

# Slowly Unfolding Disasters

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## Abstract

We develop a model that endogenously generates slowly unfolding disasters not only in the macroeconomy, but also in financial markets. Due to imperfect information, disaster periods in the model are not fully identified by investors *ex ante* at the onset, but *ex post* using the peak-to-trough approach as in the data. Bayesian learning leads to a gradual reaction of equity prices to persistent consumption declines in periods of disasters. We show that modeling realistic equity dynamics during disasters is crucial to explaining the VIX, variance risk premium, and risk premia on put-protected portfolios, addressing the shortcomings of traditional disaster risk models.

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# 1 Introduction

Do investors fear the possibility of another Great Depression? This question encapsulates how models with rare economic disasters account for the high equity premium and the low risk-free rate in the postwar period (Rietz, 1988; Barro, 2006). Typically, disaster risk models assume a small probability of an instantaneous drop in aggregate consumption, whose frequency and size distribution are calibrated to match large peak-to-trough consumption declines compiled across various countries (Barro and Ursúa, 2008).

A significant challenge to Rietz-Barro type models with instantaneous disasters is that disasters in the historical data unfold slowly over multiple periods of time not only in the macroeconomy, but also in financial markets.<sup>1</sup> Granted, it is possible to assume slowly unfolding macroeconomic disasters by introducing a persistent low-probability regime with negative growth rates (Nakamura, Steinsson, Barro, and Ursúa, 2013).<sup>2</sup> However, even in such a model, the stock market fully responds instantaneously as soon as the economy enters the disaster regime, due to the forward-looking nature of equity. Once investors become aware of being in the disaster state, they immediately incorporate the low growth expectations into prices, which leads to a large instantaneous price decline in the market.

We argue that this inability to capture gradual stock market declines during disaster periods is actually what is behind the criticism on disaster risk raised by Welch (2016) and recently re-emphasized by Cochrane (2017). Welch (2016) observes that a one-month put-protected strategy with 85% moneyness cannot lose more than 15% in any month, and therefore, should not bear a high risk premium if large instantaneous price declines during disasters are the true source of the equity premium. However, Welch (2016) finds that the

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<sup>1</sup>Barro and Ursúa (2017) document that macroeconomic disasters, defined in terms of peak-to-trough declines, last for a varying number of years with an average of 4.1 years. Barro and Ursúa (2017) also find that stock market disasters, similarly defined as large peak-to-trough declines, unfold slowly with an average duration of 3.2 years.

<sup>2</sup>Other examples of multi-period disaster models include Gourio (2012), Tsai and Wachter (2015), and Petrosky-Nadeau, Zhang, and Kuehn (2018).

put-protected portfolio still earns a premium close to the aggregate equity premium. We claim that this failure is not due to the rare disaster mechanism itself. To illustrate, Figure 1 plots the monthly price declines in the CRSP value-weighted portfolio during the Great Depression (September 1929 to June 1932). While this period represents a cumulative stock market decline of 86%, we observe only a few months with a monthly decline larger than 15%. As a result, rolling over a one-month put option does not provide much insurance against the Great Depression, resulting in a -80% cumulative return, even ignoring the insurance cost. This number is in stark contrast with a loss of 15%, the maximum loss of the put-protected portfolio if the Great Depression were an isolated instantaneous drop as in Rietz-Barro type models. That is, the real issue at hand is the behavior of the stock market during disaster periods, which in turn determines how the possibility of disasters affects asset prices in normal times.

In this paper, we develop a model that generates slowly unfolding disasters both in the macroeconomy and in financial markets. In our model, disaster periods are not fully identifiable by investors *ex ante* at the onset, but only identified *ex post* using the peak-to-trough approach as in the data. The consumption jump intensity follows an autoregressive process where its long-run mean is subject to a rare but persistent shift from its low value in the “normal” state to a very high value in the “depression” state. Such an increase can cause a potentially extended period of negative consumption growth due to more frequent negative jump realizations that cumulatively constitute a macroeconomic disaster. We assume that the representative investor with recursive utility is fully informed about the probability of a jump within the next month. However, the long-run mean of the jump intensity is not observable. Therefore, the agent does not know with certainty whether a high instantaneous jump intensity is due to a transitory increase in risk or due to a persistent shift to the depression state that may result in a prolonged disaster period.

The information structure in our model deviates from existing models and generates a slowly unfolding stock market response to consumption disasters. We use standard Bayesian

learning to model the agent's assessment of the probability of being in the depression state. Hence, a long period of high jump risk leads to a higher posterior probability of being in the depression state. This probability is a state variable itself for asset prices, and depends on the path of instantaneous jump risk that the investor observes. Even after the economy enters the depression state, it takes time for the investor to recognize this and change her belief accordingly. As a result, equity prices react to persistent declines in consumption slowly.

A prime example that illustrates the economic intuition behind our model's mechanism is the bankruptcy of Lehman Brothers in 2008, which was arguably the most terrifying moment in the U.S. economy since the Great Depression. It seems reasonable to assume that investors were aware of an increase in immediate economic risk upon Lehman's default. However, investors did not know with certainty if this event would trigger a prolonged economic crisis with a similar severity as the Great Depression, or if the economy would recover from the crisis more quickly. In our model, the first scenario is represented by a regime shift to the depression state, and the latter by a transitory increase in the jump intensity. The 2008 financial crisis did not turn out to become a macroeconomic disaster like the Great Depression, but potentially it could have. More importantly, at that moment, investors were not able to know with certainty whether or not it would.<sup>3</sup>

While our model differs from existing disaster risk models in its mechanism and its empirical implications, it is possible to nest them within our framework. To illustrate these differences quantitatively, we calibrate our model and compare it with the time-varying disaster risk model of Wachter (2013), in which stock prices decline instantaneously at the onset of macroeconomic disasters. Both models are consistent with standard asset pricing moments, such as the equity premium, the risk-free rate, stock market volatility, and the predictability of excess returns as well as consumption and dividend moments.

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<sup>3</sup>One can equivalently assume that investors that observed the stock market crash of 20% in October 1929 were not aware that the economy was entering the Great Depression period. Therefore, the cumulative stock market decline of 86% had not been realized at the onset.

However, a clear distinction between the two models arises when their implications for put-protected portfolios of Welch (2016) and the variance risk premium are examined. Unlike instantaneous disaster risk models, our model is capable of accounting for the risk premia on put-protected strategies with various moneyness values ranging from 75% to 90%. This is attributable to the slowly unfolding nature of disasters in the stock market, which our model is able to capture through the realistic identification of disaster episodes. The source of the equity premium in our model is not the prospect of an instantaneous disaster but the variation in the anticipation of experiencing a prolonged disaster period in the future. We also discover that the special case of our model with perfect information fails to explain the risk premia on these portfolios, highlighting the importance of information frictions in generating realistic stock price dynamics during disaster times.

We also show that disaster risk models with instantaneous equity price declines during disasters imply unrealistically high values of the variance risk premium, and hence, the VIX. This is because abrupt drops in prices give rise to an unrealistically high quadratic variation, which increases the expected payoffs from out-of-the-money put options and variance swaps to extreme levels. In our model, the realistic behavior of the equity price path during disasters produces a level of the VIX and the variance risk premium that is consistent with the data. In sum, a model that accurately depicts how disasters unfold in the macroeconomy and financial markets is crucial to establishing the consistency of short-term contingent claim prices with the rare disaster mechanism. While instantaneous rare disaster models are inconsistent with put-protected portfolio premia, the VIX, and the variance risk premium, we show that this does not invalidate the mechanism itself. Instead, we emphasize the importance of modeling the realistic joint behavior of consumption and financial markets during disasters.

A distinctive feature of our model is that the size distribution of consumption disasters is a model outcome, not a model input. In our model, negative jumps in consumption are calibrated to be much smaller compared to what a typical disaster risk model assumes. However, huge peak-to-trough declines in consumption can still be endogenously generated,

collectively from small jumps in the depression regime. Model simulations show that if disasters are identified *ex post* using the peak-to-trough approach of Barro and Ursúa (2008), we obtain an average consumption decline of 18% and an average duration of 4.5 years for disaster periods. These results are fairly close to what we observe in the data (21% and 4.1 years). This suggests that our model is immune to the common criticism that disaster risk may overstate consumption risk by treating a peak-to-trough decline in consumption as if it happened within a unit period (Constantinides, 2008). Our resolution of this criticism is rooted in recognizing that disasters in the data unfold slowly both in consumption and in equity prices.

Our paper contributes to the literature on rare economic disasters. Since its introduction by Rietz (1988) and Barro (2006), the rare disaster mechanism has been extended and refined to explain various aspects of macroeconomic and financial data. Gabaix (2012), Gourio (2012), and Wachter (2013) incorporate variable disaster risk to explain dynamic patterns in the data, such as high stock market volatility in normal times and significant return predictability by valuation ratios.<sup>4</sup> Nakamura, Steinsson, Barro, and Ursúa (2013) consider multi-period disasters followed by periods of fast recovery to produce a more realistic representation of consumption dynamics. Hasler and Marfe (2016) demonstrate that considering disaster recovery in disaster models helps generate a downward sloping term structure of equity risk premia and an upward sloping term structure of interest rates. Berkman, Jacobsen, and Lee (2011) and Manela and Moreira (2017) provide empirical evidence supporting the significance of time-varying disaster risk for asset pricing. Our work shows that the apparent shortcomings of the rare disaster mechanism can be resolved by modeling slowly unfolding financial market disasters, as in the data.

