0, t) + U(p_t, k_t)).$$

Denote by $B(p_0, t)$ the necessary bonus needed to retain the manager of type p_0 until time t. Then for a given $\tilde{p}_0$ it is optimal to offer bonus B at time t such that

$$\max_t \left[\int_0^t \left(p_0 + (1 - p_0)e^{-rs}\right) e^{-rs} fA(\lambda q_s - f)\, ds - \left(p_0 + (1 - p_0)e^{-rt}\right) e^{-rt} B(p_0, t) \right].$$

Substituting the expression for $B(p_0, t)$ we obtain

$$\max_t \left[\int_0^t \left(p_0 + (1-p_0)e^{-\lambda s}\right) e^{-rs} f A(\lambda q_s - f)\, ds + \left(p_0 + (1-p_0)e^{-\lambda t}\right) e^{-rt} U(p_t, k_t) - U(p_0, k_0) \right].$$

This implies that extending analysis to dynamic contracts does not change the equilibrium dynamics. □

B.1 Agent Moral Hazard Analysis

Manager's Dynamic Outside Option

Suppose the manager leaves the fund family having a private belief $\tilde{p}$, while the outside investors perceive him as having expected skill k. Our goal is to characterize the manager's expected utility $U(\tilde{p}, k)$ in that situation. Investors form their beliefs along the equilibrium path. In a sparating equilibrium $\tilde{p} = k$ implying that they expect that there is no residual uncertainty about the manager once he leaves the fund family. Investors then expect the manager to exert effort $e(k_t)$ along the equilibrium path, resulting in a belief process k_t given by

$$\dot{k}_t = \lambda(1 - e(k_t))k_t(1 - k_t) \tag{B.1}$$

while $N_t^\theta = 0$ and satisfying $k_0 = k$. Define

$$m(k_t) = k_t + e(k_t) - k_t e(k_t). \tag{B.2}$$

Then

$$U(\tilde{p}, k) = \sup_{\hat{e}} \mathrm{E}_{\tilde{p},\hat{e}} \left[\int_0^{\hat{\tau}} e^{-rt} \Big(hA(\lambda m(k_t) - h) - c(\hat{e}_t) \Big)\, dt \right]. \tag{B.3}$$

where process $\{k_t\}_{t \geq 0}$ is defined in (B.1). Suppose there exists effort process $\tilde{e}$ that maximizes the right hand side of (B.3). Define $U_\theta(\tilde{p}, k)$ as the expected value of the manager conditional on the realized value of θ

$$U_\theta(\tilde{p}, k) = \mathrm{E}_{\theta,\tilde{e}} \left[\int_0^{\hat{\tau}} e^{-rt} \Big(hA(\lambda m(k_t) - h) - c(\tilde{e}_t) \Big)\, dt \right]. \tag{B.4}$$

Then by Law of Iterated Expectation

$$U(\tilde{p}, k) = \tilde{p} \cdot U_1(\tilde{p}, k) + (1 - \tilde{p}) \cdot U_0(\tilde{p}, k).$$

In what follows we show that the incentives of the manager to exert effort are captures by value function $U_0(\tilde{p}, k)$. First, we characterize the on-path equilibrium effort $e(k)$, which the manager is exerting along the equilibrium path. We then proceed to characterize the optimal effort $\tilde{e}$ exerted by the manager out-of equilibrium.

Lemma B.1 (On-Path Effort)**.** *Suppose $\tilde{p} = k$. Manager's effort $e(k)$ when he leaves the family along the equilibrium path is unique and solves*

$$c'(e(k)) = \lambda(1-k) \cdot (U_0(k,k) - L),$$

where $U_0(k,k)$ is the solution to the differential equation

$$rU_0(k,k) = hA(\lambda m(k) - h) + c(e(k)) + \lambda(1-e(k))k(1-k)\frac{d}{dk}U_0(k,k) + \lambda(1-e(k))(L - U_0(k,k)), \quad \text{(B.5)}$$

subject to the boundary condition $U_0(1,1) = \frac{hA(\lambda-h)}{r+\lambda}$.

