The equilibrium consequences of indexing*

Philip Bond† Diego García‡

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Abstract

We develop a benchmark model to study the equilibrium consequences of indexing in a standard rational expectations setting (Grossman and Stiglitz (1980); Hellwig (1980); Diamond and Verrecchia (1981)). Individuals must incur costs to participate in financial markets, and these costs are lower for individuals who restrict themselves to indexing strategies. Individuals' participation decisions exhibit strategic complementarity. As indexing becomes cheaper (1) indexing increases, while individual stock trading decreases; (2) aggregate price efficiency falls, while relative price efficiency increases; (3) the welfare of relatively uninformed traders increases; (4) for well-informed traders, the share of trading gains stemming from market timing increases, and the share of gains from stock selection decreases; (5) market-wide reversals become more pronounced. We discuss empirical evidence for these predictions.

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†Philip Bond, University of Washington, Email: `apbond@uw.edu`; Webpage: `http://faculty.washington.edu/apbond/`.

‡Diego García, University of Colorado Boulder, Email: `diego.garcia@colorado.edu`; Webpage: `http://leeds-faculty.colorado.edu/garcia/`.

1 Introduction

The standard investment recommendation that academic financial economists offer to individual retail investors it to purchase a low-fee index mutual fund or exchange-traded fund (ETF). An increasing number of such products are available, and are increasingly inexpensive and accessible, and an increasing number of investors follow this advice. In this paper, we develop a benchmark model to analyze the equilibrium consequences of an increase in "indexing," paying particular attention to consequences for welfare.[1]

To preview our results: First, and unsurprisingly, a reduction in the cost of indexing leads to more indexing, and less trading of individual stocks. Second, this reduces price efficiency of the aggregate stock market (i.e., market price movements become more divorced from cash flow news), but increases the efficiency of the relative prices of individual securities. Third, the welfare of relatively uninformed "retail" investors increases, precisely because price efficiency has fallen. Fourth, and in contrast, the welfare of relatively well-informed "institutional" investors may either increase or decrease: they gain more from aggregate market timing trades, but less from individual security trades. Fifth, as indexing increases, the extent to which high market-wide prices today forecast low market returns in the future (reversal) increases. As we review below, these predictions are largely consistent with empirical evidence.

The indexing recommendation is frequently justified by the observation that retail investors are unlikely to be the most informed traders in the market.[2] Because we want to

[1] Our focus on welfare differentiates us from related papers on index futures and exchange traded funds by Subrahmanyam (1991), Cong and Xu (2016), Bhattacharya and O'Hara (2017), all of which employ models with exogenous "noise" trade.

[2] For example, Cochrane (2013) writes: "The average investor theorem is an important benchmark: The average investor must hold the value-weighted market portfolio. Alpha, relative to the market portfolio, is by definition a zero-sum game. For every investor who over-weights a security or invests in a fund that earns positive alpha, some other investor must underweight the same security and earn the same negative alpha. Collectively, we cannot even rebalance. And each of us can protect ourselves from being the negative-alpha mark with a simple strategy: hold the market portfolio, buy or sell only the portfolio in its entirety, and refuse to trade away from its weights, no matter what price is offered. If every uninformed trader followed this strategy, informed traders could never profit at our expense." French (2008) makes a similar argument. At the same time, note also that Pedersen (forthcoming) documents that even indexing requires a considerable amount of trading, consistent with the prediction of our model that even indexing agents care about price

characterize welfare, a necessary step in our analysis is to characterize welfare in an economy of the type introduced by Diamond and Verrecchia (1981). Like Grossman and Stiglitz (1980) and Hellwig (1980), these authors analyze trade between differentially informed agents, but different from these papers, there are no exogenous "noise" or "liquidity" trades. Instead, agents have heterogeneous and privately observed exposures to risk. Consequently, financial markets hold the potential to increase welfare by allowing agents to redistribute risk. Perhaps surprisingly, and although a model of this type has been analyzed by a significant number of authors, results on welfare are scarce, as we review below.

Specifically, we analyze an economy of this type with participation costs and heterogeneous precision levels of private signals, so that agents with relatively imprecise signals correspond to retail investors. As one would expect, only agents with sufficiently precise private signals participate, and a fall in participation costs increases participation by reducing the cutoff level of precision associated with participation. Increased participation by relatively uninformed traders in turn reduces price efficiency. Our main analytical result is to show that increased participation leads to a Pareto improvement in welfare, precisely because of the fall in price efficiency. Consequently, individuals' participation decisions exhibit strategic complementarity: the more other relatively uninformed agents participate, the greater the gains of participation. This result is related to the so-called "Hirshleifer effect" (Hirshleifer (1971)), but does not follow directly from it (see subsection 4.1 below, and also related discussions in Marín and Rahi (1999) and Dow and Rahi (2003)).

We then use these results to analyze what happens when the cost of indexing declines (and so the marginal cost of fully participating in financial markets, as opposed to simply indexing, increases). Doing so generates the results listed earlier in the introduction.

In Section 5 we review empirical support for our main predictions. As described above, the main economic effects in our analysis operate via changes in price efficiency. Consistent with our predictions, there is evidence that price efficiency is lower for assets inside an

efficiency.

index than outside, and morevover, that the difference increases as indexing increases. The relative price efficiency of different assets also generates predictions for where we expect to see relatively informed traders make trading profits; in particular, as indexing increases, and the price efficiency of broad market indexes decreases while the price efficieny of individual stocks increases, we expect to see a greater share of informed trading profits stem from broad "timing" strategies as opposed to stock-selection startagies. Moreover, and in common with the class on models on which we build, our analysis naturally generates reversion in asset prices (high current prices correlated with low future returns), with the degree of reversion determined by price efficiency.

Asides from its implications for the equilibrium effects of indexing, our paper also contributes to the wider debate of the extent to which the financial sector contributes to social welfare (see, e.g, Baumol (1965)). In particular, we work with a canonical model in which a financial market exists because it facilitates risk-sharing, and show that informed trading generally worsens this risk-sharing function, while uninformed trading improves it. While we believe there is considerable value in isolating the effect of informed trading on a specific function of the financial sector, we also fully acknowledge that our analysis is silent on how informed trading affects other possible functions of the financial sector. For example, we do not speak to the question of whether information produced be financial markets is valuable in incentive contracts or in guiding resource allocation decisions (see Bond, Goldstein, and Edmans (2012) for a survey).

Related literature: In addition to the papers noted in footnote 1, our paper is also related to Stambaugh (2014), who considers the implications of a decline in noise trade in individual assets. In our paper, a somewhat analogous decline in trading by relatively uninformed agents occurs. In contrast to Stambaugh's paper, this decline is an endogenous reaction to a decline in the cost of indexing strategies. Moreover, many of our results relate to welfare, which is absent in Stambaugh's analysis.

In an independent, contemporaneous, and complementary paper, Baruch and Zhang

(2017) likewise study the equilibrium consequence of indexing, though from a very different perspective. They consider a multi-asset version of Grossman (1976), so that without indexing prices fully reveal agents' private signals. In this setting they show that an exogenous increase in indexing reduces the amount of information prices contain about individual assets, while the amount of information prices contain about aggregates is unaffected.

One contribution of our paper is to the understanding of welfare in economies of the type introduced by Diamond and Verrecchia (1981). As we noted, such results are surprisingly scarce. One significant algebraic complication in characterizing welfare is that, when combined with the asset price, each agent's private exposure shock contains information about expected asset payoffs. To avoid this complication, Verrecchia (1982) and Diamond (1985) both consider a sequence of economies in which the variance of each individual's exposure shock grows with the number of agents, and directly study the limit of this sequence. In the limit economy, each agent's exposure shock has infinite variance, and so expected utility prior to the realization of the exposure shock is undefined, in turn making it impossible to analyze participation decisions prior to the realization of exposure shocks.[3]

In an independent, contemporaneous, and complementary paper, Kawakami (2017) also makes progress in characterizing welfare in a setting along the lines of Diamond and Verrecchia (1981). Whereas we focus on an economy with a continuum of agents and allow for heterogeneity in the precision of signals about cash flows that agents observe, thereby allowing us to consider the effect an increase in participation by relatively uninformed agents, Kawakami instead considers a finite-agent economy with homogeneous signal precisions, in which an increase in the size of the market is associated with better diversification of individual exposure shocks. Analytically, we make more explicit use than Kawakami of market-clearing conditions, which allows us to incorporate heterogeneity in signal precisions in a tractable way.

Marín and Rahi (1999) obtain welfare results in a relatively specialized setting: there are

[3]If instead one modeled participation decisions as being made after the exposure shock, then almost all agents would participate, since their exposure shocks are so large.

two classes of agents, one class of which sees identical signals about asset payoffs and private endowments, and another class of completely uninformed agents. Moreover, the traded asset is in zero net supply. Dow and Rahi (2003) also analyze welfare, and obtain some tractability by inserting a risk-neutral market maker into the economy, which reduces the applicability of the model for analyzing aggregate financial markets.

In closely related settings, Vives and Medrano (2004) argue that "the expressions for the expected utility of a hedger ... are complicated," whereas Kurlat and Veldkamp (2015) write that "there is no closed-form expression for investor welfare." The complications, common to our model as well, stem from the role of exposure shocks as signals about asset cash flows, on top of the standard risk sharing role that motivates trade.

2 The model

2.1 Preferences, assets, endowments, information

We work with a version of Diamond and Verrecchia (1981) in which there are a continuum of agents (see Ganguli and Yang (2009) and Manzano and Vives (2011)), indexed by the unit interval, $i \in [0, 1]$, and multiple assets. We emphasize that this is a canonical setting, in which risk-sharing benefits lead to gains from trade, which in turn allows for informed trading.

Each agent i has preferences with constant absolute risk aversion (CARA) over terminal W_i, and a coefficient of absolute risk aversion of γ.

There are m risky assets available for trading. Each asset $k \in \{1, \ldots, m\}$ produces a normally distributed payoff $\tilde{X}_k$. The payoffs of different assets are independent, and all assets have the same mean, and same variance τ_X^{-1}. The price of the asset k is $\tilde{P}_k$, which is determined in equilibrium. Agents are small relative to the market, and act as price-takers; we characterize a competitive equilibrium of the economy.

Each asset k is in positive net supply, where agent i's initial endowment of asset k is given by $\tilde{s}_{ik}$. The average per-capita endowment of each asset k is $\tilde{S} = \int_0^1 s_{ik} di$, and is equal across different assets. We denote by $\tilde{\theta}_{ik}$ agent i's trade in asset k.

In addition, agents also have other sources of income (e.g., labor income) that are correlated with the cash flows of the risky assets. For simplicity, we assume the correlation is perfect, and write agent i's income from sources other than the risky asset as

$$\sum_{k=1}^{m} \left(\tilde{Z}_k + \tilde{u}_{ik} \right) \tilde{X}_k. \tag{1}$$

Here, $\tilde{Z}_k + \tilde{u}_{ik}$ represents agent i's non-financial exposure to the cash flow risk $\tilde{X}_k$. Agent i (privately) observes the sum $\tilde{Z}_k + \tilde{u}_{ik}$, but not its individual components $\tilde{Z}_k$ and $\tilde{u}_{ik}$. So agent i knows his own income exposures $\tilde{Z}_k + \tilde{u}_{ik}$, but remains uncertain about the aggregate component of other agents' exposures, $\tilde{Z}_k$. Both $\tilde{Z}_k$ and $\tilde{u}_{ik}$ are randomly distributed normal variables, which are independent across assets k, and in the case of $\tilde{u}_{ik}$ independent across agents i also. The variances of $\tilde{Z}_k$ and $\tilde{u}_{ik}$ are τ_Z^{-1} and τ_u^{-1} respectively, and the mean of Z is 0. We assume throughout that, for all agents i,

$$\tilde{s}_{ik} + E\left[\tilde{u}_{ik}\right] = \tilde{S}. \tag{2}$$

Hence while some agents may have greater endowments of the financial asset k, and other agents may have more non-financial exposure to cash flow risk $\tilde{X}_k$, the net exposure of all agents is the same. This assumption is important in allowing us to tractably analyze expected utilities and participation decisions.

Note that agents' differential and privately observed exposures $\tilde{Z}_k + \tilde{u}_{ik}$ are the source of gains from trade in the financial market, since these differences lead agents to seek to improve risk sharing. It is also worth noting that it is possible to put a more behavioral interpretation on $\tilde{Z}_k + \tilde{u}_{ik}$; looking ahead to agents' optimal trade (13), $\tilde{Z}_k + \tilde{u}_{ik}$ can be interpreting simply as a shock to agent i's desired holding of asset k, independent of the source of this shock.