Due to the rare nature of economic disasters, incorporating information frictions and investors' learning is particularly relevant for disaster risk models.<sup>5</sup> Gillman, Kejak, and Pakoš

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<sup>4</sup>Related, Farhi and Gabaix (2015) show that their open economy model with variable disaster risk can address a series of puzzles in exchange rates.

<sup>5</sup>Our paper is also related to the vast literature that studies implications of investors' learning for asset

(2014) show that uncertainty about the length of a consumption disaster can address important features of equity and bond market data. Wachter and Zhu (2019) consider a model in which investors learn about the current probability of disasters from past realizations of disasters. In our paper, we do not just focus on the implications of investors' learning for asset prices but for the joint dynamics of financial markets and the macroeconomy.

In recent work, Collin-Dufresne, Johannes, and Lochstoer (2016) investigate the role of parameter uncertainty in explaining standard asset pricing moments. As one exercise, the authors evaluate how uncertainty about each parameter in a two-state disaster risk model contributes to the equity premium. Consequently, they conclude that particularly important is imperfect information about the transition probability from the disaster state to the normal state, which determines the duration of a disaster. Similar to their work, we also highlight investors' uncertainty about how long and severe future economic disasters will be. However, we create this uncertainty through a different mechanism: in our model, investors are unaware of the true economic state, not the model parameters. Therefore, unlike the model of Collin-Dufresne, Johannes, and Lochstoer (2016), investors cannot tell whether the economy is on the verge of a disaster. This imperfect information plays a key role in generating slow responses of equity prices to persistent consumption declines, which helps us account for several dimensions of the data including forward looking volatility, the variance risk premium, and put-protected equity index returns.

The rest of the paper proceeds as follows. Section 2 describes our disaster risk model with learning. Section 3 discusses the model calibration procedure. Section 4 provides results from the model and compares them with the data. Section 5 concludes.

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pricing in different contexts. See, for example, Timmermann (1996), Veronesi (2000), Brandt, Zeng, and Zhang (2004), Veronesi (2004), Lettau, Ludvigson, and Wachter (2008), Pastor and Veronesi (2009a), Pastor and Veronesi (2009b), Benzoni, Collin-Dufresne, and Goldstein (2011), Ju and Miao (2012), and Johannes, Lochstoer, and Mou (2016).

## 2 Model

### 2.1 Model Setup

We consider an infinitely-lived representative agent in an endowment economy with complete markets. The agent has recursive preferences of Epstein and Zin (1989) and Weil (1989), which leads to the following stochastic discount factor  $M_{t+1}$ :

$$M_{t+1} = \exp\left(\theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1)r_{c,t+1}\right), \quad (1)$$

where  $\Delta c_{t+1} = \log\left(\frac{C_{t+1}}{C_t}\right)$  represents the logarithm of aggregate consumption growth and  $r_{c,t+1}$  denotes the log return on the consumption claim. The coefficient  $\theta = \left[\frac{1-\gamma}{1-1/\psi}\right]$  captures the agent's attitudes toward the timing of uncertainty resolution, and the parameters  $\delta$ ,  $\psi$ , and  $\gamma$  are the rate of time preference, the elasticity of intertemporal substitution (EIS), and relative risk aversion, respectively.

We assume that aggregate consumption growth evolves according to the following process:

$$\Delta c_{t+1} = \mu_c + \sigma_c \epsilon_{t+1}^c + J_{t+1}, \quad \text{where } \epsilon_{t+1}^c \stackrel{i.i.d.}{\sim} N(0, 1). \quad (2)$$

In other words, log-consumption growth is subject to a constant drift  $\mu_c$ , an i.i.d. normal shock with a standard deviation of  $\sigma_c$ , and a compound Poisson jump  $J_{t+1} = \left[\sum_{j=1}^{N_{t+1}} Z_j\right]$ . Each jump  $Z_j$  has a time-invariant distribution whose moment generating function is denoted as  $\Phi_Z(u) = \mathbb{E}[e^{uZ}]$ . The Poisson process  $N_{t+1}$  counts the number of jumps between times  $t$  and  $t+1$  and depends on time-varying intensity  $\lambda_t$ . Similar to Wachter (2013), this intensity is observable and follows a discrete-time version of a mean-reverting square root process:

$$\lambda_{t+1} = a_{\lambda,t+1} + \rho_\lambda \lambda_t + \sigma_\lambda \sqrt{\lambda_t} \epsilon_{t+1}^\lambda, \quad \text{where } \epsilon_{t+1}^\lambda \stackrel{i.i.d.}{\sim} N(0, 1). \quad (3)$$

Our setup exhibits two important deviations from the model of Wachter (2013). First,

in line with Seo and Wachter (2018a), we allow the long-run mean of  $\lambda_t$  to vary over time. For parsimony, we assume that the variable  $a_{\lambda,t+1}$ , which determines the long-run mean of the Poisson intensity at time  $t + 1$ , takes a low value  $a_L$  during “normal” times and a high value  $a_H$  during “depression” times. By introducing a Markov-switching process  $s_{t+1}$ , which switches back and forth between the values of zero (the normal regime) and one (the depression regime), we can express  $a_{\lambda,t+1} = [(1 - s_{t+1})a_L + s_{t+1}a_H]$ . Moreover, we assume that the value of  $a_{\lambda,t+1}$  is unobservable, which clearly distinguishes our model from conventional disaster risk models. In other words, the agent has imperfect information about the stochastic long-run mean of the jump intensity process and, therefore, tries to learn about it.

A subtle yet critical distinction also exists in the calibration and interpretation of  $\lambda_t$ . In models with instantaneous disasters, this jump intensity represents the risk of disasters. Therefore,  $\lambda_t$  is calibrated to be very small, reflecting the rarity of economic disasters, and the jump size  $Z_j$  is calibrated to be large, capturing severe declines in consumption during disasters. In contrast, we do not interpret the jump intensity itself as disaster risk, but as crash risk:  $\lambda_t$  and  $Z_j$  are calibrated so that they represent more frequent but much less severe negative shocks.<sup>6</sup> A large drop in aggregate consumption can still be generated under this calibration when the economy experiences a rare transition from the normal regime to the depression regime and remains there for an extended period of time. In this case, the jump intensity  $\lambda_t$  mean-reverts to a higher value, which leads to higher jump probabilities for multiple periods. While each realization of jumps is relatively small in size, it can accumulate over time and can collectively constitute a consumption disaster.

In our model, it is not just consumption that falls in a slow manner; the stock market does so as well. Even if the economy switches to the depression regime, this information is hidden from the agent; she does not know for certain whether a high value of  $\lambda_{t+1}$  is

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<sup>6</sup>As further discussed in Section 3, we exploit the findings of the option pricing literature to calibrate the jump intensity and the size distribution.

attributable to a transitory shock  $\epsilon_{t+1}^\lambda$  or attributable to a persistent shift to the depression regime. Therefore, the agent gradually updates her belief about the current state of the economy based on Bayes' rule. Consequently, the stock market reacts to a consumption disaster in a slow manner, rather than sharply declining at once. As we demonstrate in Section 4, this feature is critical for the model in producing a reasonable level of the VIX, the variance risk premium, and put-protected strategies with various moneyness values.

## 2.2 Learning

As mentioned earlier, we assume that the representative agent is a Bayesian learner who updates her belief about the true state of the economy  $s_t$  by observing  $\lambda_t$  and its past history.<sup>7</sup> With a slight abuse of notation, let  $\lambda_{-\infty:t}$  denote the past time series of jump intensities up until time  $t$ . We define the agent's time- $t$  belief that the true state of the economy is in the depression regime as

$$\pi_t \equiv \pi_{t|t} = P(s_t = 1 | \lambda_{-\infty:t}).$$

This belief is updated to  $\pi_{t+1}$  when the agent observes the new jump intensity  $\lambda_{t+1}$  at time  $t + 1$ . It follows from Bayes' rule and the law of total probability that the dynamics of this belief updating process can be described by

$$\pi_{t+1} = P(s_{t+1} = 1 | \lambda_{-\infty:t+1}) = \frac{P(\lambda_{t+1} | s_{t+1} = 1, \lambda_{-\infty:t}) P(s_{t+1} = 1 | \lambda_{-\infty:t})}{\sum_{s \in \{0,1\}} P(\lambda_{t+1} | s_{t+1} = s, \lambda_{-\infty:t}) P(s_{t+1} = s | \lambda_{-\infty:t})}.$$

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<sup>7</sup>In our model, the agent does not learn from observing consumption growth. At time  $t$ ,  $\lambda_t$  is observable by the agent and this completely determines the conditional distribution of log consumption growth  $\Delta c_{t+1}$ .