Proof. By Ferreira, López, and Sinusía (2013) the point $k = 1$ is a regular singular point of the first order of (B.5) subject to the specified boundary condition. Thus, there exists a unique solution. □

Given effort process $e(k)$ the manager solves (B.3).

Lemma B.2 (Off-Path Behavior)**.** *There exists a unique solution to* (B.3). *Manager's effort $\tilde{e}$ when he leaves the family solves*

$$c'(\tilde{e}) = \lambda(1-\tilde{p}) \cdot (U_0(\tilde{p}, k) - L),$$

where $U_0(\tilde{p}, k)$ is a solution to the partial differential equation

$$\begin{aligned} rU_0(\tilde{p}, k) =& hA(\lambda m(k) - h) + i(\tilde{e}) + \lambda(1-\tilde{e})\tilde{p}(1-\tilde{p})\frac{\partial}{\partial \tilde{p}}U_0(\tilde{p}, k) \\ &+ \lambda(1-e(k))k(1-k)\frac{\partial}{\partial k}U_0(\tilde{p}, k) + \lambda(1-\tilde{e})(L - U_0(\tilde{p}, k)), \end{aligned} \quad \text{(B.6)}$$

subject to the boundary condition

$$U_0(1,1) = \frac{hA(\lambda-h)}{r+\lambda}.$$

Proof. For existence and uniqueness see Lemma 2 of Bonatti and Hörner (2017). The remainder is an application of Ito's Lemma for jump processes. □

A similar construction to the continuation utilities of the manager also hold for the equilibrium of the fund family.

Lemma B.3. *Suppose effort cost $c(\cdot)$ is sufficiently convex, i.e., $c''(\cdot)$ is large. Then in equilibrium higher skilled managers are more sensitive to changes in their reputation than lower skilled managers, i.e., $\frac{\partial^2}{\partial\tilde{p}\partial k}U(\tilde{p},k) > 0$.*

Proof. By Envelope Theorem in (B.3) the derivative with respect to $\tilde{p}$ is given by

$$\frac{\partial}{\partial\tilde{p}}U(\tilde{p},k) = \mathrm{E}_1\left[\int_0^\infty e^{-rt}hA(\lambda m(k_t)-h)\,dt\right] - \mathrm{E}_{0,\tilde{e}}\left[\int_0^\eta e^{-rt}hA(\lambda m(k_t)-h)\,dt\right]$$

where the first term of the right hand side is independent of manager's effort since it does not affect the high-type manager. If $c''(\cdot)$ is sufficiently large, then $m(k)$ is increasing in k and the first effect of k dominates the second effect of better management of the low type. □

Fund Family Equilibrium

Denote by $\tilde{a}_t$ the effort the fund family wishes the manager to exert. The fund family offers the manager a sequence of short-term contracts denoted by wage $w = \{w_t\}_{t\geq 0}$ and pay-for-performance sensitivity $\beta = \{\beta_t\}_{t\geq 0}$ such that the cumulative compensation received by the manager is given by

$$d\tilde{W}_t = \tilde{w}_t\,dt + \tilde{\beta}_t\left(\lambda(1-\tilde{m}_t)\,dt - dN_t^{\theta,a}\right).$$

The fund family observes manager's effort making it, indirectly, contractable. As a result, the fund family effectively chooses the effort exerted by the manager.