The terminal wealth of agent i is determined by the combination of trading profits, initial asset endowments $\tilde{s}_{ik}$, and other income (1). For notational convenience, define

$$\tilde{e}_{ik} = \tilde{s}_{ik} + \tilde{Z}_k + \tilde{u}_{ik}$$

to represent agent i's net exposure to cash flow $\tilde{X}_k$, stemming from both his initial holding of the financial asset k, and his non-financial exposure. From (2), it follows that

$$\mathbb{E}\left[\tilde{e}_{ik}|\tilde{Z}_k\right] = \tilde{S} + \tilde{Z}_k.$$

The terminal wealth of an agent who makes the vector of trades $\tilde{\theta}_i$ is

$$W_i\left(\tilde{\theta}_i\right) \equiv \sum_{k=1}^{m}\left(\tilde{\theta}_{ik}\left(\tilde{X}_k - \tilde{P}_k\right) + \tilde{s}_{ik}\tilde{X}_{ik} + \left(\tilde{Z}_k + \tilde{u}_{ik}\right)\tilde{X}_k\right) = \sum_{k=1}^{m}\left(\left(\tilde{\theta}_{ik} + \tilde{e}_{ik}\right)\left(\tilde{X}_k - \tilde{P}_k\right) + \tilde{e}_{ik}\tilde{P}_k\right). \tag{3}$$

Prior to trading, each agent i observes private signals of the form

$$\tilde{y}_{ik} = \tilde{X}_k + \tilde{\epsilon}_{ik},$$

where $\tilde{\epsilon}_{ik}$ is normally distributed with mean 0 and variance τ_i^{-1}, and independent across agents and assets.

Note that the precisions of private signals are heterogeneous across agents, so that some agents are more informed than others. Without loss, we order agents so that signal precision τ_i is decreasing in i; and for simplicity, we assume τ_i is strictly decreasing.

An agent's information set at the time of trading is hence the triple of n-vectors $\left(\tilde{y}_i, \tilde{e}_i, \tilde{P}\right)$, which consists of his signals about cash flows $\tilde{y}_i$, his own exposure $\tilde{e}_i$, and the price $\tilde{P}$.

2.2 Indexing and participation

Agents incur a cost $\kappa > 0$ of fully participating in financial markets, reflecting a combination of information collection and processing costs, psychic costs, expected trading costs, and the cost of potentially trading in a less than optimal way. Agents make participation decisions prior to observing any of $\left(\tilde{y}_i, \tilde{e}_i, \tilde{P}\right)$.

In addition to fully participating in financial markets, agents have the option of participating only via trading an "index" asset. The index covers the first $l \leq m$ of the m assets. It is mathematically convenient to focus on the case in which l is a power of 2, for reasons we explain in Section 3 below.

Since all assets have the same supply S, equal-weighted and value-weighted indices coincide. The cash flow produced by the index is hence

$$X_1 \equiv l^{-\frac{1}{2}} \sum_{k=1}^{l} X_k, \tag{4}$$

where $l^{\frac{1}{2}}$ is an index divisor, chosen so that $\text{var}\left(X_1\right) = \tau_X^{-1}$.

Formally, we denote by Θ_l the set of trade vectors that are feasible for an indexing agent, i.e., trades in which agent i buys or sells equal amounts of all assets in the index, and zero units of any asset outside the index:

$$\Theta_l = \left\{\theta_i \in \mathbb{R}^m : \tilde{\theta}_{ij} = \tilde{\theta}_{ik} \text{ for any } j, k \in \{1, \ldots, l\} \text{ and } \tilde{\theta}_{ik} = 0 \text{ for } k > l\right\}.$$

The advantage of participating in financial markets only via indexing is that the participation cost is lower, and is $\kappa_l \in (0, \kappa)$. The lower participation cost of indexing reflects lower trading costs (because of the availability of low cost index mutual funds and exchange traded funds (ETFs)); lower cognitive demands and attention costs; and lower information costs, since as our formal analysis will show, a sufficient statistic for an agent's private information if he is indexing is the sum of signals related to the assets in the index, $\sum_{i=1}^{l} y_{il}$, which can be interpreted as agent i simply paying attention to broad economic aggregates, instead of individual stocks.

Looking ahead, the main comparative static we will be interested in is a fall in κ_l, the participation cost associated with indexing. This corresponds to falling fees, greater availability, and greater awareness of products such as low-cost index funds and ETFs. It may also reflect an increase in public awareness of the standard advice given by finance academics.

Finally, if an individual does not participate in financial market at all, he pays no participation cost, but does not trade, i.e., $\tilde{\theta}_{ik} = 0$ for all assets k.

2.3 Equilibrium

The equilibrium definition is a straightforward extension of that used in competitive rational expectations models (Grossman and Stiglitz (1980), Hellwig (1980)), with the participation decision incorporated:

Definition 1 *A rational expectations equilibrium consists of non-overlapping sets of agents who choose to fully participate, N, and who choose to only index, N_l; trading strategies $\left\{\tilde{\theta}_i\right\}_{i\in N}$ consistent with these participation decisions; and a price function $\tilde{P}(\tilde{X},\tilde{Z})$ such that markets clear,*

$$\int_0^1 \tilde{\theta}_i di = 0; \tag{5}$$

and (taking prices as given), agent i's trading strategy is optimal, i.e., for fully participating agents $i \in N$,

$$\tilde{\theta}_i \in \arg\max_{\hat{\theta}_i} \mathbb{E}\left[u\left(W_i\left(\hat{\theta}_i\right)\right) | \tilde{y}_i, \tilde{e}_i, \tilde{P}\right], \tag{6}$$

while for indexing agents $i \in N_l$,

$$\tilde{\theta}_i \in \arg\max_{\hat{\theta}_i \in \Theta_l} \mathbb{E}\left[u\left(W_i\left(\hat{\theta}_i\right)\right) | \tilde{y}_i, \tilde{e}_i, \tilde{P}\right]; \tag{7}$$

and participation decisions are optimal, for $i \in N$,

$$\mathbb{E}\left[u\left(W_i\left(\tilde{\theta}_i\right)-\kappa\right)\right] \geq \max\left\{\mathbb{E}\left[\max_{\hat{\theta}_i\in\Theta_l}\mathbb{E}\left[u\left(W_i\left(\hat{\theta}_i\right)-\kappa_l\right)|\tilde{y}_i,\tilde{e}_i,\tilde{P}\right]\right],\mathbb{E}\left[u\left(\sum_{k=1}^m \tilde{e}_{ik}\tilde{X}_k\right)\right]\right\},$$

for $i \in N_l$,

$$\mathbb{E}\left[u\left(W_i\left(\tilde{\theta}_i\right)-\kappa_l\right)\right] \geq \max\left\{\mathbb{E}\left[\max_{\hat{\theta}_i}\mathbb{E}\left[u\left(W_i\left(\hat{\theta}_i\right)-\kappa\right)|\tilde{y}_i,\tilde{e}_i,\tilde{P}\right]\right],\mathbb{E}\left[u\left(\sum_{k=1}^m \tilde{e}_{ik}\tilde{X}_k\right)\right]\right\},$$

and for $i \notin N \cup N_l$,

$$\mathbb{E}\left[u\left(\sum_{k=1}^m \tilde{e}_{ik}\tilde{X}_k\right)\right] \geq \left\{\mathbb{E}\left[\max_{\hat{\theta}_i}\mathbb{E}\left[u\left(W_i\left(\hat{\theta}_i\right)-\kappa\right)|\tilde{y}_i,\tilde{e}_i,\tilde{P}\right]\right],\mathbb{E}\left[\max_{\hat{\theta}_i\in\Theta_l}\mathbb{E}\left[u\left(W_i\left(\hat{\theta}_i\right)-\kappa_l\right)|\tilde{y}_i,\tilde{e}_i,\tilde{P}\right]\right]\right\}.$$

2.4 Parameter restrictions

Throughout, we assume

$$4\gamma^2(\tau_Z^{-1}+\tau_u^{-1}) < \tau_X \tag{8}$$

$$\gamma^2 > 4\tau_0\tau_u, \tag{9}$$

where τ_0 is the precision of agent 0's information, i.e., the highest precision in the population of agents. Condition (8) ensures that expected utility is well-defined for an agent who behaves autarchicly, and does not trade. Without this condition, an autarchic agent is exposed to so much risk that his expected utility is infinitely low. Condition (9) ensures that an equilibrium exists at the trading stage (see Proposition 1 below). Loosely speaking, without this condition there is too much trading based on information relative to trading based on risk-sharing; Ganguli and Yang (2009) impose essentially the same condition.[4]

[4] The main extension in Manzano and Vives (2011) relative to Ganguli and Yang (2009) is that they allow for the error terms in the trader's signals to be correlated. Non-zero correlation eliminates the existence issues in our model. Since our focus is on welfare, we choose to study the slightly more tractable model with conditionally independent estimation errors.

3 Disentangling markets

Although the fundamentals of different assets are independent in all dimensions, the potential presence of indexing agents introduces a link between the prices of distinct assets. For example, if j and k are two distinct assets covered by the index, then an indexing agent's exposure $\tilde{e}_{ij}$ to cash flow risk $\tilde{X}_j$ affects the agent's desired trade of asset k as well as for asset j.

Because of the entanglement that indexing produces between different assets covered by the index, it is analytically very convenient to effectively change basis and study the economy in terms of set of synthetic assets that are mutually independent even in the presence of indexers.[5] We first give a simple example, and then generalize:

Example: Suppose there and $m = 5$ assets and the index covers the first $l = 4$. Then consider the following set of 5 synthetic assets, where the 1st synthetic asset pays X_1 as defined in (4); the 5th synthetic asset coincides with the underlying asset 5, i.e., it pays $X_5 = \tilde{X}_5$; and the remaining 3 synthetic assets pay X_2, X_3, X_4 defined by

$$\begin{aligned} X_2 &= \frac{1}{2}\left(\tilde{X}_1 + \tilde{X}_3 - \tilde{X}_2 - \tilde{X}_4\right), \\ X_3 &= \frac{1}{2}\left(\tilde{X}_1 + \tilde{X}_2 - \tilde{X}_3 - \tilde{X}_4\right), \\ X_4 &= \frac{1}{2}\left(\tilde{X}_1 + \tilde{X}_4 - \tilde{X}_2 - \tilde{X}_3\right). \end{aligned}$$

The 5 synthetic assets span the underlying assets $\tilde{X}_k$. Moreover, they are uncorrelated (i.e., $\operatorname{cov}(X_j, X_k) = 0$ for all j, k), and each has variance τ_X^{-1}. The first synthetic asset is simply the index, while synthetic assets $2, 3, 4$ constitute three different long-short trades of assets within the index.

We construct synthetic assets in a way that generalizes the properties in the example. In

[5] This change of basis is not essential to solve for equilibrium prices ; see Admati (1985)'s analysis of a multi-asset version of Hellwig (1980). But the change of basis allows us to characterize everything in terms of the price efficiency of individual assets, which in turn facilitates the analysis of participation decisions.

particular, the first synthetic asset corresponds to the index, while synthetic assets $2, \ldots, l$ correspond to long-short trades of assets contained in the index. Mathematically, this generalization follows from the following straightforward result in matrix algebra:

Lemma 1 *For any positive integers m and $l \leq m$ such that l is a power of 2, there exists an $m \times m$ matrix A with the following properties: A is symmetric and invertible, with $A^{-1} = A$ (i.e., A is involutory); $A_{jk} = 0$ if $j \neq k$ and either $j > l$ or $k > l$; $A_{jk} = 1$ if $j = k > l$; $A_{1k} = l^{-\frac{1}{2}}$ and $|A_{jk}| = l^{-\frac{1}{2}}$ for all $j, k \leq l$; for any j, $j' \neq j$, $\sum_{k=1}^{m} A_{jk} A_{j'k} = 0$; and $\sum_{k=1}^{l} A_{jk} = 0$ for $j = 2, \ldots, l$.*

Given the existence of a matrix A of the type established in Lemma 1, we define synthetic assets $1, \ldots, m$ as paying off

$$X_k \equiv \sum_{j=1}^{m} A_{kj} \tilde{X}_j, \tag{10}$$

or in matrix form, the vector of cash flows produced by synthetic assets is $X = A\tilde{X}$. As in the example, the synthetic assets are uncorrelated, and each has variance τ_X^{-1}. The mean of synthetic asset X_1 is $\sqrt{l}\mathbb{E}\left[\tilde{X}_1\right]$, the mean of assets $2, \ldots, l$ is 0, and the mean of assets $k > l$ is simply $\mathbb{E}\left[\tilde{X}_k\right]$.