Dividing both the numerator and the denominator by the numerator results in the following expression:

$$\pi_{t+1} = \left[ 1 + \underbrace{\frac{P(\lambda_{t+1}|s_{t+1} = 0, \lambda_{-\infty:t})}{P(\lambda_{t+1}|s_{t+1} = 1, \lambda_{-\infty:t})}}_{\substack{\text{(i) belief update due to} \\ \text{a new shock to } \lambda_{t+1}}} \times \underbrace{\frac{P(s_{t+1} = 0|\lambda_{-\infty:t})}{P(s_{t+1} = 1|\lambda_{-\infty:t})}}_{\substack{\text{(ii) belief update due} \\ \text{to transition dynamics}}} \right]^{-1}. \quad (4)$$

Equation (4) makes it explicit that belief updating comes from two sources. First, the agent updates her belief based on a new shock to the jump intensity. Due to imperfect information, the agent is incapable of exactly determining whether this shock originates from  $\epsilon_{t+1}^\lambda$  or from  $a_{\lambda,t+1}$ . A high value of  $\lambda_{t+1}$  can be due to a large transitory shock  $\epsilon_{t+1}^\lambda$  or due to a high value of  $a_{\lambda,t+1}$  under the depression regime. Therefore, the agent considers which scenario is more likely: if the new value of the jump intensity is so much larger than its previous value that it is more likely to be observed under the depression regime, the value of expression (i) in equation (4) becomes smaller, which, in turn, raises  $\pi_{t+1}$ . In contrast, if a new shock to the jump intensity is small enough to make it more likely to occur under the normal regime, the agent decreases her belief about the depression regime. Specifically, expression (i) reduces to

$$\frac{P(\lambda_{t+1}|s_{t+1} = 0, \lambda_{-\infty:t})}{P(\lambda_{t+1}|s_{t+1} = 1, \lambda_{-\infty:t})} = \exp\left(-\frac{(a_H - a_L)(\lambda_{t+1} - \rho_\lambda \lambda_t - \frac{a_H + a_L}{2})}{\sigma_\lambda^2 \lambda_t}\right), \quad (5)$$

because conditional on knowing the true regime  $s_{t+1}$ ,  $\lambda_{t+1}$  follows a normal distribution with a mean of  $[(1 - s_{t+1})a_L + s_{t+1}a_H] + \rho_\lambda \lambda_t$  and a variance of  $[\sigma_\lambda^2 \lambda_t]$ .

We identify three parameters that control the speed of learning from new information. Equation (5) suggests that  $\pi_{t+1}$  becomes more sensitive to a new shock to  $\lambda_{t+1}$  when  $[a_H - a_L]$  is larger,  $\sigma_\lambda$  is smaller, and  $\rho_\lambda$  is smaller. This is intuitive. A large magnitude of  $[a_H - a_L]$  makes it easier to distinguish between the two regimes. Similarly, it is easier to detect the depression state when the volatility of a transitory shock  $\sigma_\lambda$  is smaller. Lastly, as  $\rho_\lambda$  gets

closer to zero, the jump intensity process mean-reverts to its long-run mean at a faster pace, which helps the agent determine whether the jump intensity process is headed toward the long-run mean under the normal regime or the one under the depression regime.

The second source of belief updating is the transition dynamics of  $s_{t+1}$ . Even without any news from the jump intensity, the agent still updates her belief because the true state of the economy alternates between the normal and depression regimes exogeneously according to the transition probabilities  $p_{ss'} = P(s_{t+1} = s' | s_t = s)$ . If the economy is in the normal regime (depression regime), the probability of staying in the same regime in the next period is  $p_{00}$  ( $p_{11}$ ) and the probability of switching to the other regime is  $p_{01}$  ( $p_{10}$ ). Reflecting these transition dynamics, the belief at time  $t$  evolves into the belief at time  $t + 1$  even when there is no extra information from  $\lambda_{t+1}$ .<sup>8</sup>

This is essentially what expression (ii) in equation (4) represents: given today's belief  $\pi_t$ , if the transition dynamics suggest that the depression regime is more likely at time  $t + 1$ , expression (ii) becomes smaller, and thus,  $\pi_t$  becomes larger. It follows from the law of total probability that the time- $t$  conditional probability of being in the depression regime at time  $t + 1$  equals

$$\begin{aligned} \pi_{t+1|t} \equiv P(s_{t+1} = 1 | \lambda_{-\infty:t}) &= p_{01}P(s_t = 0 | \lambda_{-\infty:t}) + p_{11}P(s_{t+1} = 1 | \lambda_{-\infty:t}) \\ &= 1 - p_{00} + (p_{00} + p_{11} - 1)\pi_t. \end{aligned} \tag{6}$$

We define this conditional probability as  $\pi_{t+1|t}$  in the above equation. Note that  $P(s_{t+1} = 0 | \lambda_{-\infty:t})$  is simply  $[1 - \pi_{t+1|t}]$ .

Finally, plugging equations (5) and (6) into equation (4) explicitly shows how the agent's

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<sup>8</sup>In the absence of new information from  $\lambda_{t+1}$ ,  $\pi_t$  mean-reverts to the belief in the steady state:  $\mathbb{E}[s_t] = [(1 - p_{00}) / (2 - p_{00} - p_{11})]$ .

belief evolves over time:

$$\pi_{t+1} = \left[ 1 + \exp \left( - \frac{(a_H - a_L) (\lambda_{t+1} - \rho_\lambda \lambda_t - \frac{a_H + a_L}{2})}{\sigma_\lambda^2 \lambda_t} \right) \frac{p_{00} - (p_{00} + p_{11} - 1)\pi_t}{1 - p_{00} + (p_{00} + p_{11} - 1)\pi_t} \right]^{-1}. \quad (7)$$

Namely, the agent's future belief ( $\pi_{t+1}$ ) is a function of the current belief ( $\pi_t$ ) as well as the current and future values of the jump intensity ( $\lambda_t$  and  $\lambda_{t+1}$ ).

### 2.3 Solving the Model

We first solve for the wealth-consumption ratio. Define  $P_{c,t}$  as the price of the consumption claim, and let  $pc_t = \log(P_{c,t}/C_t)$  denote the log wealth-consumption ratio. Under this notation, the log return on the consumption claim is expressed as  $r_{c,t+1} = [\log(1 + e^{pc_{t+1}}) - pc_t + \Delta c_{t+1}]$ . In equilibrium, this return should satisfy the following Euler equation:

$$\mathbb{E}_t [\exp(\log M_{t+1} + r_{c,t+1})] = 1,$$

where  $\mathbb{E}_t$  denotes the expectation conditional on the agent's time- $t$  information set. In Appendix A, we show that this Euler equation leads to the following recursive relation between  $pc_t$  and  $pc_{t+1}$ :

$$pc_t = \frac{1}{\theta} \left[ \theta \log \delta + (1 - \gamma)\mu_c + \frac{1}{2}(1 - \gamma)^2 \sigma_c^2 + \lambda_t [\Phi_Z(1 - \gamma) - 1] + \log \mathbb{E}_t \left[ (1 + e^{pc_{t+1}})^\theta \right] \right]. \quad (8)$$

Due to the Markov property,  $pc_t$  is a function of  $\lambda_t$  and  $\pi_t$ , namely  $pc_t = pc(\lambda_t, \pi_t)$ . However, a closed-form expression for this function does not exist in our model due to the nonlinearity of the learning dynamics. Following Lettau, Ludvigson, and Wachter (2008), we numerically find this function over a two-dimensional grid of  $\lambda$  and  $\pi$ . The key idea is that  $pc(\lambda, \pi)$  is a fixed point of equation (8). Under the current values of  $pc$ , we can find the “new” value of  $pc$  for each set of  $(\lambda, \pi)$  by calculating the right-hand side of equation (8).

This is straightforward because

$$\mathbb{E}_t \left[ (1 + e^{pc_{t+1}})^\theta \right] = \sum_{s \in \{0,1\}} P(s_{t+1} = s | \lambda_{-\infty:t}) \mathbb{E} \left[ (1 + e^{pc(\lambda_{t+1}, \pi_{t+1})})^\theta | s_{t+1} = s, \lambda_t, \pi_t \right],$$

where  $P(s_{t+1} = 0 | \lambda_{-\infty:t}) = 1 - \pi_{t+1|t}$  and  $P(s_{t+1} = 1 | \lambda_{-\infty:t}) = \pi_{t+1|t}$ .<sup>9</sup> We continue updating the values of  $pc$  by repeating this procedure until the function converges and finds the fixed point  $pc$ .<sup>10</sup>

Now we turn to a claim that pays the aggregate dividend. Following Abel (1990), we assume that the aggregate dividend is levered consumption  $D_t = C_t^\phi$ . We define the price of the dividend claim as  $P_{d,t}$  and the log price-dividend ratio as  $pd_t = \log(P_{d,t}/D_t)$ . Then, the log return on the dividend claim (or equity) is expressed as  $r_{d,t+1} = [\log(1 + e^{pd_{t+1}}) - pd_t + \phi \Delta c_{t+1}]$ . It follows from the Euler equation that the log price-dividend ratio satisfies the following equation:

$$pd_t = \theta \log \delta + (\phi - \gamma)\mu_c + \frac{1}{2}(\phi - \gamma)^2 \sigma_c^2 + \lambda_t [\Phi_Z(\phi - \gamma) - 1] - (\theta - 1)pc_t + \log \mathbb{E}_t \left[ (1 + e^{pc_{t+1}})^{\theta-1} (1 + e^{pd_{t+1}}) \right]. \quad (9)$$

See Appendix A for more details. Since the function  $pc$  is already known, equation (9) recursively characterizes the function for the log price-dividend ratio. We apply the same numerical procedure we use for the wealth-consumption ratio to find  $pd_t = pd(\lambda_t, \pi_t)$  as a fixed point of equation (9).