Lemma B.4 (Managerial Incentives)**.** *In equilibrium the fund managers' expected utility given private belief $\tilde{p}_t$ and outside belief k_t is given by $U(\tilde{p},k)$. The fund manager exerts effort*

$$c'(\tilde{a}_t) = \lambda(1-\tilde{p}_t)\cdot\left(\tilde{\beta}_t + U_0(\tilde{p}_t,k_t) - L\right). \tag{B.7}$$

The necessary compensation required to retain the manager given effort $\tilde{a}_t$ is given by

$$\tilde{w}_t = hA(\lambda k_t - h) + c(\tilde{a}_t) - c(\tilde{e}_t) + \lambda(1-\tilde{p}_t)(\tilde{a}_t - \tilde{e}_t)\cdot(L - U_0(\tilde{p}_t,k_t)) - \gamma_t\cdot\frac{\partial}{\partial k}U(\tilde{p}_t,k_t). \tag{B.8}$$

Proof. The fund family observes the manager's effort implying that his expected utility is going to be exactly equal to $U(\tilde{p},k)$. □

Denote by $V_t^\theta(\tilde{p})$ the value to the fund family after t periods of employing the fund manager and having a private belief $\tilde{p}$ about him. If the manager exerts effort $\tilde{a}$ it is given by

$$V_t^\theta(\tilde{p}) = \mathrm{E}_{\theta,\tilde{a}}\left[\int_t^\tau e^{-r(s-t)}(fA(\lambda m_t - f) - w_t(\tilde{p}_t))\,dt + e^{-r(\tau-s)}M\right].$$

The fund family's expected value is given by

$$V_t(\tilde{p}) = \tilde{p}V_t^1(\tilde{p}) + (1-\tilde{p})V_t^0(\tilde{p}) = \mathrm{E}_{\tilde{a}}\left[\int_t^\tau e^{-r(s-t)}(fA(\lambda m_t - f) - w_t(\tilde{p}_t))\,dt + e^{-r(\tau-t)}M\right].$$

Lemma B.5 (Family Value Functions)**.** *The fund family chooses performance sensitivity process* $\tilde{\beta} = (\tilde{\beta}_t)_{t\geq 0}$ *so that the associated effort* $\tilde{a}$ *which solves* (B.7) *to maximize*

$$\max_{\tilde{a}} \mathrm{E}_{\tilde{p},\tilde{a}}\left[\sum_{n=1}^{\infty}\left(\int_{\tau_{n-1}}^{\tau_n} e^{-rt} fA(\lambda m_t - f) + e^{-r\tau_n}(U(\tilde{p}_{\tau_n}, k_{\tau_n}) - I\right)\right], \tag{B.9}$$

which we denote by M*. Given effort* M*, the effort* $\tilde{a}$ *solving* (B.9) *satisfies*

$$c'(\tilde{a}) = \lambda\left(V_t^0(\tilde{p}) - M + U^0(\tilde{p},k) - L\right),$$

which implies a performance-sensitive contract $\tilde{\beta}_t = V_t^0(\tilde{p}_t) - M$*.*

Proof. The fund family internalizes the manager's cost of effort and, thus, maximizes surplus between it and the manager. The result of the Lemma can, thus, be viewed as a corollary to B.4. □

Lemma B.6 (Necessary equilibrium Dynamics)**.** *The rate of separation* γ_t *along the equilibrium path is given by*

$$\gamma_t = \frac{rM + hA(\lambda m(k_t) - h) - fA(\lambda m(q_t) - f)}{\frac{\partial}{\partial k}U(\tilde{p}_t, k_t)\Big|_{\tilde{p}_t = k_t}}.$$

The evolution of the cutoff type for $t \geq t^*$ *follows*

$$\dot{k}_t = \lambda(1 - e(k_t))k_t(1-k_t) + \gamma_t.$$

Proof. Once separation begins, the cutoff type updates quicker since the low type manager is not exerting effort. This implies that k_t converges to q_t quicker implying that Assumption 2 is satisfied even more strongly. Note that this assumption needs to be satisfied conditional on the equilibrium effort of the agent $e(k)$. □

The following existence Lemma is the first time we are relying on the binary structure of the limiting equilibrium.