For all exogenous variables, we use tildes to denote the underlying fundamental quantity, and the absence of a tilde to denote a variable constructed analogously to (10). For example, we define $Z_k \equiv \sum_{j=1}^{m} A_{kj} \tilde{Z}_j$. Note that $\mathbb{E}\left[e_{i1} | Z_1\right] = \sqrt{l}\tilde{S} + Z_1$; $\mathbb{E}\left[e_{ik} | Z_k\right] = Z_k$ for $k = 2, \ldots, l$; and $\mathbb{E}\left[e_{ik} | Z_k\right] = \tilde{S} + Z_k$ for $k > l$. Accordingly, define $S_1 = \sqrt{l}\tilde{S}$, $S_k = 0$ for $k = 2, \ldots, l$; and $S_k = \tilde{S}$ for $k > l$.

Let θ_i denote the m-vector of agent i's trades of the synthetic assets. This delivers income $X^T \theta_i$ to agent i. Note that $X^T \theta_i = \left(A\tilde{X}\right)^T \theta_i = \tilde{X}^T A^T \theta_i = \tilde{X}^T A \theta_i$. Hence the trade θ_i of synthetic assets corresponds to the trade $A\theta_i$ of the underlying assets. Hence $\theta_i = A\tilde{\theta}_i$, or equivalently, $\tilde{\theta}_i = A^{-1}\theta_i = A\theta_i$.

Similarly, let P denote the m-vector of prices of the synthetic assets. So a trade θ_i of the synthetic assets costs $P^T \theta_i$, which equals $P^T A \tilde{\theta}_i = \left(A^T P\right)^T \tilde{\theta}_i = (AP)^T \tilde{\theta}_i$. So the price vectors P and $\tilde{P}$ are related by $P = A\tilde{P}$.

The terminal wealth of agent i is (in matrix form) is $\left(\tilde{X}^T - \tilde{P}^T\right)\left(\tilde{\theta}_i + \tilde{e}_i\right) + \tilde{P}^T \tilde{e}_i$. Using the properties of the matrix A, by straightforward manipulation this expression equals $\left(X^T - P^T\right)(\theta_i + e_i) + P^T e_i$.

Consequently, to solve for equilibrium prices and welfare, we work directly with the synthetic assets described above. By construction, the synthetic assets are independent of each other in all respects. Moreover, and importantly, indexing corresponds simply to the constraint that an agent can trade only synthetic asset 1, with no trade of any of the other synthetic assets. This means we can analyze the equilibrium in the market for each synthetic asset in isolation.

4 Informed trading and welfare in each asset market

Given Section 3, we analyze the equilibrium for the market in each of the synthetic assets $X_1, \ldots, X_m$ separately, along with the expected utility associated with the corresponding market. For notational ease, throughout this section we omit the asset subscript k.

4.1 Welfare benchmarks

It is useful to consider a couple of welfare benchmarks. First, in the (symmetric)[6] unconstrained solution to the social planner's problem each agent i has terminal wealth

$$W_i = (S + Z)\, X. \tag{11}$$

That is, the aggregate endowment $(S + Z)\, X$ is simply split equally among agents. This is the outcome that would be obtained if agents could pool risk before knowing their exposures e_i, and if contracts could be written contingent on the realizations of e_i.

[6] In non-symmetric solutions, each agent has terminal wealth $W_i = (S + Z)\, X + K_i$, where K_i is a constant, and $\int K_i di = 0$.

A second useful benchmark is the case in which all signals about X are public, with all other aspects the same as in the model described above (in particular, exposures e_i are private information, and trade occurs only after agents observe these exposures). In this case, all agents have the same posterior of X at the trading stage, and so trades θ_i must satisfy (see (13) below)

$$\theta_i + e_i = S + Z.$$

So each agent's terminal wealth is

$$W_i = e_i P + (S + Z)(X - P) = (u_i + s_i - S) P + (S + Z) X. \tag{12}$$

Note that in this second benchmark, each agent is exposed to an additional risk term, $(u_i + s_i - S) P$.

The comparison of these two benchmarks illustrates the challenge of characterizing how information about X affects welfare. For example, comparing (11) and (12), one can see that ceteris paribus agents prefer the price P to have low variance. In turn, P has low variance if it relatively unaffected by both the realization of the cash flow X and the aggregate exposure Z. But higher-precision signals about X may increase P's dependence on X (increasing the variance of P, and the Hirshleifer effect) but decrease P's dependence on Z (reducing the variance of P), so that the overall effect is unclear.[7]

[7] It is worth noting that the limiting case of perfect information about X is straightforward. In this case, the price P simply equals X, and so (12) reduces to $W_i = e_i X$, which is the autarchy outcome. Hence welfare is minimized by perfect information about X, since in this case the financial market cannot provide any risk sharing (the Hirshleifer effect). Our analysis below concerns the more relevant non-limit case. Moreover, note that Diamond (1985) characterizes how welfare changes as the precision of public information changes, though with the mathematical compromises discussed earlier. Finally, in an online appendix we show that welfare in this benchmark case indeed monotonically declines in the precision of public information, though the proof is non-trivial, consistent with the discussion above.

4.2 Basic equilibrium properties

As standard in the literature, to characterize an equilibrium we first conjecture key equilibrium characteristics, and then verify that an equilibrium with these characteristics indeed exists. More concretely, we characterize linear equilibria (i.e., the price is a linear function of the cash flow X and the aggregate endowment Z) in which all agents with sufficiently precise signals participate. In such equilibria, there is a cutoff agent n such that all agents $i \leq n$ participate, and agents $i > n$ do not participate. In this subsection we establish some key equilibrium properties that hold in any equilibrium of this type. To maximize transparency, we establish these properties as directly as possible, making use primarily of the market clearing condition (5).

In a linear equilibrium, each agent i's optimal trade has the standard mean-variance form,

$$\theta_i + e_i = \frac{1}{\gamma} \frac{\mathbb{E}\left[X - P | y_i, e_i, P\right]}{\operatorname{var}\left(X - P | y_i, e_i, P\right)} = \frac{1}{\gamma} \frac{\mathbb{E}\left[X | y_i, e_i, P\right] - P}{\operatorname{var}\left(X | y_i, e_i, P\right)}. \tag{13}$$

The form of the optimal trade (13) indicates that the reciprocal of conditional variance, $\operatorname{var}\left(X | y_i, e_i, P\right)^{-1}$, is an important quantity. The following result uses market clearing (5) to derive a useful relation between the average reciprocal of conditional variance in the economy, and the (endogenous) covariance between returns $X - P$ and cash flows X.

Lemma 2 *In any linear equilibrium,*

$$\frac{1}{n} \int_0^n \frac{1}{\operatorname{var}\left(X | y_i, e_i, P\right)} di = \frac{1}{\operatorname{cov}\left(X - P, X\right)}. \tag{14}$$

Note that Lemma 2 nests the special case in which no agent has any information about the cash flow X, so that the price is unrelated to X, and so for any agent i, $\operatorname{var}\left(X | y_i, e_i, P\right) = \operatorname{var}\left(X\right) = \operatorname{cov}\left(X - P, X\right)$.

Lemma 2 turns out to be very important in our analysis, and since its proof is short, we give it here. Differentiation of market clearing (5) with respect to X gives

$$\frac{\partial}{\partial X}\int_0^n \theta_i di = 0.$$

Substituting in the portfolio θ_i from (13); recalling the property of multivariate normality that conditional variances do not depend on the realizations of random variables; and noting that $\frac{\partial P}{\partial X} = \frac{\text{cov}(P,X)}{\text{var}(X)}$, it follows that

$$\int_0^n \frac{\frac{\partial}{\partial X}\mathbb{E}\left[X|y_i,e_i,P\right]}{\text{var}\left(X|y_i,e_i,P\right)}di = \int_0^n \frac{\frac{\text{cov}(P,X)}{\text{var}(X)}}{\text{var}\left(X|y_i,e_i,P\right)}di. \tag{15}$$

Because the information set (y_i, e_i, P) consists of a set of normally distributed random variables,

$$\frac{\partial}{\partial X}\mathbb{E}\left[X|y_i,e_i,P\right] = 1 - \frac{\text{var}\left(X|y_i,e_i,P\right)}{\text{var}\left(X\right)}. \tag{16}$$

(Note that the right hand side (RHS) of (16) is simply the R^2 of regressing cash flows on (y_i, e_i, P). See Lemma A-1 for a formal proof of (16).) Substitution of (16) into (15) yields

$$\left(1 - \frac{\text{cov}\left(P,X\right)}{\text{var}\left(X\right)}\right)\int_0^n \frac{1}{\text{var}\left(X|y_i,e_i,P\right)}di = \int_0^n \frac{1}{\text{var}\left(X\right)}di,$$

which is equivalent to (14), completing the proof of Lemma 2.

Among other things, we use Lemma 2 to characterize the equilibrium risk premium $\mathbb{E}\left[X-P\right]$. Taking the unconditional expectation of (13) gives

$$E\left[\theta_i\right] + S = \frac{1}{\gamma}\frac{\mathbb{E}\left[X-P\right]}{\text{var}\left(X|y_i,e_i,P\right)}.$$

Combined with market clearing (5) (specifically, $\frac{1}{n}\int_0^n E\left[\theta_i\right]di = 0$), we obtain:

Corollary 1 *In a linear equilibrium,*

$$\mathbb{E}\left[X-P\right] = \gamma S \text{cov}\left(X-P,X\right).$$

As for Lemma 2, it may help to note that Corollary 1 nests the special case in which no agent has any information, and so $\mathbb{E}[X - P] = \gamma S \text{var}(X)$.

Both prices and endowment shocks play two distinct roles in determining an agent's demand: they directly affect demand, and separately, they also affect an agent's beliefs about the cash flow X, thereby indirectly affecting demand. To clarify this dual role, write $\theta_i\left(y_i, e_i, \hat{e}_i, P, \hat{P}\right)$ for the demand of an agent who has exposure e_i and can trade at price P, but who evaluates his conditional distribution over X using the the information set $\left(y_i, \hat{e}_i, \hat{P}\right)$. Even though $\hat{e}_i = e_i$ and $\hat{P} = P$, keeping separate track of the two roles of prices and endowment shocks is conceptually useful.

To further exploit the equilibrium market clearing condition (5), note that differentiation with respect to X and Z respectively yields:

$$\frac{\partial P}{\partial X}\int_N \frac{\partial \theta_i}{\partial P} di + \frac{\partial P}{\partial X}\int_N \frac{\partial \theta_i}{\partial \hat{P}} di + \int_N \frac{\partial \theta_i}{\partial Y_i} di = 0 \qquad (17)$$

$$\frac{\partial P}{\partial Z}\int_N \frac{\partial \theta_i}{\partial P} di + \frac{\partial P}{\partial Z}\int_N \frac{\partial \theta_i}{\partial \hat{P}} di + \int_N \frac{\partial \theta_i}{\partial e_i} di + \int_N \frac{\partial \theta_i}{\partial \hat{e}_i} di = 0. \qquad (18)$$

One immediate consequence is that, in equilibrium, $\frac{\partial P}{\partial Z} \neq 0$. To see this, note that if instead $\frac{\partial P}{\partial Z} = 0$, then Z and e_i provide no information about the cash flow X, so that $\frac{\partial \theta_i}{\partial \hat{e}_i} = 0$ for all agents. In contrast, the non-informational effect of endowment shocks on the portfolio decision is certainly negative (formally, see Lemma 3). But then the lefthand side of (18) is strictly negative, a contradiction.

As typical for this class of models, an important equilibrium quantity is the relative sensitivity of price to X and Z, which we denote by ρ:

$$\rho \equiv -\frac{\frac{\partial P}{\partial X}}{\frac{\partial P}{\partial Z}}.$$

We refer to ρ as the price efficiency of the risky asset, since

$$\text{var}(X|P)^{-1} = \tau_X + \rho^2 \tau_Z \quad (19)$$

$$\text{var}(X|y_i, e_i, P)^{-1} = \tau_X + \rho^2(\tau_Z + \tau_u) + \tau_i. \quad (20)$$

These expressions (derived in the proof of Lemma 3) measure the ability of an outside observer and agent i, respectively, to forecast the cash flow X.

The following properties of individual demand follow only from Bayesian updating. They hold whenever price P is a linear function of X and Z, regardless of whether P is an equilibrium price.