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<sup>9</sup>Given  $s_{t+1}$ , conditioning on the agent's full information set simply reduces to conditioning on  $\lambda_t$  and  $\pi_t$  because these two variables, together with  $s_{t+1}$ , fully determine the conditional distribution of  $(\lambda_{t+1}, \pi_{t+1})$ .

<sup>10</sup>To assure high precision, we use a very fine grid with 20,000 grid points. We speed up the convergence toward the fixed point by applying the secant method to each grid point, which results in a substantial reduction in the number of iterations.

## 2.4 Conditional Moments and Option Prices

Once we obtain the wealth-consumption ratio and the price-dividend ratio as functions of  $\lambda_t$  and  $\pi_t$ , various conditional moments can be computed in a semi-analytical way. Typically, calculating time- $t$  conditional moments requires computing a time- $t$  conditional expectation of a function with the following form:

$$F_{t+1} = F\left(\Delta c_{t+1}, \lambda_{t+1}, \pi_{t+1}, pc(\lambda_{t+1}, \pi_{t+1}), pd(\lambda_{t+1}, \pi_{t+1})\right).$$

As illustrated in Section 2.3, the conditional expectation of this general function can be written as

$$\mathbb{E}_t[F_{t+1}] = \sum_{s \in \{0,1\}} P(s_{t+1} = i | \lambda_{-\infty:t}) \mathbb{E}[F_{t+1} | s_{t+1} = s, \lambda_t, \pi_t]. \quad (10)$$

Conditional on  $(s_{t+1}, \lambda_t, \pi_t)$ , the distribution of  $F_{t+1}$  is completely characterized by the distributions of  $\epsilon_{t+1}^\lambda$  and  $\Delta c_{t+1}$ . Specifically, under this conditioning, a shock to the jump intensity  $\epsilon_{t+1}^\lambda$  pins down  $\lambda_{t+1}$  and  $\pi_{t+1}$  and, thus,  $pc_{t+1}$  and  $pd_{t+1}$ , as can be seen in equations (3) and (7). Therefore, the value of the expectation  $\mathbb{E}[F_{t+1} | s_{t+1} = s, \lambda_t, \pi_t]$  in equation (10) can be found using a double integral with respect to  $\epsilon_{t+1}^\lambda$  and  $\Delta c_{t+1}$ .<sup>11</sup> Note that the conditional density of  $\lambda_{t+1}$  is simply the standard normal density function. In Appendix B, we show that the conditional density of  $\Delta c_{t+1}$  can be derived in terms of the Gauss error function, which is, in turn, expressed in terms of a normal distribution function.

As our first example, we calculate the risk-free rate  $r_{f,t}$ . The Euler equation implies that

$$\begin{aligned} r_{f,t} &= -\log \mathbb{E}_t[M_{t+1}] \\ &= -\theta \log \delta + \gamma \mu_c - \frac{1}{2} \gamma^2 \sigma_c^2 - \lambda_t [\Phi_Z(-\gamma) - 1] + (\theta - 1) pc_t - \log \mathbb{E}_t \left[ (1 + e^{pc_{t+1}})^{\theta-1} \right]. \end{aligned}$$

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<sup>11</sup>Alternatively, this expectation can also be estimated quickly using Monte Carlo simulations.

The expression  $\left[(1 + e^{pc_{t+1}})^{\theta-1}\right]$  is a special case of  $F_{t+1}$  in which the function does not depend on  $\Delta c_{t+1}$ . Hence, its conditional expectation is simply computed as a one-dimensional integral with respect to  $\epsilon_{t+1}^\lambda$ . Similarly, the time- $t$  expected equity return is also calculated as a one-dimensional integral because

$$\mathbb{E}_t [r_{d,t+1}] = \mathbb{E}_t [\log(1 + e^{pd_{t+1}})] - pd_t + \phi\mu_c + \phi\mu_Z\lambda_t,$$

where  $\mu_Z = \Phi'_Z(0)$  is the mean jump size.

Equation (10) allows us to calculate not only the first moments of returns, but also their higher moments. For instance, the conditional equity return variance is calculated as:

$$\text{Var}_t(r_{d,t+1}) = \mathbb{E}_t \left[ \left( \log(1 + e^{pd_{t+1}}) - pd_t + \phi\Delta c_{t+1} - \mathbb{E}_t[r_{d,t+1}] \right)^2 \right].$$

That is, it is possible to semi-analytically calculate any moment by first expressing it in terms of a time- $t$  conditional expectation and applying the general formula derived in equation (10).

Lastly, we consider put option prices, as our analyses in Section 4 concern put-protected strategies. It follows from the pricing relation that  $O_t$ , the one-month put option price with strike price  $K$ , equals

$$O_t(K) = \mathbb{E}_t [M_{t+1} (K - P_{d,t+1})^+].$$

By dividing both sides of the equation by the equity price  $P_{d,t}$ , we obtain the expression for the normalized option price  $O_t^n = O_t/P_{d,t}$ :

$$O_t^n(k) = \mathbb{E}_t \left[ e^{\theta \log \delta - \gamma \Delta c_{t+1} + (\theta-1)[\log(1+e^{pc_{t+1}}) - pc_t]} \left( k - e^{pd_{t+1} - pd_t + \phi \Delta c_{t+1}} \right)^+ \right], \quad (11)$$

where  $k = K/P_{d,t}$  is the moneyness of the option. Clearly, the expression inside the conditional expectation in equation (11) is a special case of  $F_{t+1}$ . Therefore, the normalized option

price  $O_t^n(k) = O^n(k; \lambda_t, \pi_t)$  can also be calculated using the general formula in equation (10).

### 3 Calibration

We calibrate the model at a monthly frequency. Our calibration of the preference parameters is standard. We set risk aversion  $\gamma$  to 5, which is sufficiently lower than 10, consistent with Mehra and Prescott (1985) and subsequent papers in the equity premium puzzle literature. The EIS  $\psi$  is chosen as 1.5, implying that the representative agent prefers early resolution of uncertainty (see, e.g., Bansal, Kiku, and Yaron, 2012). The time discount factor  $\delta$  is 0.999, which is equivalent to an annual rate of time preference of 1.2% in Wachter (2013).

We calibrate the transition dynamics, characterized by  $p_{10}$  and  $p_{01}$ , by targeting the duration of historical consumption disasters in the U.S. The average duration of consumption disasters in the data is roughly 4 years. This suggests that it is reasonable to choose  $p_{10}$ , the transition probability from the depression regime to the normal regime, equal to  $0.25\Delta t$ , where  $\Delta t = 1/12$ . Then, we set  $p_{01}$ , the transition probability from the normal regime to the depression regime, to 2% per annum ( $0.02\Delta t$  monthly) so that the unconditional probability of a year being in the depression state  $[p_{01}/(p_{01} + p_{10})]$  is approximately 7%. This is consistent with what we observe in the U.S. consumption time series: over the last 185 years, the U.S. experienced three disasters with a total duration of 13 years, indicating that the fraction of disaster periods relative to the total years considered is  $(13/185) = 7.02\%$ . Note that Barro and Ursúa (2008) adopt the same approach to calibrate these switching probabilities, yet with international data, which implies a higher disaster probability (3.63%), mainly due to a much larger portion of disaster years (12%). By focusing on U.S. consumption disasters, we pick  $p_{01}$  conservatively in an attempt to make it clear that our model's success is not driven by an overstatement of disaster likelihood.<sup>12</sup>

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<sup>12</sup>In our model, shifting to the depression state does not necessarily mean an occurrence of a disaster; the economy can quickly switch back to the normal state, resulting in a consumption drop that is less than 10% in the peak-to-trough sense. Therefore, the unconditional probability of disasters is even lower than 2%

Now we turn to the calibration of jump risk in the model. We assume that the size of each log consumption jump  $Z_j$  follows the negative of an exponential distribution with mean  $\mu_Z$ . Note that in our model, dividends are defined as levered consumption, which causes dividends and equity prices to drop by  $\phi\mu_Z$ , on average, in response to jumps in consumption. This relation enables us to calibrate  $\mu_Z$  using the results from the option pricing literature, rather than relying on the historical time series of consumption. Consistent with Eraker (2004), we target -6% stock market jumps that occur every other year on average during non-disaster periods.<sup>13</sup> We choose  $\phi = 3$ , which is standard in the literature (see, e.g, Bansal and Yaron, 2004). As a result, the mean consumption jump size  $\mu_Z$  is calibrated as -2%.

The persistence of the jump intensity process is determined by the autoregressive coefficient  $\rho_\lambda$  in equation (3). We follow Wachter (2013) and set the parameter equal to  $[1 - 0.08\Delta t]$ , which corresponds to a mean reversion rate of 8%. Given  $\rho_\lambda$ , choosing  $a_L$  and  $a_H$  is equivalent to choosing  $\lambda_L = \left[\frac{a_L}{1-\rho_\lambda}\right]$  and  $\lambda_H = \left[\frac{a_H}{1-\rho_\lambda}\right]$ , the long-run means of the jump intensity under the two regimes. As discussed above, we set  $\lambda_L = 0.5\Delta t$  so that, on average, the equity market experiences negative jumps once in 2 years under the normal regime.

This leaves us two free parameters:  $\lambda_H$  and  $\sigma_\lambda$ . We calibrate these two parameters to match the equity premium and the stock market volatility in the data. Specifically, under our calibration, the long-run mean of the jump intensity increases tenfold when the economy falls into the depression state ( $\lambda_H = 5\Delta t$ ). Lastly, based on the calibrated jump process, we pick the values of  $\mu_c$  and  $\sigma_c$  so that the model matches the postwar mean and volatility of log consumption growth.