The conditional types follow

$$\dot{p}_t^L = \lambda(1 - a_t^L)p_t^L(1 - p_t^L),$$
$$\dot{p}_t^H = \lambda(1 - a_t^H)p_t^H(1 - p_t^H).$$

The remaining mass of low and high types that survive is

$$\alpha_t^L = \alpha_0^L \cdot \frac{p_0^L}{p_t^L}, \qquad \alpha_t^H = \alpha_0^H \cdot \frac{p_0^H}{p_t^H},$$

where $\alpha_0^L + \alpha_0^H = 1$. The conditional average is then

$$q_t = \frac{\alpha_t^L \cdot p_t^L + \alpha_t^H \cdot p_t^H}{\alpha_t^L + \alpha_t^H} = \frac{\alpha_0^L p_0^L + \alpha_0^H p_0^H}{\alpha_0^L \cdot \frac{p_0^L}{p_t^L} + \alpha_0^H \cdot \frac{p_0^H}{p_t^H}}.$$

Lemma B.7 (Existence)**.** *There exists a limiting equilibrium if the manager is exerting effort.*

Proof. We proove existence by construction of the solution to the system of differential equations pinning down the equilibrium dynamics. It is useful to redefine $t^* = 0$ while the game begins at some time $-t^*$. Suppose that at the point when separation begins, i.e., $t = 0$, the value function of the fund family conditional on $\theta = 0$, and the manager is a high type, is equal to $V_0^0(p_0^H) = v$. Similarly, suppose that the private belief of the high-type manager is p_0^H. Similarly, the posterior belief about the low-type manager when the separation begins is p_0^L. In what follows, we show that there exists the triple (v, p_0^H, p_0^L) satisfying two initial conditions, the separation condition, and the terminal condition that the fund family receives M in the continuation.

1. Separation start. It must be the case that the fund family prefers to begin to let go of the low type fund managers at $t = 0$ implying

   $$fA(\lambda m(v, p_0^H, p_0^L) - f) = hA(\lambda m(p_0^L) - h) + rM.$$

   This implies that for a given v and p_0^H there exists a p_0^L at which separation begins. Note that the left hand side of the above equation depends on p_0^H and p_0^L since it is used to compute the average belief about the manager at the beginning of separation. Also, there is implicit dependency on $\alpha, \underline{p}, \overline{p}$. This implies that there are only two equilibrium values which we need to identify: v and p_0^H.

2. Initial belief of the low type manager. When separation begins, the value function of the family conditional on the low type is exactly equal to M. It implies that for $[-\infty, 0]$ we can solve back for the belief about the low type. Since conditional on no negative shock arrival the process of beliefs is monotonically increasing, it means that there exists a $-t^*$ at which $p_{-t^*}^L = \underline{p}$. This, effectively, pins down the separation time t^* and does not present an additional boundary condition. Note that it must be the case that $p_0^L \geq \underline{p}$.

3. Initial belief of the high type manager. Given the value of v at $t = 0$ there exists a value $p^H(v)$ such that $p^H_{-t^*} = \overline{p}$. Thus, the initial condition $\overline{p}$, implies a continuous mapping for the value of the belief about the high type at the time of separation given the value to the family v conditional on $\theta = 0$. By considering the highest value $p^H(v)$ satisfying the initial condition of $p^H_{-t^*} = \overline{p}$ we can ensure that the mapping $p^H(\cdot)$ is continuous.

4. Terminal condition of the principal's value function. Given the value v at $t = 0$ and the associated $p^H(v)$ and $p^L(v)$ we can construct the equilibrium forward. Define

$$T(v) = \inf\{t : p^H_t = k_t\}$$

where p^H_t and k_t evolve according to the laws of motion described above. The relevant boundary condition is that the associated $V^0_{T(v)} = M$. We need to show that there exists a solution to this equation. We show this by continuity: if $v = 0$, then $V^0_{T(v)} < 0 < M$. If $v \to \infty$, then $V^0_{T(v)} \to \infty$ since $T(v)$ is going to remain bounded. This implies there exists v^* such that $V^0(T(v^*)) = M$.

Now that we have shown that, given M, an equilibrium exists, we can similarly characterize the fixed point for M as the function of the search cost I. As long as this search cost is not too large, the the equilibrium exists. □