Lemma 3 *If P is a linear function of X and Z then the effects of non-informational factors on demand θ_i are given by*

$$\frac{\partial \theta_i}{\partial e_i} = -1; \quad \frac{\partial \theta_i}{\partial P} = -\frac{1}{\gamma} \frac{1}{\text{var}\left(X|y_i, e_i, P\right)};$$

while the effects of informational factors on demand θ_i satisfy

$$\frac{\partial \theta_i}{\partial y_i} = \frac{\tau_i}{\gamma}; \quad \frac{\partial \theta_i}{\partial \hat{e}_i} = \frac{\rho}{\gamma}\tau_u; \quad \frac{\partial P}{\partial Z}\frac{\partial \theta_i}{\partial \hat{P}} = -\frac{\rho}{\gamma}\left(\tau_Z + \tau_u\right); \quad \frac{\partial P}{\partial X}\frac{\partial \theta_i}{\partial \hat{P}} = \frac{\rho^2}{\gamma}\left(\tau_Z + \tau_u\right).$$

Note that the informational effects of $\tilde{e}_i$ and Z (as reflected in the price $\tilde{P}$) on demand are related by the precisions of the idiosyncratic and aggregate components of exposure shocks:

Corollary 2 *In a linear equilibrium, for any agent i,*

$$\frac{\frac{\partial P}{\partial Z}\frac{\partial \theta_i}{\partial \hat{P}}}{\frac{\partial \theta_i}{\partial \hat{e}_i}} = -\frac{\tau_Z + \tau_u}{\tau_u}, \quad (21)$$

$$\frac{\partial P}{\partial Z}\frac{\partial \theta_i}{\partial \hat{P}} + \frac{\partial \theta_i}{\partial \hat{e}_i} = -\frac{\rho}{\gamma}\tau_Z. \quad (22)$$

In equilibrium $\frac{\partial P}{\partial Z} < 0$ and $\rho \geq 0$ (see Lemma 4 immediately below). So a higher Z is associated with lower prices, which in turn are associated with lower estimates of X. Holding u_i fixed, a higher Z also leads to a higher value of e_i, thereby raising an agent's estimate of Z, and hence (given equilibrium prices) an agent's estimate of X. Equation (22) shows that the first of these effects dominates.

We next establish the basic result that aggregate demand for the risky asset is decreasing in the price. Because of the informational content of prices, this is not completely obvious. At the same time, we show that the price is increasing in the asset's payoff, and the price is decreasing in the aggregate endowment shock. We highlight that the proof of this result makes use only of the market clearing condition (5), along with the *signs* (but not magnitudes) established in Lemma 3.

Lemma 4 *In a linear equilibrium, the aggregate demand curve slopes down, i.e.,*

$$\int_0^n \frac{\partial \theta_i}{\partial P} di + \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di < 0, \tag{23}$$

and the price is an increasing function of X, and a strictly decreasing function of Z,

$$\frac{\partial P}{\partial X} \geq 0 \text{ and } \frac{\partial P}{\partial Z} < 0,$$

and so in particular $\rho \geq 0$.

A further immediate implication of Lemmas 2 and 3 is the following expression for the non-informational effect of prices on aggregate demand:

Corollary 3 *In a linear equilibrium, the non-informational effect of prices on aggregate demand, i.e., $\int_0^n \frac{\partial \theta_i}{\partial P} di$, satisfies*

$$\frac{1}{n} \int_0^n \frac{\partial \theta_i}{\partial P} di = -\frac{1}{\gamma} \frac{1}{\operatorname{cov}(X - P, X)}. \tag{24}$$

Knowledge of individual exposure e_i contains information about X only because it helps agent i interpret the price (for example, it provides information about whether a high price is due to a high cash flow X or a low aggregate exposure Z). Because the information in exposure is subsidiary to the information in prices, it is intuitive that prices contain more information than exposures, as formalized in the following result:

Lemma 5 *In a linear equilibrium, the ratio of the informational to non-informational effect of prices on demand exceeds the ratio of the informational to non-informational effect of exposures on demand,*

$$\frac{\left|\int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di\right|}{\left|\int_0^n \frac{\partial \theta_i}{\partial P} di\right|} > \frac{\left|\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di\right|}{\left|\int_0^n \frac{\partial \theta_i}{\partial e_i} di\right|}. \tag{25}$$

Lemma 5 turns out to be critical to the analysis of welfare below. We again highlight that its proof makes use only of basic equilibrium properties established above, along with the *signs* (but not magnitudes) established in Lemma 3 and Corollary 2.

4.3 Equilibrium at the trading stage

Ganguli and Yang (2009) and Manzano and Vives (2011) give conditions for equilibrium existence, taking the set of trading agents as given. We follow Manzano and Vives (2011) and focus on the equilibrium with lower price efficiency, since the one with higher price efficiency is unstable.[8] The following result contains a minor extension of these papers to cover heterogeneity in information precision, and then uses results from the previous section to give an explicit expression for the risk premium $\mathbb{E}[X - P]$.

[8]Manzano and Vives (2011) give a mathematical definition of stability. One way to think about stability is in terms of condition (A-7) in the proof of Proposition 1. The RHS describes agents' demands, which in turn depend on price efficiency (this can be seen explicitly from (A-8)). The left hand side (LHS) of (A-7) describes how prices must behave to clear the market, given agents' demands on the RHS. Equilibrium price efficiency is a fixed point of this relation. Moreover, the RHS is increasing in ρ, at least in the neighborhood of any solution. If the RHS crosses the 45^o line from below, the corresponding equilibrium is unstable in the following sense: A small upwards perturbation in agents' beliefs about price efficiency affects agents' demands and shifts the RHS up (for any value of ρ). To preserve market clearing, this then pushes ρ up, and precisely because the RHS crosses the 45^o line from below, the change in ρ is greater than the original perturbation in agents' beliefs about ρ, i.e., instability.

Proposition 1 *There is unique stable linear equilibrium, in which price efficiency ρ is given by*

$$\rho = \frac{\gamma}{2\tau_u} - \sqrt{\left(\frac{\gamma}{2\tau_u}\right)^2 - \frac{1}{\tau_u}\frac{1}{n}\int_0^n \tau_i di}. \tag{26}$$

Price efficiency ρ is decreasing in participation n. The unconditional risk premium is given by

$$\mathbb{E}[X - P] = \frac{\gamma S}{\tau_X + \rho^2(\tau_Z + \tau_u) + \frac{1}{n}\int_0^n \tau_i di}. \tag{27}$$

From Proposition 1, price efficiency is determined by the average information precision of agents who actively trade. As participation increases, newly participating agents lower this average. Even though these agents bring more information to the market, they also bring more trade motivated by risk-sharing concerns, which function in the same way as noise.

(Note that the comparative static in Proposition 1 would be reversed if we had instead assumed that all agents observe cash flow signals with the same precision, but have heterogeneous variances of exposure, i.e., if non-financial income related to $\tilde{X}_k$ were $a_i\left(\tilde{Z}_k + \tilde{u}_{ik}\right)\tilde{X}_k$, with a_i varying across agents. in this case, increased participation would correspond to a reduction in the average non-financial exposure of participating agents, and this would increase rather than decrease price efficiency.)

The risk premium $\mathbb{E}[X - P]$ is driven by the amount of aggregate risk, as measured by S and τ_X, as well as the risk tolerance in the economy, γ. The risk premium also reflects the average conditional precision (see (20)) of participating agents. As participation increases, this decreases, both directly, as discussed immediately above, and also indirectly, as price efficiency falls. Consequently, agents bear more risk when they hold the asset, and the risk premium increases with participation.

4.4 Welfare and participation

We next turn to agents' participation decisions. To do so, we first characterize an agent's expected utility from participation. As noted in the introduction, a concise representation of expected utility in economies of this type has proved challenging to obtain in related work. Looking ahead, a concise representation is important for analyzing comparative statics in participation costs.

Proposition 2 *In a linear equilibrium, agent i's expected utility from participation, conditional on the realization of his exposure shock* e_i*, is given by*

$$\mathbb{E}\left[u(W_i-\kappa)|e_i\right]=-\left(d_iD\right)^{-\frac{1}{2}}\exp\left(-\gamma e_i\mathbb{E}\left[X\right]+\frac{\gamma^2{e_i}^2}{2\tau_X}-\frac{1}{2}\Lambda\left(e_i-S\right)^2+\gamma\kappa\right) \tag{28}$$

where

$$d_i = \frac{\mathrm{var}(X|e_i,P)}{\mathrm{var}(X|y_i,e_i,P)}, \tag{29}$$

$$D = \frac{\mathrm{var}\left(X-P|e_i\right)}{\mathrm{var}\left(X|e_i,P\right)}, \tag{30}$$

$$\Lambda = \frac{\left(\frac{\mathrm{cov}(P,e_i)}{\mathrm{var}(e_i)}+\gamma\mathrm{cov}(X-P,X)\right)^2}{\mathrm{var}(X-P|e_i)}. \tag{31}$$

The key step in the proof of Proposition 2 is the substitution of Corollary 1's expression for the unconditional risk premium $\mathbb{E}[X-P]$.

To interpret Proposition 2, note that an agent's expected utility from non-participation is simply

$$\mathbb{E}\left[u\left(e_iX\right)|e_i\right]=-\exp\left(-\gamma e_i\mathbb{E}\left[X\right]+\frac{\gamma^2{e_i}^2}{2\tau_X}\right).$$

So an agent's expected gain from participation is reflected in the benefits D and Λ which stem from risk-sharing and are the same for all agents, no matter how precise or imprecise their private information; and the advantages stemming from more precise private information, represented by d_i. These gains must be balanced against the cost of participation, κ.

An immediate and intuitive implication of Proposition 2 is that expected utility is increasing in the precision of an agent's information, τ_i. So consistent with our initial conjecture, linear equilibria are characterized by some n such that agents $i \leq n$ participate, and agents

$i > n$ do not.

Next, we show that agents' individual participation decisions exhibit strategic complementarity:

Proposition 3 *As participation n increases, each individual agent's gain from participation increases.*

Economically, the key driving force behind strategic complementarity is that, as participation n increases, price efficiency ρ drops (see Proposition 1 and related discussion). Loosely speaking, lower price efficiency increases the amount of risk-sharing that the financial market enables. Specifically, the risk sharing function of the financial market is to enable agents with high idiosyncratic exposures u_i to share cash flow risk $u_i X$ with other agents with low idiosyncratic exposures.

Lower price efficiency corresponds to agents having less information about the cash flow X, which makes risk sharing easier to sustain as in Hirshleifer (1971). However, and as discussed in the context of the public information benchmark of subsection 4.1, the risk sharing benefits of less efficient prices must be compared to the potential costs that arise from more volatile prices, since if prices are less efficient, they are relatively more exposed to the aggregate exposure shock Z (essentially, the discount rate), and this can easily lead to greater volatility. Proposition 3 establishes that the benefits of lower price efficiency always dominate the potential costs.

Because Proposition 3 is central to our analysis, we give the structure of the argument here, focusing as much as possible on interpretable economic properties, while relegating algebraic details to the appendix.

To establish the result, we show that each of the three terms d_i, D, and Λ are increasing in participation n. We start with the term d_i, which corresponds to an agent's expected gains from the precision of his private signal y_i. Substitution of (20) delivers

$$d_i = \frac{\tau_X + \rho^2(\tau_Z + \tau_u) + \tau_i}{\tau_X + \rho^2(\tau_Z + \tau_u)}.$$

Because price efficiency ρ is decreasing in participation n, the private gains from information, d_i, are increasing in participation. Economically, when prices convey less information about cash flows X, an agent's private information about X is more valuable.

Next, we consider the risk-sharing terms Λ and D. As a first step, note that straightforward algebraic manipulation and the basic equilibrium property Corollary 3 combine to deliver

$$\Lambda = \frac{\frac{\text{var}(Z)^2}{\text{var}(e_i)^2}\left(\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di - \frac{\text{var}(e_i)}{\text{var}(Z)}\right)^2}{\frac{1}{\gamma^2}\frac{1}{\text{var}(X)} + \left(\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di\right)^2 \text{var}\left(Z|e_i\right)} \tag{32}$$

$$D = 1 + \frac{\left(-\rho + \gamma \text{var}\left(Z|e_i\right)\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di\right)^2}{\gamma^2 \text{var}\left(Z|e_i\right)\text{var}\left(X\right)\left(\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P}di\right)^2}. \tag{33}$$

Given these expressions, it is clearly important to understand the term $\frac{1}{n}\frac{\partial P}{\partial Z}\int \frac{\partial \theta_i}{\partial P}di$. On the one hand, from Lemma 3 and (20),

$$\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P}di = -\frac{1}{\gamma}\frac{1}{n}\int_0^n \frac{1}{\text{var}\left(X|y_i, e_i, P\right)}di = -\frac{\tau_X + \rho^2(\tau_Z + \tau_u) + \frac{1}{n}\int_0^n \tau_i di}{\gamma}.$$

So as participation n increases, $\left|\frac{1}{n}\int_N \frac{\partial \theta_i}{\partial P}di\right|$ declines, both because price efficiency declines, as discussed above; and because the average signal precision of participating agents, $\frac{1}{n}\int_N \tau_i di$, declines. Economically, both forces mean that the average participating agent is exposed to more risk when they trade, and so $\left|\frac{\partial \theta_i}{\partial P}\right|$ increases. On the other hand, greater price efficiency is typically associated with prices that depend less on Z, corresponding to a reduction in $\left|\frac{\partial P}{\partial Z}\right|$. However, substitution of (22) of Corollary 2 into the market clearing condition (18), along with the very basic property that the non-informational effect of exposure shocks on demand is $\frac{\partial \theta_i}{\partial e_i} = -1$, yields

$$\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di = 1 + \frac{\rho}{\gamma}\tau_Z. \tag{34}$$

Hence the former effect is the dominant one, and $\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di$ is increasing in price efficiency

ρ, and hence (by Proposition 1) is decreasing in participation n.