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under our calibration.

<sup>13</sup>In the option pricing literature, the estimated jump size ranges from -2% to -10%. Typically, when the estimated jump size is small, the frequency of jumps is estimated to be high (2-3 times a year). When the estimated jump size is large, the frequency of jumps is low (once in 2-3 years). Examples include, but are not limited to, Eraker, Johannes, and Polson (2003), Eraker (2004), and Broadie, Chernov, and Johannes (2007).

## 4 Model Results

In this section, we present and discuss the quantitative implications of our model. Section 4.1 describes the model mechanism based on a simulated sample path and examines the properties of the equilibrium solution. Section 4.2 evaluates our model’s implications for standard asset pricing moments as well as the VIX, the variance risk premium, and put-protected portfolio premia. Finally, Section 4.3 delineates the characteristics of disaster realizations in our model and compares them to the data.

### 4.1 Inspecting the Model Mechanism

How does our model generate slowly unfolding disasters in consumption and in equity prices? In Figure 2, we illustrate the mechanism using a sample path of the model that includes a consumption disaster. Panel A plots the path of the true state  $s_t$  and the agent’s belief  $\pi_t$ . Once the economy switches to the depression state with  $s_t = 1$ , instantaneous jump risk represented by  $\lambda_t$  enters an upward trend (Panel B). This is because  $\lambda_t$  now mean-reverts to a higher value under the depression regime, compared to the normal regime.

The agent learns about  $s_t$  from the historical path of  $\lambda_t$ . Yet, the belief dynamics in Panel A indicate that learning is slow and imperfect. It takes more than a year for the agent to raise her subjective probability of being in the depression state to a level close to one. Furthermore, the belief is not stable as  $\lambda_t$  is a noisy signal and sometimes points to the “wrong” direction. For instance, the perceived depression state probability drops below 50% around month 35 when  $\lambda_t$  is hit by a few consecutive negative shocks  $\epsilon_t^\lambda$ .

Panel C of Figure 2 plots the path of annual consumption. Because it takes time for  $\lambda_t$  to increase due to persistence, the consumption disaster starts later than the onset of the depression state. Lastly, Panel D plots the trajectory of the aggregate equity price along the path. The financial market disaster starts prior to the consumption disaster. The trough of the equity price is observed toward the end of the consumption disaster so that the

peak-to-trough duration of the disaster in the equity market is close to the duration of the consumption disaster. This particular disaster represents a cumulative equity index decline of about 80%, similar to the Great Depression, while the consumption decline is about 25%. Once the economy exits the depression state, equity prices start recovering in a slow manner. This is because not only does the jump intensity follow a persistent process, but the speed of learning is also slow for high values of  $\lambda_t$ .

The driver of these realistic equity price dynamics during disasters is the presence of imperfect information. At the beginning of and throughout the disaster period, the representative investor does not know the true state of the economy with certainty. This uncertainty has profound implications for equity prices: while it is a source of risk itself, it drives the extent to which the agent incorporates disaster risk into pricing. If the agent were fully aware of being at the start of a consumption disaster, equity prices would fully react right away. Through the learning mechanism, our model breaks this unrealistic link between a shift to the disaster state and an instant price reaction.

The U.S. economic history features two prime examples in which investors did not immediately recognize the state of the economy at the onset of exceptionally bad periods: the Great Depression and the Great Recession. During the Great Depression, the cumulative stock market decline of over 80% spans roughly 4 years. At the beginning of the Depression in 1929, however, the stock market decline remained much smaller. The reason is arguably because investors did not know *ex ante* that the recession they were experiencing would become as severe as it turned out to be *ex post*. The Great Recession is an opposite example. At the Lehman default in 2008, the market feared the possibility of severe and lengthy economic downturns similar to the Great Depression, which was not quite the case. In sum, at the beginning of a recession, investors face uncertainty about how long and how bad economic downturns will be.

Our model is able to capture this phenomenon. A significant increase in consumption risk in our model can arise due to either a large transitory shock or a shift to the depression

regime. Moreover, even if the economy transitions to the depression regime, it is possible to quickly switch back to the normal regime. These features create uncertainty about the duration and severity of economic recessions, which the representative investor incorporates into equity prices.

Figures 3 and 4 present model-implied quantities as a function of one of the two state variables,  $\lambda_t$  or  $\pi_t$ .<sup>14</sup> Panel A of Figure 3 illustrates that the log wealth-consumption ratio and log price-dividend ratio have an almost linear relationship with  $\lambda_t$ , as in conventional variable disaster risk models such as Wachter (2013). Panel B of Figure 3 shows that these valuation ratios decline as  $\pi_t$  increases, but so does the rate of decrease, making the two graphs convex. This convexity appears because uncertainty about the true state of the economy, which the agent dislikes under the model's preference configuration, is highest when  $\pi_t$  is 50%.<sup>15</sup> As a result, for small values of  $\pi_t$ , the valuation ratios decrease with  $\pi_t$  at a relatively faster speed as (i) the risk of being in the bad regime and (ii) uncertainty about the true economic state both rise. In contrast, for high values of  $\pi_t$ , the valuation ratios decrease with  $\pi_t$  at a relatively lower speed because (ii) reduces, partially offsetting the effect from a rise in (i).

Figure 4 plots the risk-free rate, equity premium, and conditional equity return volatility as a function of  $\lambda_t$  or  $\pi_t$ . Panels A, C, and E show that for medium to high values of  $\lambda_t$ , the behavior of asset prices in terms of  $\lambda_t$  is similar to that of conventional disaster risk models. That is, as  $\lambda_t$  increases, the risk-free rate decreases due to the precautionary savings motive. The conditional equity premium and return volatility increase due to the heightened risk of a joint decline in consumption and dividends as well as higher conditional volatility of  $\lambda_t$ .

However, there is another source of variation that plays a significant role, especially when  $\lambda_t$  is low. As can be seen in equation (7),  $\lambda_t$  affects the conditional volatility of  $\pi_{t+1}$ . Specifically, as  $\lambda_t$  decreases, the speed of learning becomes faster, which raises the conditional

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<sup>14</sup>When examining the relationship between each state variable and a model quantity, we set the other state variable at its median value.

<sup>15</sup>Note that  $\text{Var}_t(s_t) = \pi_t(1 - \pi_t)$  is maximized when  $\pi_t$  equals 0.5.

volatility of  $\pi_{t+1}$  by making it more responsive to shocks to the jump intensity process. This channel pushes the risk-free rate downward and the conditional equity premium and return volatility upward, as the agent dislikes higher uncertainty about the future belief  $\pi_{t+1}$ . For small values of  $\lambda_t$ , this effect dominates the effect described in the previous paragraph, producing U-shaped/hump-shaped patterns of the risk-free rate, conditional equity premium, and return volatility in Panels A, C, and E.

Panels B, D, and F of Figure 4 illustrate the behavior of the risk-free rate, equity premium, and conditional equity return volatility in terms of  $\pi_t$ . Under our calibration,  $\pi_t$  remains fairly close to zero most of the time because the depression state is unlikely. In such cases, an increase in  $\pi_t$ , which implies a higher perceived probability of being in the depression state, leads to a decrease in the risk-free rate and an increase in the equity premium and return volatility, like in conventional models. However, if  $\pi_t$  moves significantly away from zero, we observe U-shaped/hump-shaped patterns. As discussed above, this is because uncertainty about the hidden state of the economy is highest when  $\pi_t$  is 50%.

We also observe that in Panels B, D, and F, the risk-free rate dips and the equity premium/return volatility peaks at a value of  $\pi_t$  that is smaller than 50%. For instance, the conditional equity premium peaks around  $\pi_t = 30\%$ . Why is this the case, given that uncertainty about the hidden state, namely  $\text{Var}_t(s_t) = [\pi_t(1 - \pi_t)]$ , peaks at 50%? There are two additional forces at play here. On one hand, an increase in  $\pi_t$  raises the perceived probability of being in the depression state and leads to a higher equity premium. On the other hand, an increase in  $\pi_t$  also implies a lower sensitivity of the price-dividend ratio with respect to  $\pi_t$  (as can be seen in Figure 3) and leads to a lower equity premium. While the first effect pushes the peak of the curve to the right beyond 50%, the second effect pushes the peak of the curve to the left below 50%. Under our calibration, the latter dominates the former, and, as a result, the peak of the equity premium is located at  $\pi_t < 50\%$ . A similar argument applies to the risk-free rate and the return volatility.

## 4.2 Asset Pricing Moments

### 4.2.1 Simulation Procedure and Identification of Disasters

We simulate 70-year-long samples from our model at a monthly frequency and compare the resulting model moments to their data counterparts in the postwar U.S. data. For some moments that involve option prices, we rely on 28-year-long simulated paths instead, as the options data are only available from 1990 to 2017. We juxtapose the asset pricing implications of our model with those of Wachter (2013)'s variable disaster risk model, referred to as the instantaneous disaster risk model hereafter.<sup>16</sup> To investigate the role of learning in isolation, we also compute the results from a special case of our model with perfect information and report them in Appendix C.