It then clear from (32) that Λ is indeed increasing in participation n if

$$\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P} di - \frac{\text{var}\,(e_i)}{\text{var}\,(Z)} < 0. \tag{35}$$

This is indeed the case, since (21) of Corollary 2 and $\frac{\partial \theta_i}{\partial e_i} = -1$ imply that (35) is equivalent to fact that the prices contain more information than endowments in the sense of (25), as established in Lemma 5.

Turning to the determinant term D, it is immediate that the denominator $\text{var}\,(X|e_i, P)$ is decreasing price efficiency ρ. Loosely speaking, one would also expect the numerator $\text{var}\,(X - P|e_i)$ to fall: as prices become more efficient, P is more closely related to X, and so $X - P$ is less volatile. Below, we establish that the numerator indeed falls, and moreover, that effect dominates the fall in $\text{var}\,(X|e_i, P)$.

Substituting for $\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P} di$ in (33) using (34), and noting that $\text{var}\,(Z|e_i) = \frac{1}{\tau_Z + \tau_u}$, we obtain

$$D = 1 + \frac{(\gamma - \rho\tau_u)^2\, \text{var}\,(Z|e_i)}{\gamma^2 \text{var}\,(X)\left(\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P} di\right)^2}. \tag{36}$$

As discussed above, both price efficiency ρ and the sensitivity of demand to price, $\left|\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P} di\right|$, are decreasing in participation n. So D is indeed decreasing in participation n if $\gamma - \rho\tau_u > 0$, which using the equality of (33) and (36) and the fact that $\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P} di = -\frac{\rho}{\gamma \text{var}(Z|e_i)}$ (see Lemma 3) is equivalent to the aggregate demand curve sloping down, i.e., inequality (23), as is established in Lemma 4.

Finally, and for use below, note that the above arguments establish that the benefits to agent i of financial market participation depend on the participation decisions of other agents, summarized by n, only via the the average precision of participating agents, $\frac{1}{n}\int_0^n \tau_i di$, and the associated price efficiency ρ (see Proposition 1). Accordingly, define

$$T(n) \equiv \frac{1}{n}\int_0^n \tau_i di$$
$$f(n) \equiv \frac{\gamma}{2\tau_u} - \sqrt{\left(\frac{\gamma}{2\tau_u}\right)^2 - \frac{T(n)}{\tau_u}}.$$

5 The effect of declining indexing costs

We are now in a position to address our main question: How does a decline in the cost of indexing, as represented by the parameter κ_l, affect equilibrium outcomes?

5.1 Equilibrium in participation levels

We first need to solve for equilibrium participation levels for a given cost of indexing κ_l. As we describe below, this is a straightforward fixed point problem. The effect of a decline in a cost of indexing then follows by monotone comparative statics arguments (see Milgrom and Roberts (1994)).

As noted previously, an immediate implication of Proposition 2 is that the gains to participating in a financial market are increasing in the precision τ_i of an agent's private information about cash flows. Consequently, in any equilibrium the full participation set is of the form $N = [0, n]$, the indexing participation set of the form $N_l = (n, n_l]$, and agents $i > n_l$ do not participate at all, where $0 \leq n \leq n_l \leq 1$.

A very useful property to note is that the equilibrium participation levels n_l and n can be solved for recursively. First, the indexing-non-participation boundary n_l depends only on the price efficiency of the index, ρ_1, and the price efficiency of the index depends only on the total number of agents who trade it, n_l, regardless of whether these agents also trade other assets, i.e. regardless of the location of n. So the equilibrium level of n_l can be characterized in isolation; and then, given n_l, the equilibrium full participation level $n \in [0, n_l]$ can be characterized.

More formally, to find the indexing participation level n_l, define the function

$$g_l(n_l;\kappa_l) = \sup\{i \in [0,1] : \text{agent } i \text{ prefers indexing to non-participation if the average signal precision of agents is trading the index asset 1 is } T(n_l) \text{ and price efficiency is } f(n_l)\}.$$

That is, $g_l(n_l;\kappa_l)$ gives agents' indexing participation decisions if the average signal precision in the index market is $T(n_1)$, and price efficiency is $\rho = f(n_l)$. By Proposition 3, $g(n_l;\kappa_l)$ is increasing in the argument n_l, while clearly it is decreasing in the cost argument κ_l. The equilibrium indexing participation level n_l is then simply a solution to the fixed point relation $g_l(n_l;\kappa_l) = n_l$, where the existence of at least one fixed point follows from Tarski's fixed point theorem.[9]

Given an equilibrium value of n_l, the equilibrium level of full participation is then the fixed point of the analogously defined function $g(n;\kappa)$, where

$$g(n;\kappa) = \sup\{i \in [0,1] : \text{agent } i \text{ prefers indexing to full participation to indexing if the average signal precision of full participating agents is } T(n) \text{ and price efficiency in non-index assest is } f(n)\}.$$

As before, Proposition 3 implies that g is increasing in the argument n, and the existence of a fixed point follows from Tarski.

5.2 Comparative statics in the cost of indexing

Given the above fixed-point arguments for equilibrium existence, the effect of a change in the cost of indexing (κ_l) then follows from monotone comparative statics arguments. We

[9]Given strategic complementarity of participation decisions, it is possible that there are multiple fixed points, corresponding to multiple equilibrium levels of participation. In such cases, the comparative statics below should be understood in the sense of Milgrom and Roberts (1994), i.e., as applying to the equilibria with minimum and maximum participation.

highlight that our main interest in this paper is in reductions in the cost of indexing, which imply an *increase* in the marginal cost of full participation, $\kappa - \kappa_l$. In practice, it is possible that the cost of full participation has declined at the same time that the cost of indexing has declined. However, the qualitative implications of the following results are unchanged provided that κ falls more slowly than κ_l.

Proposition 4 *An equilibrium exists. As the indexing participation cost, κ_l, decreases:*

(i) The number of agents, n_l, who trade the index asset increases. Moreover, the number of agents who index, $n_l - n$, increases.

(ii) The number of agents, n, who trade non-index assets falls.

(iii) The price efficiency of the index, ρ_1, falls.

(iv) Price efficiency of non-index assets increases.

(v) Expected utility for existing indexers increases. The effect on the expected utility of agents who fully participate is ambiguous; however, the share of welfare gains that stem from trading the index asset increase.

The fall in the cost of indexing increases the number of agents who trade the index both directly, and also indirectly, since once more people are trading it, the benefits of doing so are higher (strategic complementarity, Proposition 3), in turn leading to yet more participation. Moreover, recall that the reason that greater participation increases the benefits of participation is that it reduces price efficiency.

Because the marginal cost of full participation increases, fewer agents trade non-index assets. Again, this is driven by both the direct effect (higher cost), and the indirect effect that as agents drop out of trading non-index assets, the benefits from doing so decline, because price efficiency increases.

Consequently, the number of indexers, $n_l - n$, increases. There is entry into indexing at both margins—some people who did not previously participate start trading the index, and some people who previously traded non-index assets switch to trading just the index.

The expected utility of indexers increases both because the cost of indexing drops, and because the welfare gains to participating in a less-price efficient market are higher. The expected utility of people who fully participate is subject to two offsetting forces. On the one hand, they gain more from trading the indexing asset, since price efficiency is lower. But on the other hand, they gain less from trading other assets, where price efficiency is higher.

The clearest empirical predictions of our analysis relate to price efficiency. In particular, our analysis predicts that (i) assets inside the index have lower price efficiency than assets outside the index;[10] (ii) as indexing becomes more prominent, price efficiency of assets in the index will decline relative to assets outside the index; (iii) the relative price efficiency[11] of assets in the index will rise.

A number of recent empirical papers have studying related predictions, especially in regard to ETFs. Farboodi, Matray, and Veldkamp (2018) and Coles, Heath, and Ringgenberg (2017) present results consistent with (i).[12] Israeli, Lee, and Sridharan (2017)[13] and Brogaard, Ringgenberg, and Sovich (2018) present results consistent with (ii). Bai, Philippon, and Savov (2016) and Farboodi, Matray, and Veldkamp (2018) both show that the relative price efficiency of assets in the S&P500 has increased since 1960, which is consistent with (iii), though not tied specifically to the indexing channel.[14]

[10]For assets inside the index, price efficiency is determined by the combination of price efficiency of the index (ρ_1), and price efficiency of long-short positions among industry assets ($\rho_2, \ldots, \rho_l$). For assets $k > l$ that are outside the index, price efficiency is simply ρ_k. Our analysis predicts $\rho_1 < \rho_2 = \ldots = \rho_m$, and hence that price efficiency is lower for assets inside the index. (Note that the exact equality of price efficiency of all assets other than the index assets stems from our symmetry assumptions, and is not a robust prediction of our analysis.)

[11]By relative price efficiency, we mean the extent to which the difference in prices of two assets predicts the difference in future cash flows, i.e., $\text{var}\left(\tilde{X}_1 - \tilde{X}_2 | \tilde{P}_1 - \tilde{P}_2\right)^{-1}$.

[12]In contrast, Huang, O'Hara, and Zhong (2018) "find that the industry ETF membership ... reduces the market reaction to the firm's earnings surprise," suggesting higher rather than lower price efficiency for assets in the index. However, this paper has a less obvious strategy for controlling for which assets are included in the ETF.

[13]For full disclosure, we note that Glosten, Nallareddy, and Zou (2016) document just the opposite. As discussed by Israeli, Lee, and Sridharan (2017), the difference in results stems from the time period over which changes in ETF ownership is measured.

[14]Farboodi, Matray, and Veldkamp (2018) also find a decrease in relative price efficiency outside the index, which is not consistent with our results. In general, Farboodi, Matray, and Veldkamp (2018) argue for the importance of accounting for difference in form characteristics, which (given our symmetry assumptions) is outside the scope of our model.

Price efficiency also determines in which markets relatively informed agents make trading profits. With the caveat that expected utilities are distinct objects from expected profits, Proposition 4(v) implies that as indexing increases, informed trading profits will stem increasingly from "timing" strategies based on the entire index, rather than individual asset trades. There is at least some empirical evidence supporting this prediction. AQR document that the correlation between hedge fund returns and market returns has risen from 0.6 to 0.9 over the last two decades.[15] Related, Stambaugh (2014) documents a decline in asset-selection strategies by active mutual funds over the same period. Also related, and using data since 2000, Gerakos, Linnainmaa, and Morse (2017) show that a significant fraction of returns generated by active mutual funds stem from market timing strategies.

(Our model can also be used to study the effect of index inclusion on correlation etc. For example, index inclusion will generally lead to greater correlation of returns. But these predictions are further from our core mechanisms.)

Our model also implies reversal in stock prices, with the degree of reversal determined by price efficiency, as we discuss next.

5.3 Reversals, informed trades, and indexing

A direct implication of market-clearing (5) and agents' trading decisions (13) is that

$$\frac{1}{\gamma}\mathbb{E}\left[X_1 - P_1 | P_1\right] \frac{1}{n_l} \int_0^{n_l} \frac{1}{\operatorname{var}\left(X_1 | y_{i1}, e_{i1}, P_1\right)} = S_1 + \mathbb{E}\left[Z_1 | P_1\right], \tag{37}$$

with analogous identities for other assets. Moreover, from Lemma 4, we know $\mathbb{E}\left[Z_1 | P_1\right]$ is decreasing in P_1, i.e., prices and exposure shocks are negatively correlated. Consequently, our framework naturally generates a reversal pattern in prices, with high prices today associated with lower expected returns.[16]

[15] See "Hedge fund correlation risk alarms investors," Financial Times, June 29th, 2014.

[16] Consequently, if an investor observes a low price for an asset, and has no exposure to economic shocks, he should take a long position in the asset, since its conditional expected return is high. That is, an investor can profit from buying "value" stocks. Although this point is often overlooked, it is nonetheless a standard

Economically, high prices are more likely when the average exposure Z is high. Consequently, agents who observe a high asset price are unable to fully infer whether the high price indicates a high future cash flow, or a high value of Z (i.e., high aggregate unwillingness to buy the asset). Although all agents face this inference problem, agents with more precise private signals are better able to resolve it, and to shy away from the asset when future cash flows are in fact low. To express this formally, fix an arbitrary $\hat{n} \in (0, n_l)$, so that agents $[0, \hat{n}]$ can be though of as representing "smart money," while agents $[\hat{n}, n_l]$ correspond to "dumb money." Then a direct application of the formulas for agents' optimal trades implies that the difference in the average position of smart money from dumb money is given by

$$\frac{1}{\gamma}\left(\frac{1}{\hat{n}}\int_0^{\hat{n}} \tau_i di - \frac{1}{n_l - \hat{n}}\int_{\hat{n}}^{n_l} \tau_i di\right)(X_1 - P_1). \tag{38}$$

That is, smart money owns a disproportionately high share of the asset precisely when returns are high, and a disproportionately low share precisely when returns are low.

Moreover, combining (37) and (38) implies that, empirically, one would expect to see relatively uninformed ownership of an asset increase when prices are high, and for this to be followed by low subsequent returns. This is consistent with the empirical evidence in Ben-Rephael, Kandel, and Wohl (2012) for the market as a whole (i.e., the index asset), and in Jiang, Verbeek, and Wan (2017) in the cross-section.

The strength of reversals is captured by the steepness of the negative slope of $\mathbb{E}[X - P|P]$, i.e., when this relation is strongly negative, the expected returns following high prices are much lower than following low prices. This is determined by price efficiency, and hence in turn by participation decisions:

Lemma 6 $\frac{\partial}{\partial P_1} E[X_1 - P_1|P_1]$ *is negative, and decreases (i.e., becomes further from* 0*) as price efficiency* ρ_1 *declines, and hence as the cost of index-participation* κ_l *decreases.*

This is consistent with the empirical findings of Baltussen, van Bekkum, and Da (2017).

implication of models of the type we consider here (see, e.g., Biais, Bossaerts, and Spatt, 2010).

6 Discussion

So far, we have analyzed what we believe is the most direct impact of a decline in the costs of indexing, namely those that stem from the entry of new participants into financial markets in response to lower costs, along with substitution of other traders away from full participation to index-only strategies.

Nonetheless, our analysis inevitably omits other potentially important forces. In particular, we have held constant the precision of agents' private signals (beyond the basic question of whether or not to acquire information at all, which is one possible interpretation of participation decisions). In an earlier draft of this paper we fully analyzed a model that includes this force. In brief, consider the consequences of an exogenous increase in indexing among agents with low-precision private signals. The direct effect is an increase in the price efficiency of the spread asset, with no effect on price efficiency of the market asset (because, by assumption, the increase in indexing simply consists of agents with low-precision signals stopping trading the spread asset, with no increase in trade of the market asset). Allowing agents to reoptimize the precision of their private signals, these changes in price efficiency in turn induce agents to acquire less precise signals about the spread asset, since this information is now less valuable; and in turn to substitute their information collection activities towards acquiring information about economic aggregates. The net effect is price efficiency increases in both the market and spread assets. The same economic forces as in our current analysis then lead to a reduction in expected utility, since (exactly as in Proposition 2) agents prefer to participate in financial markets where price efficiency is low.

So to summarize: by themselves, information acquisition decisions in response to an exogenous rise in indexing end up reducing rather than increasing the welfare of agents with low-precision private signals, who can be interpreted as retail investors.

More generally, our analysis highlights that the welfare consequences of shifts in indexing—or, indeed, or other changes to financial markets—depend critically on how such shifts affect price efficiency. At least when the gains from trade that underpin financial markets are driven by the benefits from risk sharing, as is the case of many standard models, agents generally prefer low levels of price efficiency.

Of course, price efficiency may be desirable for other reasons, such as for the efficient allocation of capital in primary financial markets, or because of information conveyed by markets that provides incentives and guides "real" productive decisions.[17] Including such affects in our analysis would add new elements to the welfare analysis.

7 Conclusion

We develop a benchmark model to study the equilibrium consequences of indexing in a standard rational expectations setting (Grossman and Stiglitz (1980); Hellwig (1980); Diamond and Verrecchia (1981)). Individuals must incur costs to participate in financial markets, and these costs are lower for individuals who restrict themselves to indexing strategies. Individuals' participation decisions exhibit strategic complementarity. As indexing becomes cheaper (1) indexing increases, while individual stock trading decreases; (2) aggregate price efficiency falls, while relative price efficiency increases; (3) the welfare of relatively uninformed traders increases; (4) for well-informed traders, the share of trading gains stemming from market timing increases, and the share of gains from stock selection decreases; (5) market-wide reversals become more pronounced. We discuss empirical evidence for these predictions.

[17] See Bond, Goldstein, and Edmans (2012) for a survey of the literature. See also Bond and Goldstein (2015) for an example of how to embed such effects into a model along the lines of Hellwig (1980) in a relatively tractible way.

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Appendix

Results omitted from main text

Lemma A-1 *Suppose that the information set $\mathcal{F}_i$ consists of a set of normally distributed random variables. Then*

$$\frac{\partial}{\partial X}\mathbb{E}\left[\tilde{X}|\mathcal{F}_i\right] = 1 - \frac{\operatorname{var}\left(\tilde{X}|\mathcal{F}_i\right)}{\operatorname{var}\left(\tilde{X}\right)}.$$

Proof of Lemma A-1 Let Σ_{22} be the variance matrix of the random variables in $\mathcal{F}_i$; and Σ_{12} be the row vector of covariances between X and the random variables in $\mathcal{F}_i$. By the properties of multivariate normality

$$\begin{aligned}
\frac{\partial}{\partial X}\mathbb{E}\left[\tilde{X}|\mathcal{F}_i\right] &= \Sigma_{12}\Sigma_{22}^{-1}\frac{\Sigma_{12}'}{\operatorname{var}\left(\tilde{X}\right)} \\
\operatorname{var}\left(\tilde{X}|\mathcal{F}_i\right) &= \operatorname{var}[\tilde{X}] - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}'.
\end{aligned}$$

Combining these two equations yields the result, completing the proof.

Lemma A-2 *Let $\xi \in \mathbb{R}^n$ be a normally distributed random vector with mean μ and variance-covariance matrix Σ. Let $b \in \mathbb{R}^n$ be a given vector, and $A \in \mathbb{R}^{n\times n}$ a symmetric matrix. If $I - 2\Sigma A$ is positive definite, then $\mathbb{E}\left[\exp(b^\top\xi + \xi^\top A\xi)\right]$ is well defined, and given by:*

$$\mathbb{E}\left[\exp\left((b^\top\xi + \xi^\top A\xi\right)\right] = |I - 2\Sigma A|^{-1/2}\exp\left(b^\top\mu + \mu^\top A\mu + \frac{1}{2}(b + 2A\mu)^\top(I - 2\Sigma A)^{-1}\Sigma(b + 2A\mu)\right). \tag{A-1}$$

Proof of Lemma A-2: Standard result.

Proofs of results stated in main text

Proof of Lemma 1: We focus on cases $l = m$, since the generalization to $l < m$ is trivial. The proof is inductive: Given the existence of an $m \times m$ matrix A with the stated properties, we construct a $2m \times 2m$ matrix B with the same properties. Specifically, define

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} A & A \\ A & -A \end{pmatrix}.$$

With the exception of the symmetry and inversion properties, it is straightforward to see that B has the desired properties. To establish that B is involutory, simply note that

$$BB = \frac{1}{2}\begin{pmatrix} A & A \\ A & -A \end{pmatrix}\begin{pmatrix} A & A \\ A & -A \end{pmatrix} = \frac{1}{2}\begin{pmatrix} AA + AA & AA - AA \\ AA - AA & AA + AA \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2I_m & 0_m \\ 0_m & 2I_m \end{pmatrix} = I_{2m},$$

where I_m denotes the $m \times m$ identity matrix and 0_m denotes the $m \times m$ matrix in which all entries are zero. To establish that B is symmetric, simply note that

$$B^T = \frac{1}{\sqrt{2}}\begin{pmatrix} A^T & A^T \\ A^T & -A^T \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} A & A \\ A & -A \end{pmatrix} = B.$$

This completes the proof.

Proof of Lemma 3: Consider first the case in which $\frac{\partial P}{\partial X} \neq 0$. The information content of (y_i, e_i, P) is the same as the information content of

$$\left(y_i, \frac{P - \mathbb{E}[P]}{\frac{\partial P}{\partial X}} + \mathbb{E}[X], \frac{P - \mathbb{E}[P]}{\frac{\partial P}{\partial X}} + \mathbb{E}[X] + \rho^{-1}(e_i - S)\right) = \left(X + \epsilon_i, X - \rho^{-1} Z, X + \rho^{-1}(u_i + s_i - S)\right).$$

Since ϵ_i, Z, and $u_i + s_i - S$ are independent and all have mean 0, the conditional variance expressions (19) and (20) follow by standard normal-normal updating. Using $\frac{\partial P}{\partial X} = -\rho \frac{\partial P}{\partial Z}$, the corresponding conditional expectation $\mathbb{E}[X|y_i, e_i, P]$ is given by

$$\frac{\mathbb{E}[X|y_i, e_i, P]}{\text{var}(X|y_i, e_i, P)} = \tau_X \mathbb{E}[X] + \rho(\tau_Z + \tau_u)\left(\rho \mathbb{E}[X] - \frac{P - \mathbb{E}[P]}{\frac{\partial P}{\partial Z}}\right) + \rho \tau_u (e_i - S) + \tau_i y_i. \quad \text{(A-2)}$$

Finally, if $\frac{\partial P}{\partial X} = 0$ then neither the price nor the endowment e_i contains any information about X; and $\rho = 0$; so (19), (20), and (A-2) are all immediate.

The expressions in Lemma 3 are then immediate from the demand equation (13), completing the proof.

Proof of Lemma 4: From Lemma 3, $\frac{\partial \theta_i}{\partial P} < 0$ for all agents.

If $\frac{\partial P}{\partial X} = 0$, then P contains no information about X, so $\int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di = 0$, and (23) is then immediate.

If instead $\frac{\partial P}{\partial X} \neq 0$, then $\frac{\partial P}{\partial X} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di > 0$ by Lemma 3. By (17) and Lemma 3,

$$\frac{\partial P}{\partial X} \int_0^n \frac{\partial \theta_i}{\partial P} di + \frac{\partial P}{\partial X} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di = -\int_0^n \frac{\partial \theta_i}{\partial Y_i} di < 0. \quad \text{(A-3)}$$

Hence $\frac{\partial P}{\partial X} > 0$, which (again using (A-3)) implies (23).

Note that the above arguments also establish that $\frac{\partial P}{\partial X} \geq 0$.

The main text establishes that $\frac{\partial P}{\partial Z} \neq 0$. So to establish $\frac{\partial P}{\partial Z} < 0$, suppose to the contrary that $\frac{\partial P}{\partial Z} > 0$. Then $\rho \leq 0$, and Lemma 3 implies $\int_0^n \frac{\partial \theta_i}{\partial e_i} di < 0$ and $\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di \leq 0$. Combined with (23), this in turn implies that the lefthand side of (18) is strictly negative. The contradiction completes the proof.

Proof of Lemma 5: At various points in the proof, we make use of $\rho < 0$ and $\frac{\partial P}{\partial Z} < 0$ (by Lemma 4), and $\frac{\partial \theta_i}{\partial \hat{P}} > 0$ (by Lemma 3).