We first simulate the consumption process given in equations (2) and (3), and the endogenous belief process given in equation (7).<sup>17</sup> Once  $\lambda_t$  and  $\pi_t$  are simulated, we find the corresponding paths of the wealth-consumption and price-dividend ratios, the risk-free rate, the equity premium, the conditional return volatility, and option prices by interpolating their values over the grid. For each simulation, we compute our moments of interest using the simulated time series of the model quantities.

This procedure is repeated 10,000 times so that we obtain the model-implied distribution of each moment. Since the postwar U.S. data do not include any macroeconomic disasters, we report the results from no-disaster samples. No-disaster samples constitute 57% of the entire simulated samples in the case of 70-year simulations and 78% in the case of 28-year simulations. We also investigate the population properties of the model by simulating a long path of 1,000,000 years.

A distinguishing feature of our approach is in the identification of disaster periods. In

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<sup>16</sup>To facilitate comparison, we implement the model of Wachter (2013) in discrete time with monthly intervals.

<sup>17</sup>In each simulation, we set the starting values of  $\lambda_t$  and  $\pi_t$  to their long-run means and consider a 100-year burn-in period.

conventional disaster risk models, an occurrence of a disaster is identified as soon as a Poisson process jumps or as soon as the economy enters into a disaster state, depending on the model. In contrast, disaster periods in our model are not fully identified *ex ante* using a state variable, but *ex post*.

The reason is twofold. First, entering the depression state with  $s_t = 1$  does not ensure that the economy will remain in the depression state long enough to accumulate a large consumption decline; it is possible for the economy to quickly switch back to the normal state. Furthermore, the true economic state is not directly observable, and the agent cannot perfectly distinguish the onset of a disaster from an increase in transitory risk. We establish full consistency between the model and the data by following Barro and Ursúa (2008) and identifying macroeconomic disasters in our model as peak-to-trough consumption declines that are larger than 10%.

#### 4.2.2 Standard Asset Pricing Moments

Table 2 reports the standard asset pricing moments from our model (Panel A) and from the instantaneous disaster risk model (Panel B). A general observation from comparing Panels A and B is that both models perform fairly well in terms of explaining the high equity premium, the high stock market volatility, and the low and smooth risk-free rate, while matching a low consumption growth volatility in the postwar U.S. data. The model with time-varying instantaneous disaster risk is designed to account for these aspects of the data, providing a joint solution to the equity premium, risk-free rate, and stock market volatility puzzles. While our model operates through a different mechanism, it is reassuring that it can address these puzzles, meeting the necessary requirements to be regarded as a quantitatively serious macro-finance model.

A striking difference between our model and the model with instantaneous disasters is in the discrepancy between asset pricing moments in no-disaster samples versus in population. For instance, the median no-disaster consumption growth volatility in our model is 1.80%

and its population value is 2.50%. The difference is much larger in the instantaneous disaster risk model: a median of 1.62% in no-disaster samples versus 5.47% in population. This gap is also manifested in the volatilities of the equity return and the risk-free rate.

Why does the instantaneous disaster risk model exhibit a larger discrepancy between no-disaster sample moments and population moments? In such a model, consumption, dividends, and equity prices all fall within a unit period (i.e. one month) during a disaster. As a result, volatilities and other higher moments are greatly amplified due to the instantaneous nature of disasters. In contrast, our model generates disasters that unfold slowly over multiple periods of time both in consumption and prices, consistent with the data. This brings the asset pricing moments in population much closer to those in no-disaster samples.

Now we turn to the predictability of returns by the price-dividend ratio. Panel B of Table 3 illustrates that the instantaneous disaster risk model produces the empirical predictive relation between equity returns and the price-dividend ratio. This is possible because the price-dividend ratio is decreasing in disaster risk whereas the conditional equity premium is increasing in disaster risk, resulting in a negative predictive relation between the price-dividend ratio and future excess returns.

We find that our model also generates predictable returns, consistent with the data. However, the relation between the price-dividend ratio and the equity premium is not as straightforward: as illustrated in Figures 3 and 4 and explained in Section 4.1, the price-dividend ratio is monotonically decreasing in  $\lambda_t$  and  $\pi_t$  whereas the conditional equity premium is not. Therefore, it is a question of calibration whether the model would spend enough time in a region where the correlation between the price-dividend ratio and the conditional equity premium is negative. Panel A of Table 3 shows that this is indeed the case: the price-dividend ratio predicts future excess equity returns with a negative coefficient. The magnitude of model-implied coefficients and  $R^2$  values are close to the data.

Lastly, Tabel 4 demonstrates that the two models of interest account for the absence of consumption growth predictability in the data. Note that in both models, expected con-

sumption growth exhibits persistent variation due to time variation in jump risk. In the case of the instantaneous disaster risk model, however, this does not lead to consumption growth predictability because no-disaster samples, by definition, do not contain jump realizations (i.e. disasters). In contrast, in our model, disaster samples can potentially include a series of jumps that are not large enough to constitute a disaster. Despite this possibility, we discover that the predictability of consumption growth by the price-dividend ratio is relatively small, containing the postwar U.S. data within the confidence bands of our model.

All in all, our model and Wachter (2013)'s instantaneous disaster risk model are capable of explaining standard asset pricing facts. In what follows, we discuss additional moments that can sharply contrast these two models: the VIX and the variance risk premium (Section 4.2.3), and risk premia on put-protected portfolios (Section 4.2.4).

### 4.2.3 The VIX and the Variance Risk Premium

An important distinction between our model and the instantaneous disaster risk model is the source of the equity premium, which brings about crucial empirical implications. In our model, the agent fears disasters because disasters are prolonged depression periods that are slowly revealed over time. On the contrary, the instantaneous disaster risk model attributes the equity premium almost entirely to the risk of a joint tail event in consumption, dividends, and equity prices. We argue that an unrealistically quick financial market response to macroeconomic disasters significantly distorts the model implications for short-term contingent claim prices.

We start with the model implications for implied variance and the variance risk premium. Following Drechsler and Yaron (2011), we calculate one-month implied variance ( $IV_t$ ) and the variance risk premium ( $VRP_t$ ) as follows:

$$\begin{aligned}
 IV_t &= \mathbb{E}_t^Q[\text{Var}_{t+1}^Q(r_{d,t+2})] \\
 VRP_t &= \mathbb{E}_t^Q[\text{Var}_{t+1}^Q(r_{d,t+2})] - \mathbb{E}_t[\text{Var}_{t+1}(r_{d,t+2})],
 \end{aligned}$$

where  $Q$  denotes the risk-neutral measure. Consistent with the standard convention in the literature, both quantities are expressed in monthly percentage squared terms. Note that under this unit, the VIX is simply equal to  $\sqrt{12 \times IV_t}$ .

Table 5 compares the resulting implied variance and variance risk premium from the two models with their data counterparts.<sup>18</sup> Panel B shows that the instantaneous disaster risk model significantly overstates the level and volatility of implied variance as well as of the variance risk premium.<sup>19</sup> For instance, the average implied variance in the data between 1990 and 2017 is 36.05. The corresponding value in the median no-disaster sample is 165.96 in the instantaneous disaster risk model.

This discrepancy is deeply rooted in how disasters unfold in financial markets. When the economy enters the disaster state with a, say 20%, drop in consumption, equity prices immediately fall by 60% (20% times a leverage parameter of three) in the model. As a result, the expected variance within a month is extremely high since all of the 60% decline in equity is expected to happen during that month. This is inconsistent with the way in which disasters actually develop in the real world, and results in unrealistically high values of the expected variance.<sup>20</sup>

The variance risk premium is also entangled in the same issue. States with abrupt disasters are associated with extremely high marginal utility, which leads to a very high average variance risk premium in the instantaneous disaster risk model. The average variance risk premium in the data is 16.20, but its counterparts in the model are much higher: the median of the average variance risk premium in the model is 134.96, and the lower end of

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<sup>18</sup>In the data, implied variance, which is simply the squared VIX multiplied by 12, is computed as the value of a portfolio of S&P 500 index option prices. We obtain the time series of implied variance and the variance risk premium from Hao Zhou's website. For details, see Zhou (2018).

<sup>19</sup>To mitigate the issue of extremely high values of the VIX in disaster risk models, Dew-Becker, Giglio, Le, and Rodriguez (2017) and Seo and Wachter (2018b) adopt an ad-hoc approach and apply an upper bound on the maximum instantaneous decline in equity prices during disasters. There is no need to make such an assumption in our model as it generates an endogenously slow decline in prices, consistent with the data.

<sup>20</sup>For instance, the largest one-month decline in the aggregate equity during the Great Depression is 30% while the cumulative decline is 86%. Assuming that the cumulative decline occurs within a month causes a major distortion in the underlying price process and implied one-month option prices.

the confidence band is 65.21, which is still substantially larger than the data value.

Our model resolves this inconsistency by generating realistic disaster dynamics in financial markets. While disasters still occur, they unfold over multiple periods, resulting in a lower expected variance. Panel A of Table 5 reveals that our model implies realistic levels of implied variance and the variance risk premium. Specifically, the medians of the average implied variance and variance risk premium in no-disaster samples are 40.16 and 16.38, respectively, which are very close to the data. While the volatilities of implied variance and the variance risk premium are unrealistically high in the instantaneous disaster risk model, our model performs well in these aspects of the data: 39.41 for the volatility of implied variance (versus 33.52 in the data) and 17.89 for the volatility of the variance risk premium (versus 20.57 in the data).

In Appendix C, we also consider a special case of our model with perfect information in an attempt to highlight the importance of learning. This model features slow consumption disasters but not slow financial disasters; since the transition to the depression state is directly observable, the agent can fully react to this bad news. While this model produces a lower level of implied variance and the variance risk premium compared to the instantaneous disaster risk model, the model values are still too high compared to the data. (See Panel A of Table C.1.) That is, the instantaneous reaction of prices to the depression state still results in unrealistically high values of short-term variance and its market price. We conclude that information frictions and learning dynamics are crucial in generating a realistically slow reaction of prices to macroeconomic disasters, which translates into realistic values of expected short-term variance and the risk premium attached to it.

Given the slow nature of disasters in our model, it is worth revisiting the criticism against the rare disaster pricing mechanism regarding the timing of shocks to the stochastic discount factor and returns. Constantinides (2008) points out that although the Euler equation is based on concurrent moves in consumption and asset prices, Rietz-Barro type models implicitly violate this premise by treating slow consumption disasters in the data as one-shot

shocks. This simplifying assumption may artificially amplify risk premia in these models due to a counterfactually strong connection between consumption declines and financial markets.

Our analysis shows that the assumption of instantaneous disasters is indeed not innocuous when one considers the consistency between the equity premium and the variance risk premium. As discussed above, the instantaneous disaster risk model implies a reasonable equity premium yet a too large variance risk premium. Assuming slowly unfolding consumption disasters alone does not resolve this issue: in the special case of our model with perfect information, the variance risk premium is still high.

The solution lies in modeling a reasonable joint behavior of consumption and equity prices during disasters. Our model achieves an equity premium close to the data due to the pricing of the time variation in the prospect of future disasters. The variance risk premium is realistically low because disasters are not identifiable *ex ante*, preventing equity prices from plummeting too abruptly. Essentially, addressing the criticism of Constantinides (2008) is at the heart of reconciling the equity premium and the variance risk premium under the rare disaster mechanism: both can be resolved by introducing slowly unfolding consumption and financial disasters.

#### **4.2.4 Risk Premia on Put-Protected Portfolios**

Another testable implication of disaster risk models concerns the source of the equity premium. Welch (2016) provides an intuitive empirical exercise to test whether the equity premium can be explained by the possibility of rare disasters. If the equity premium is due to the likelihood of very large jumps in the equity market, a portfolio that is protected against the risk of large jumps should not carry a significant premium. Welch (2016) builds a portfolio that consists of the S&P 500 index and a one-month out-of-the-money (OTM) put option with 85% moneyness and finds that this portfolio still earns a significant premium close to the full equity premium. This result leads him to conclude that a major portion of the equity premium cannot be attributed to rare disaster events.

Our framework provides a way to reconcile Welch (2016)'s evidence with the presence of rare disasters in the data. Before we discuss the pricing mechanism, we point out that one-month put options with moneyness 85% would not have provided significant protection against the most prominent disaster, the Great Depression. Panel B of Figure 1 shows that a put-protected strategy with 85% moneyness would have lost 80% throughout the Great Depression, even ignoring the cost of purchasing put options. Given that the stock market experienced a cumulative decline of 86%, the insurance against a monthly drop larger than 15% would not have served as an effective hedge against the disaster. From this episode, we can understand that modeling slowly unfolding disasters in equity prices, rather than instantaneous price reactions, is critical in explaining risk premia on put-protected portfolios.

We first compute option prices in our model based on equation (11). Then, the return on a put-protected portfolio is calculated as  $\left[ \frac{\max(P_{d,t+1}, K) + D_{t+1}}{P_{d,t} + O_t(K)} \right]$ , where  $K$  is the strike price of the one-month OTM put option in the portfolio. This expression is straightforward. The denominator represents the initial investment in the portfolio at the beginning of each month at time  $t$ , which is the sum of the equity value  $P_{d,t}$  and the option price  $O_t(K)$ . The numerator represents the payoff from this portfolio after a month at time  $t + 1$ , which consists of (i) the equity value  $P_{d,t+1}$ , (ii) the dividend payment  $D_{t+1}$ , and (iii) the put option payoff  $[\max(K - P_{d,t+1}, 0)]$ . By dividing both the denominator and the numerator by  $P_{d,t}$ , we express the log return on the put-protected portfolio in terms of moneyness  $k = K/P_{d,t}$ :

$$r_{d,t+1}^k = \log \left( \frac{\max [P_{d,t+1}/P_{d,t}, k] + D_{t+1}/P_{d,t}}{1 + O_t^n(k)} \right).$$

Table 6 presents the resulting risk premia in the model and in the data for various moneyness values  $k = 0.75, 0.80, 0.85, \text{ and } 0.90$ .<sup>21</sup> To evaluate the relative performance among different models, we report the results in the following two ways: (i) the difference

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<sup>21</sup>To obtain option prices at fixed values of moneyness and time to maturity from the data, we use a second-order polynomial to express implied volatility as a function of moneyness and time to maturity, following Dumas, Fleming, and Whaley (1998), Christoffersen and Jacobs (2004), and Christoffersen, Heston, and Jacobs (2009). S&P 500 option prices are obtained from OptionMetrics.

between the full equity premium and put-protected portfolio premium,  $\mathbb{E}[r_d - r_d^k]$ , and (ii) the put-protected portfolio premium as a fraction of the full equity premium,  $\frac{\mathbb{E}[r_d^k - r_f]}{\mathbb{E}[r_d - r_f]}$ .

In the data, the premium difference between the equity index and the put-protected portfolio with  $k = 0.85$  is approximately 2% per annum, which is consistent with the evidence from Welch (2016). The premium difference further shrinks to 0.75% when the moneyness value is 0.75. As can be seen from Table 6, this put-protected portfolio bears almost 89% of the entire equity premium, suggesting that monthly price declines that are larger than 25% are not the main source of the equity premium, at least over a one-month horizon.

As anticipated by Welch (2016), the instantaneous disaster risk model fails to capture the high premium on put-protected portfolios: the put-protected portfolio with  $k = 0.75$  earns a premium that is 59% of the full equity premium in the median no-disaster sample (Panel B). That is, the instantaneous disaster risk model relies too much on extremely large shocks to explain the equity premium, and this is inconsistent with the data.<sup>22</sup> In Appendix C, we also find that the model with perfect information cannot generate the patterns of put-protected portfolio premia in the data because this model still features the instantaneous reaction of equity prices to consumption disasters. (See Panel B of Table C.1.)

Now we turn to the results from our model with slow consumption and financial disasters. Panel A of Table 6 demonstrates that our model performs well in explaining put-protected portfolio premia across various moneyness values, which is in sharp contrast to the other two models. For instance, in the median no-disaster sample, the risk premium on the put-protected portfolio with moneyness  $k = 0.75$  is 0.73% lower than the full equity premium (versus 0.75% in the data), occupying 90% of the full equity premium (versus 89% in the data). This is possible because our model produces realistic price dynamics during disasters, which, in turn, results in put-protected portfolio returns that are consistent with the data. In our model, it is the prospect of slow and prolonged disasters in equity prices that gives rise

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<sup>22</sup>See also Beason and Schreindorfer (2019) and Schneider (2019) for detailed analyses about the sources of the equity premium using options data.

to the equity premium, not the prospect of an extremely severe crash in the equity market.

### 4.3 Model-Implied Macroeconomic and Financial Disasters

The size distribution of consumption disasters, constructed by Barro and Ursúa (2008) using the peak-to-trough approach, plays an important role in the calibration of disaster risk models. Panel A of Figure 5 shows this empirical distribution of consumption declines during disasters. In typical models with instantaneous disasters, this distribution is used as a time-invariant jump size distribution (see, for example, Barro and Jin (2011), Gabaix (2012), and Wachter (2013)).

In our model, the size distribution of consumption disasters is not a model input, but an important model outcome. We do not assume that disasters occur exogenously following a certain size distribution; rather, disasters occur slowly over time as a series of negative shocks to consumption accumulates. Therefore, the size distribution of consumption disasters endogenously arises as an implication of our model.

To examine the severity of consumption disasters under our model, we simulate the consumption process for 1,000,000 years and identify disasters using the peak-to-trough approach. Panel B of Figure 5 displays the model-implied disaster size distribution in population. We can observe that this implied distribution is fairly close to its empirical counterpart in Panel A. While the average consumption decline during disasters is 21% in the data, it is 18% in our model. Furthermore, our analysis shows that the model also accounts for the average duration of consumption disasters: 4.5 years in the model versus 4.1 years in the data. In sum, we conclude that our model produces consumption disasters that are consistent with historical consumption data.

The key feature of our model is that not only consumption but also equity prices fall gradually in periods of disasters. We show that generating realistic equity price dynamics during disasters is essential in explaining a number of empirical patterns, as discussed in Sections 4.2.3 and 4.2.4. We investigate whether the average size and duration of stock

market disasters in the model are indeed comparable to the data. Following Barro and Ursúa (2017), we identify stock market disasters as peak-to-trough price declines that are larger than 25%. In our model, stock market disasters exhibit a 46% cumulative decline and a 2.5-year duration, on average. These numbers are close to their data counterparts: 46% and 3.2 years.<sup>23</sup> Hence, we confirm that our model’s characterization of financial disasters is in line with the data.