Re-arranging market-clearing (18) gives

$$0 = \frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial P} di + \int_0^n \frac{\partial \theta_i}{\partial e_i} di + \left(\frac{\frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di}{\int_0^n \frac{\partial \theta_i}{\partial \tilde{e}_i} di} + 1 \right) \int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di. \tag{A-4}$$

By Lemma 4 and market-clearing (18), we know

$$\int_0^n \frac{\partial \theta_i}{\partial e_i} di + \int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di < 0. \tag{A-5}$$

From (22) of Corollary 2, $\frac{\partial P}{\partial Z} \frac{\partial \theta_i}{\partial \hat{P}} + \frac{\partial \theta_i}{\partial \hat{e}_i} < 0$, which together with $\frac{\partial \theta_i}{\partial \hat{e}_i} > 0$ implies

$$\frac{\frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di} + 1 < 0.$$

So substituting (A-5) into (A-4) gives

$$\begin{aligned} 0 \quad &> \quad \frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial P} di + \int_0^n \frac{\partial \theta_i}{\partial e_i} di - \left(\frac{\frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di} + 1 \right) \int_0^n \frac{\partial \theta_i}{\partial e_i} di \\ &= \quad \frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial P} di - \frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di \frac{\int_0^n \frac{\partial \theta_i}{\partial e_i} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di}. \end{aligned} \tag{A-6}$$

Using $\frac{\partial P}{\partial Z} < 0$ and the signs established in Lemma 3, inequality (A-6) is equivalent to

$$\frac{\int_0^n \frac{\partial \theta_i}{\partial P} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di} > \frac{\int_0^n \frac{\partial \theta_i}{\partial e_i} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di},$$

which is in turn equivalent to (25), completing the proof.

Proof of Proposition 1: From (17) and (18),

$$-\frac{\frac{\partial P}{\partial X}}{\frac{\partial P}{\partial Z}} = -\frac{\int_0^n \frac{\partial \theta_i}{\partial Y_i} di}{\int_0^n \frac{\partial \theta_i}{\partial e_i} di + \int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di}. \tag{A-7}$$

Substituting in from Lemma 3,

$$\rho = \frac{\frac{1}{\gamma}\int_0^n \tau_i di}{\int_0^n \left(1 - \frac{\rho}{\gamma}\tau_u\right) di}, \tag{A-8}$$

and so

$$\rho^2 \tau_u - \gamma\rho + \frac{1}{n}\int_0^n \tau_i di = 0,$$

leading to (26). The comparative statics of price efficiency is immediate from the fact that $\frac{1}{n}\int_0^n \tau_i di$ is decreasing in n. To derive (27), note that Corollaries 1 and 3 combine to give

$$\mathbb{E}\left[X - P\right] = -\frac{S}{\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P} di},$$

and so by Lemma 3,

$$\mathbb{E}\left[X - P\right] = \frac{\gamma S}{\frac{1}{n}\int_0^n \frac{1}{\mathrm{var}(X|y_i, e_i, P)} di}.$$

Substituting in (20) completes the proof.

Proof of Proposition 2: The final wealth of agent i, given optimal trading (13), is

$$W_i = e_i P + \frac{\mathbb{E}[X - P|y_i, e_i, P]\,(X - P)}{\gamma \mathrm{var}(X|y_i, e_i, P)}.$$

So by the standard expression for the expected utility of an agent with CARA utility facing normally shocks, combined with simple manipulation, agent i's expected utility at the time of trading is

$$\mathbb{E}\left[u\left(W_i - \kappa\right)|y_i, e_i, P\right] = -\exp\left(-\gamma\left(e_i P + \frac{1}{2}\frac{\mathbb{E}[X - P|y_i, e_i, P]^2}{\gamma \mathrm{var}(X|y_i, e_i, P)} - \kappa\right)\right). \tag{A-9}$$

To obtain (28), we use (A-9), and proceed in two stages. First, we integrate out possible realizations of the private signal y_i. Second, we integrate out possible realizations of the price P. Note that the first stage is relatively standard, and closely related algebraic arguments can be found in the related literature. Readers familiar with these arguments should proceed directly to the second stage.

For the first stage, define $\xi_i = \mathbb{E}\left[X - P|y_i, e_i, P\right]$ and $A_i = -1/\left(2\text{var}(X|y_i, e_i, P)\right)$. Minor algebraic manipulation of Lemma A-2 implies

$$\mathbb{E}\left[\exp\left(\xi_i^2 A_i\right)|e_i, P\right] = \left(1 - 2A_i\text{var}(\xi_i|e_i, P)\right)^{-\frac{1}{2}} \exp\left(\frac{A_i}{1 - 2A_i\text{var}(\xi_i|e_i, P)}\mathbb{E}[\xi_i|e_i, P]^2\right). \tag{A-10}$$

By the law of total variance,

$$\text{var}(X - P|e_i, P) = \text{var}(\mathbb{E}[X - P|y_i, e_i, P]|e_i, P) + \mathbb{E}[\text{var}(X - P|y_i, e_i, P)|e_i, P]$$

which implies

$$\text{var}(\xi_i|e_i, P) = \text{var}(X|e_i, P) - \text{var}(X|y_i, e_i, P)$$

and so

$$1 - 2A_i\text{var}(\xi_i|e_i, P) = \frac{\text{var}(X|e_i, P)}{\text{var}(X|y_i, e_i, P)} = d_i,$$

where d_i is as defined in (29). Substitution and straightforward manipulation implies that expression (A-10) equals

$$d_i^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\frac{\mathbb{E}[X - P|e_i, P]^2}{\text{var}(X|e_i, P)}\right),$$

and so

$$\mathbb{E}\left[u\left(W_i - \kappa\right)|e_i, P\right] = -d_i^{-\frac{1}{2}} \exp\left(-\gamma e_i P - \frac{1}{2}\frac{\mathbb{E}[X - P|e_i, P]^2}{\text{var}(X|e_i, P)} + \gamma\kappa\right), \tag{A-11}$$

completing the first stage.

In the second stage, we integrate out possible realizations of P. Since $P = (P - \mathbb{E}[P|e_i]) + \mathbb{E}[X|e_i] - \mathbb{E}[X - P|e_i]$, the expression in the exponent of (A-11) equals

$$-\frac{1}{2}\frac{\mathbb{E}[X - P|e_i, P]^2}{\text{var}(X|e_i, P)} - \gamma e_i(P - \mathbb{E}[P|e_i]) - \gamma e_i\mathbb{E}[X|e_i] + \gamma e_i\mathbb{E}[X - P|e_i] + \gamma\kappa. \tag{A-12}$$

Denote the expected return $X - P$ given exposure e_i by α_e, i.e.,

$$\alpha_e \equiv \mathbb{E}[X - P|e_i] = \mathbb{E}[X - P] - \frac{\text{cov}(P, e_i)}{\text{var}(\mathrm{e_i})}\left(e_i - S\right), \tag{A-13}$$

and note that

$$\mathbb{E}[X - P|e_i, P] = \frac{\text{cov}\left(X - P, P|e_i\right)}{\text{var}\left(P|e_i\right)}\left(P - \mathbb{E}[P|e_i]\right) + \alpha_e. \tag{A-14}$$

By substitution and Lemma A-2, the expectation of (A-11) conditional on e_i is given by

$$\begin{aligned}\mathbb{E}\left[u\left(W_i-\kappa\right)|e_i\right] &= -d_i^{-\frac{1}{2}}D^{\frac{1}{2}}\exp\left(-\gamma e_i\left(\mathbb{E}\left[X\right]-\alpha_e\right)+\gamma\kappa\right)\\ &\times \exp\left(-\frac{1}{2}\frac{\alpha_e^2}{\mathrm{var}(X|e_i,P)}+\frac{1}{2}\left(\frac{\alpha_e\mathrm{cov}(X-P,P|e_i)}{\mathrm{var}(P|e_i)\mathrm{var}(X|e_i,P)}+\gamma e_i\right)^2\frac{\mathrm{var}(P|e_i)}{D}\right)\end{aligned} \tag{A-15}$$

where

$$D=1+\frac{\mathrm{cov}\left(X-P,P|e_i\right)^2}{\mathrm{var}\left(X|e_i,P\right)\mathrm{var}\left(P|e_i\right)}. \tag{A-16}$$

The law of total variance and (A-14) together yield

$$\mathrm{var}\left(\mathrm{E}\left[X-P|e_i,P\right]|e_i\right)=\mathrm{var}\left(X-P|e_i\right)-\mathrm{var}\left(X|e_i,P\right)=\frac{\mathrm{cov}\left(X-P,P|e_i\right)^2}{\mathrm{var}\left(P|e_i\right)}, \tag{A-17}$$

and substitution into (A-16) delivers (30).

For use below, note also that (A-17) implies that

$$\begin{aligned}\frac{\mathrm{var}\left(P|e_i\right)}{D}=\frac{\mathrm{var}\left(P|e_i\right)\mathrm{var}(X|e_i,P)}{D\mathrm{var}(X|e_i,P)} &= \frac{\mathrm{var}\left(P|e_i\right)\mathrm{var}\left(X-P|e_i\right)-\mathrm{cov}\left(X-P,P|e_i\right)^2}{D\mathrm{var}(X|e_i,P)}\\ &= \frac{\mathrm{var}\left(X|e_i\right)\mathrm{var}\left(X-P|e_i\right)-\mathrm{cov}\left(X-P,X|e_i\right)^2}{D\mathrm{var}(X|e_i,P)}\\ &= \mathrm{var}\left(X|e_i\right)-\frac{\mathrm{cov}\left(X-P,X|e_i\right)^2}{D\mathrm{var}(X|e_i,P)}\end{aligned} \tag{A-18}$$

where the penultimate equality follows from the fact that for any random variables r_1 and r_2,

$$\mathrm{cov}\left(r_1-r_2,r_1\right)^2-\mathrm{cov}\left(r_1-r_2,r_2\right)^2=\mathrm{var}\left(r_1-r_2\right)\left(\mathrm{var}\left(r_1\right)-\mathrm{var}\left(r_2\right)\right),$$

and the final equality follows from (30).

By expanding and then re-arranging the quadratic, (A-15) equals

$$
\begin{aligned}
& -\left(d_i D\right)^{-\frac{1}{2}} \exp\left(-\gamma e_i \mathbb{E}\left[X\right] + \gamma\kappa + \frac{1}{2}\frac{\alpha_e^2}{\mathrm{var}(X|P,e_i)}\left(\frac{\mathrm{cov}(X-P,P|e_i)^2}{D\mathrm{var}(P|e_i)\mathrm{var}(X|e_i,P)} - 1\right)\right) \\
\times\ & \exp\left(\gamma e_i \alpha_e\left(\frac{\mathrm{cov}(X-P,P|e_i)}{D\mathrm{var}(X|e_i,P)} + 1\right) + \frac{1}{2}\gamma^2 e_i^2 \frac{\mathrm{var}(P|e_i)}{D}\right) \\
=\ & -\left(d_i D\right)^{-\frac{1}{2}} \exp\left(-\gamma e_i \mathbb{E}\left[X\right] + \gamma\kappa - \frac{1}{2}\frac{\alpha_e^2}{D\mathrm{var}(X|e_i,P)}\right) \\
\times\ & \exp\left(\gamma e_i \alpha_e \frac{\mathrm{cov}(X-P,X|e_i)}{D\mathrm{var}(X|e_i,P)} + \frac{1}{2}\gamma^2 e_i^2\left(\mathrm{var}(X|e_i) - \frac{\mathrm{cov}(X-P,X|e_i)^2}{D\mathrm{var}(X|e_i,P)}\right)\right) \\
=\ & -\left(d_i D\right)^{-\frac{1}{2}} \exp\left(-\gamma e_i \mathbb{E}\left[X\right] + \frac{\gamma^2 {e_i}^2}{2\tau_X} + \gamma\kappa - \frac{1}{2}\frac{\left(\alpha_e - \mathrm{cov}(X-P,X)\gamma e_i\right)^2}{D\mathrm{var}(X|e_i,P)}\right). \qquad \text{(A-19)}
\end{aligned}
$$

where the first equality follows from (A-16), (30), and (A-18). Substituting Corollary 1's expression for $\mathbb{E}[X-P]$ into (A-13)'s expression for α_e and then in turn substituting into (A-19) yields (31). This completes the proof.

Proof of Proposition 3: The main steps are given in the main text. Here, we give the details for a few of the steps. For use below, recall that $\frac{\partial P}{\partial X} = \frac{\mathrm{cov}(X,P)}{\mathrm{var}(X)}$, $\frac{\partial P}{\partial Z} = \frac{\mathrm{cov}(Z,P)}{\mathrm{var}(Z)}$, and note that

$$
\begin{aligned}
\mathrm{var}\left(P|e_i\right) &= \frac{\mathrm{cov}\left(X,P\right)^2}{\mathrm{var}\left(X\right)^2}\mathrm{var}\left(X\right) + \frac{\mathrm{cov}\left(Z,P\right)^2}{\mathrm{var}\left(Z\right)^2}\mathrm{var}\left(Z|e_i\right), \qquad \text{(A-20)} \\
\mathrm{var}\left(X-P|e_i\right) &= \left(\frac{\mathrm{cov}\left(X-P,X\right)}{\mathrm{var}\left(X\right)}\right)^2 \mathrm{var}\left(X\right) + \left(\frac{\mathrm{cov}\left(P,Z\right)}{\mathrm{var}\left(Z\right)}\right)^2 \mathrm{var}\left(Z|e_i\right). \qquad \text{(A-21)}
\end{aligned}
$$

Details for the argument that Λ is decreasing in price efficiency ρ: Substituting (A-21) into (31) gives

$$
\begin{aligned}
\Lambda &= \frac{\frac{\mathrm{var}(Z)^2}{\mathrm{var}(e_i)^2}\left(-\frac{\mathrm{cov}(P,e_i)}{\mathrm{var}(Z)} - \gamma\mathrm{cov}\left(X-P,X\right)\frac{\mathrm{var}(e_i)}{\mathrm{var}(Z)}\right)^2}{\left(\frac{\mathrm{cov}(X-P,X)}{\mathrm{var}(X)}\right)^2 \mathrm{var}\left(X\right) + \left(\frac{\mathrm{cov}(P,Z)}{\mathrm{var}(Z)}\right)^2 \mathrm{var}\left(Z|e_i\right)} \\
&= \frac{\frac{\mathrm{var}(Z)^2}{\mathrm{var}(e_i)^2}\left(-\frac{\mathrm{cov}(P,Z)}{\mathrm{var}(Z)}\frac{1}{\gamma\mathrm{cov}(X-P,X)} - \frac{\mathrm{var}(e_i)}{\mathrm{var}(Z)}\right)^2}{\frac{1}{\gamma^2}\frac{1}{\mathrm{var}(X)} + \left(-\frac{\mathrm{cov}(P,Z)}{\mathrm{var}(Z)}\frac{1}{\gamma\mathrm{cov}(X-P,X)}\right)^2 \mathrm{var}\left(Z|e_i\right)}. \qquad \text{(A-22)}
\end{aligned}
$$

Substitution (24) of Corollary 3 into (A-22) yields (32).

By Corollary 2 and $\frac{1}{n}\int_0^n \frac{\partial\theta_i}{\partial e_i}di = -1$,

$$\frac{\text{var}(e_i)}{\text{var}(Z)} = \frac{\tau_Z + \tau_u}{\tau_u} = -\frac{\frac{\partial P}{\partial Z}\int_0^n \frac{\partial\theta_i}{\partial\hat{P}}di}{\int_0^n \frac{\partial\theta_i}{\partial\hat{e}_i}di} = \frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial\theta_i}{\partial\hat{P}}di\frac{\int_0^n \frac{\partial\theta_i}{\partial e_i}di}{\int_0^n \frac{\partial\theta_i}{\partial\hat{e}_i}di}. \tag{A-23}$$

Hence (35) is equivalent to (A-6) in the proof of Lemma 5, which itself is equivalent to (25).
Details for the argument that D is decreasing in price efficiency ρ: Using the law of total variance,

$$\begin{aligned}\text{var}(X|e_i,P)\,\text{var}(P|e_i) &= \left(\text{var}(X|e_i) - \frac{\text{cov}(X,P|e_i)^2}{\text{var}(P|e_i)}\right)\text{var}(P|e_i)\\ &= \text{var}(X|e_i)\,\text{var}(P|e_i) - \text{cov}(X,P|e_i)^2.\end{aligned}$$

Substituting in (A-20) gives

$$\text{var}(X|e_i,P)\,\text{var}(P|e_i) = \frac{\text{cov}(Z,P)^2}{\text{var}(Z)^2}\text{var}(Z|e_i)\,\text{var}(X). \tag{A-24}$$

Also by (A-20), and making use of (24) of Corollary 3,

$$\begin{aligned}\text{cov}(X-P,P|e_i) &= \text{cov}(X,P|e_i) - \text{var}(P|e_i)\\ &= \frac{\text{cov}(X,P)}{\text{var}(X)}\left(\text{var}(X) - \text{cov}(X,P)\right) - \frac{\text{cov}(Z,P)^2}{\text{var}(Z)^2}\text{var}(Z|e_i) \quad \text{(A-25)}\\ &= \left(\frac{\frac{\text{cov}(X,P)}{\text{var}(X)}}{\frac{\text{cov}(Z,P)}{\text{var}(Z)}} - \frac{\frac{\text{cov}(Z,P)}{\text{var}(Z)}\text{var}(Z|e_i)}{\text{cov}(X-P,X)}\right)\frac{\text{cov}(Z,P)}{\text{var}(Z)}\text{cov}(X-P,X)\\ &= \left(-\rho + \gamma\text{var}(Z|e_i)\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial\theta_i}{\partial P}di\right)\frac{\text{cov}(Z,P)}{\text{var}(Z)}\left(-\frac{1}{\gamma\frac{1}{n}\int_0^n \frac{\partial\theta_i}{\partial P}di}\right) \quad \text{(A-26)}\end{aligned}$$

Substitution of (A-24) and (A-26) into (A-16) yields (33).

To establish the equivalence of $\gamma\rho - \tau_u > 0$ with (23), note that the equality of (33) and (36) and $\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial\theta_i}{\partial\hat{P}}di = -\frac{\rho}{\gamma\text{var}(Z|e_i)}$ imply that $\gamma\rho - \tau_u > 0$ is equivalent to

$$1 + \frac{\frac{\partial P}{\partial Z}\int_0^n \frac{\partial\theta_i}{\partial P}di}{\frac{\partial P}{\partial Z}\int_0^n \frac{\partial\theta_i}{\partial\hat{P}}di} < 0,$$

which given $\frac{\partial\theta_i}{\partial\hat{P}} > 0$ is equivalent to (23).

Proof of Lemma 6: For transparency of notation, we establish the result for a single-asset economy. From (37), Lemma 3, and (34),

$$\mathbb{E}\left[X-P|P\right]=\frac{S+\mathbb{E}\left[Z|P\right]}{\frac{1}{\gamma}\frac{1}{n}\int_0^n\frac{1}{\operatorname{var}(X|y_i,e_i,P)}}=-\frac{S+\mathbb{E}\left[Z|P\right]}{\frac{1}{n}\int_0^n\frac{\partial\theta_i}{\partial P}di}=-\frac{S+\mathbb{E}\left[Z|P\right]}{1+\frac{\rho}{\gamma}\tau_Z}\frac{\partial P}{\partial Z}.$$

Note that

$$\frac{\partial}{\partial P}\mathbb{E}\left[Z|P\right]=\frac{\operatorname{cov}\left(Z,P\right)}{\operatorname{var}\left(P\right)}=\frac{\operatorname{cov}\left(Z,P\right)}{\operatorname{var}\left(Z\right)}\frac{\operatorname{var}\left(Z\right)}{\operatorname{var}\left(P\right)}=\frac{\partial P}{\partial Z}\frac{\operatorname{var}\left(Z\right)}{\left(\frac{\partial P}{\partial X}\right)^2\operatorname{var}\left(X\right)+\left(\frac{\partial P}{\partial Z}\right)^2\operatorname{var}\left(Z\right)}.$$

Hence (and using the fact that $\frac{\partial P}{\partial Z}$ is independent of Z)

$$\frac{\partial}{\partial P}\mathbb{E}\left[X-P|P\right]=-\frac{1}{\left(1+\frac{\rho}{\gamma}\tau_Z\right)\left(\rho^2\frac{\operatorname{var}(X)}{\operatorname{var}(Z)}+1\right)},$$

completing the proof.

Online appendix

Proposition A-1 *Consider the benchmark economy described in subsection 4.1, in which agents do not possess any private information about the asset's cash flow* X*, but instead all observe a public signal of the form* $Y = X + \epsilon$*, where* $\epsilon \sim \mathcal{N}\left(0, \tau_\epsilon^{-1}\right)$*. In such a setting, each agent's expected utility is decreasing in the precision of the public signal,* τ_ϵ*.*

Proof: Agent i's terminal wealth is

$$W_i = e_i P + \left(\theta_i + e_i\right)\left(X - P\right),$$

and he optimally chooses the portfolio

$$\theta_i + e_i = \frac{1}{\gamma}\frac{\mathbb{E}\left[X|Y\right] - P}{\mathrm{var}\left(X|Y\right)}.$$

So agent i's expected utility at the trading stage is

$$\begin{aligned}
\mathbb{E}\left[-\exp\left(-\gamma W_i\right)|Y, P\right] &= \mathbb{E}\left[-\exp\left(-\gamma e_i P - \frac{\mathbb{E}\left[X|Y\right] - P}{\mathrm{var}\left(X|Y\right)}\left(X - P\right)\right)|Y, P\right] \\
&= -\exp\left(-\gamma e_i P - \frac{\left(\mathbb{E}\left[X|Y\right] - P\right)^2}{\mathrm{var}\left(X|Y\right)} + \frac{1}{2}\frac{\left(\mathbb{E}\left[X|Y\right] - P\right)^2}{\mathrm{var}\left(X|Y\right)}\right) \\
&= -\exp\left(-\gamma e_i P - \frac{1}{2}\frac{\left(\mathbb{E}\left[X|Y\right] - P\right)^2}{\mathrm{var}\left(X|Y\right)}\right).
\end{aligned}$$

We evaluate

$$\mathbb{E}\left[-\exp\left(-\gamma e_i P - \frac{1}{2}\frac{\left(\mathbb{E}\left[X|Y\right] - P\right)^2}{\mathrm{var}\left(X|Y\right)}\right)|e_i\right]. \tag{A-1}$$

Expanding, this expression equals

$$\mathbb{E}\left[-\exp\left(-\gamma e_i \mathbb{E}\left[X|Y\right] + \gamma e_i\left(\mathbb{E}\left[X|Y\right] - P\right) - \frac{1}{2}\frac{\left(\mathbb{E}\left[X|Y\right] - P\right)^2}{\mathrm{var}\left(X|Y\right)}\right)|e_i\right].$$

By market clearing,

$$\frac{1}{\gamma}\frac{\mathbb{E}\left[X|Y\right] - P}{\mathrm{var}\left(X|Y\right)} = S + Z,$$

i.e.,

$$\mathbb{E}\left[X|Y\right] - P = \gamma \mathrm{var}\left(X|Y\right)\left(S + Z\right),$$

and so (A-1) equals

$$\mathbb{E}\left[-\exp\left(-\gamma e_i \mathbb{E}\left[X|Y\right]+\frac{\gamma^2 \text{var}\left(X|Y\right)}{2}\left(2e_i\left(S+Z\right)-\left(S+Z\right)^2\right)\right)|e_i\right].$$

Moreover,

$$\mathbb{E}\left[X|Y\right]=\frac{\tau_X \mathbb{E}\left[X\right]+\tau_\epsilon Y}{\tau_X+\tau_\epsilon}=\frac{\tau_\epsilon^{-1}\mathbb{E}\left[X\right]+\tau_X^{-1}Y}{\tau_X^{-1}+\tau_\epsilon^{-1}}=\frac{\left(\text{var}\left(Y\right)-\text{var}\left(X\right)\right)\mathbb{E}\left[X\right]+\text{var}\left(X\right)Y}{\text{var}\left(Y\right)}.$$

Hence (A-1) equals

$$\mathbb{E}\left[\exp\left(-\gamma e_i \mathbb{E}\left[X\right]+\frac{\gamma^2 e_i^2}{2}\frac{\text{var}\left(X\right)^2}{\text{var}\left(Y\right)}+\frac{\gamma^2 \text{var}\left(X|Y\right)}{2}\left(2e_i\left(S+Z\right)-\left(S+Z\right)^2\right)\right)|e_i\right].$$

By the law of total variance,

$$\text{var}\left(X\right)=\text{var}\left(X|Y\right)+\text{var}\left(E\left[X|Y\right]\right)=\text{var}\left(X|Y\right)+\frac{\text{var}\left(X\right)^2}{\text{var}\left(Y\right)}.$$

So (A-1) equals

$$\begin{aligned}
& \mathbb{E}\left[-\exp\left(-\gamma e_i \mathbb{E}\left[X\right]+\frac{\gamma^2 e_i^2}{2}\text{var}\left(X\right)+\frac{\gamma^2 \text{var}\left(X|Y\right)}{2}\left(2e_i\left(S+Z\right)-\left(S+Z\right)^2-e_i^2\right)\right)|e_i\right] \\
= \; & \mathbb{E}\left[-\exp\left(-\gamma e_i X-\frac{\gamma^2 \text{var}\left(X|Y\right)}{2}\left(e_i-\left(S+Z\right)\right)^2\right)|e_i\right].
\end{aligned}$$

This expression is increasing in $\text{var}\left(X|Y\right)$, completing the proof.