## 5 Conclusion

What is the source of the high equity premium in the postwar sample? As alluded by Welch (2016), risk premia on put-protected portfolios can shed light on answering this question. By considering one-month put-protected portfolios with different moneyness levels, it is possible to decompose the equity premium into multiple pieces. Specifically, we find that 50%, 29%, 17%, and 11% of the entire equity premium in the data originate from monthly price declines that are larger than 10%, 15%, 20%, and 25%, respectively. On the surface, this seems to suggest a rejection of the rare disaster mechanism because only a small portion of the equity premium over a month is attributable to shocks that are larger than 25%.

In this paper, we show that modeling realistic equity dynamics during disasters can resolve this issue. As exemplified by the Great Depression, macroeconomic disasters entail gradual but prolonged declines in the stock market, which makes it ineffective to hold a short-maturity put option as a hedge against disasters. To demonstrate this idea, we propose a model in which the true state of the economy is hidden. Investors form their beliefs about the current economic state in a Bayesian fashion, as they do not know with certainty whether today’s change in immediate risk is due to a transitory shock or a persistent regime shift.

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<sup>23</sup>Barro and Ursúa (2017) report 71 cases of stock market disasters that are associated with consumption disasters (a 53% average decline and a 3.8-year average duration), and 161 cases of stock market disasters that are not associated with consumption disasters (a 43% average decline and a 2.9-year average duration). By aggregating these two types of events, we obtain an average decline of 46% and an average duration of 3.2 years for all historical stock market disasters.

The information structure and learning dynamics of our model create a slow response of equity prices to consumption declines during disasters.

Our results emphasize that how disasters unfold in financial markets has important asset pricing implications not only for disaster periods, but also for normal periods with high valuation. While typical models with instantaneous stock market disasters struggle to account for the VIX, the variance risk premium, and risk premia on put-protected portfolios, we find that our model can explain them naturally when it is calibrated to generate reasonable stock market dynamics during disasters. After all, investors fear disasters not because they expect sharp declines in equity prices within a short period of time, but because they are concerned about an extended period of stock market depression with no clear end in sight.

# Appendix

## A Model Derivations

We start from the Euler equation for the return on the consumption claim:

$$\mathbb{E}_t [\exp (\log M_{t+1} + r_{c,t+1})] = 1.$$

The definition of the stochastic discount factor in equation (1) and the expression  $r_{c,t+1} = [\log (1 + e^{pc_{t+1}}) - pc_t + \Delta c_{t+1}]$  lead to the following equation:

$$\mathbb{E}_t \left[ \exp \left( \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \log (1 + e^{pc_{t+1}}) - \theta pc_t + \theta \Delta c_{t+1} \right) \right] = 1.$$

We can take the term  $[\exp (\theta \log \delta - \theta pc_t)]$  out of this conditional expectation, as it is time- $t$  measurable. In addition, since log consumption growth consists of the i.i.d. normal shock  $\epsilon_{t+1}^c$  and the jump process  $J_{t+1}$  that are both independent of the price-consumption ratio  $pc_{t+1}$  conditional on  $\lambda_t$ , we can split this conditional expectation into the following three:

$$\exp (\theta \log \delta - \theta pc_t) \mathbb{E}_t \left[ e^{(1-\gamma)(\mu_c + \sigma_c \epsilon_{t+1}^c)} \right] \mathbb{E}_t \left[ e^{(1-\gamma)J_{t+1}} \right] \mathbb{E}_t \left[ (1 + e^{pc_{t+1}})^\theta \right] = 1.$$

The first conditional expectation equals  $\left[ e^{(1-\gamma)\mu_c + \frac{1}{2}(1-\gamma)^2\sigma_c^2} \right]$ , because it is simply the moment generating function of a normal distribution, evaluated at  $[1 - \gamma]$ . Similarly, the second conditional expectation is the moment generating function of a Poisson jump process  $J_{t+1}$  evaluated at  $[1 - \gamma]$ , and is expressed as  $[e^{\lambda_t(\Phi_Z(1-\gamma)-1)}]$ . Therefore, the Euler equation becomes:

$$\exp \left( \theta \log \delta - \theta pc_t + (1 - \gamma)\mu_c + \frac{1}{2}(1 - \gamma)^2\sigma_c^2 + \lambda_t [\Phi_Z(1 - \gamma) - 1] \right) \mathbb{E}_t \left[ (1 + e^{pc_{t+1}})^\theta \right] = 1.$$

Taking the logarithm of both sides leads to the recursive expression for the log wealth-consumption ratio in equation (8).

The recursive expression for the log price-dividend ratio in equation (9) can also be obtained using a similar approach. The log return on the dividend claim (or equity) satisfies the Euler equation:

$$\mathbb{E}_t [\exp (\log M_{t+1} + r_{d,t+1})] = 1.$$

Plugging the expression  $r_{d,t+1} = [\log (1 + e^{pd_{t+1}}) - pd_t + \phi \Delta c_{t+1}]$  into the Euler equation results in:

$$\begin{aligned} \mathbb{E}_t [\exp (\theta \log \delta + (\phi - \gamma) \Delta c_{t+1} + (\theta - 1) \log (1 + e^{pc_{t+1}}) \\ - (\theta - 1) pc_t + \log (1 + e^{pd_{t+1}}) - pd_t)] = 1. \end{aligned}$$

We re-express this equation using the fact that  $pd_t$  is time- $t$  measurable and  $\Delta c_{t+1}$  is independent of  $pc_{t+1}$  and  $pd_{t+1}$ , given  $\lambda_t$ :

$$\exp (\theta \log \delta - (\theta - 1) pc_t - pd_t) \mathbb{E}_t [\exp ((\phi - \gamma) \Delta c_{t+1})] \mathbb{E}_t \left[ (1 + e^{pc_{t+1}})^{\theta - 1} (1 + e^{pd_{t+1}}) \right] = 1,$$

where, as explained earlier,  $\mathbb{E}_t [\exp ((\phi - \gamma) \Delta c_{t+1})]$  equals the multiplication of the moment generating function of  $\epsilon_{t+1}^c$  and that of  $J_{t+1}$ , evaluated at  $[\phi - \gamma]$ . By taking the logarithm of both sides of the above equation, we obtain the recursive expression for the log price-dividend ratio in equation (9).

## B The Conditional Distribution of Consumption Growth

In this section, we derive the semi-analytic expression for the conditional density of log consumption growth  $\Delta c_{t+1}$ . By the law of total probability, this density can be expressed as

$$\begin{aligned} P_t(\Delta c_{t+1}) &= \sum_{n=0}^{\infty} P(N_{t+1} = n | \lambda_t) P_t(\Delta c_{t+1} | N_{t+1} = n) \\ &= \sum_{n=0}^{\infty} \frac{\lambda_t^n e^{-\lambda_t}}{n!} P_t(\Delta c_{t+1} | N_{t+1} = n). \end{aligned} \quad (\text{B.1})$$

Conditional on the number of jumps (i.e.  $N_{t+1} = n$ ), consumption growth is the sum of a constant term  $\mu_c$ , i.i.d. normal innovation  $\epsilon_{t+1}^c$ , and  $n$  i.i.d. realizations of jumps. Since jump size follows the negative of an exponential distribution  $Z_j = -\eta_j$ , it follows that

$$P_t(\Delta c_{t+1} | N_{t+1} = n) = P\left(\mu_c + \sigma_c \epsilon_{t+1}^c - \sum_{j=1}^n \eta_j\right),$$

where  $\eta_j$  is an exponential random variable with mean  $\mu_\eta$ .

In the case of  $n = 0$ , the conditional distribution of log consumption growth simply reduces to normal. Hence, we consider the nontrivial case where  $n$  is larger than or equal to one. Note that the sum of  $n$  independent exponential random variables with the same mean  $\mu_\eta$  follows a Gamma (or Erlang) distribution with a shape parameter  $n$  and a scale parameter  $\nu = 1/\mu_\eta$  whose density is known as:

$$P\left(\sum_{j=1}^n \eta_j = h\right) = \frac{\nu^n h^{n-1} e^{-\nu h}}{(n-1)!}, \quad h \in [0, \infty).$$

Therefore, it suffices to find the distribution of a normal random variable minus a gamma distribution using the convolution operation:

$$P_t(\Delta c_{t+1} = c | N_{t+1} = n) = \int_0^\infty \frac{\nu^n h^{n-1} e^{-\nu h}}{(n-1)!} \frac{e^{-\frac{(h+c-\mu_c)^2}{2\sigma_c^2}}}{\sqrt{2\pi}\sigma_c} dh.$$

We apply a change of variables so that the definite integral on the right-hand side of the equation is with respect to  $u = \left[ \frac{h + \nu\sigma_c^2 + c - \mu_c}{\sqrt{2}\sigma_c} \right]$  instead of  $h$ :

$$P_t(\Delta c_{t+1} = c | N_{t+1} = n) = \frac{\nu^n e^{\frac{\nu}{2}(\nu\sigma_c^2 + 2c - 2\mu_c)}}{(n-1)!} \frac{1}{\sqrt{\pi}} \int_{\frac{\nu\sigma_c^2 + c - \mu_c}{\sqrt{2}\sigma_c}}^{\infty} \left( \sqrt{2}\sigma_c u - \nu\sigma_c^2 - c + \mu_c \right)^{n-1} e^{-u^2} du.$$

Expressing the integral in the above form enables us to exploit the definition of the Gauss (complementary) error function:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du,$$

whose values can be obtained from the cumulative standard normal distribution function. For instance, in the case of  $n = 1$ , we obtain the density of the negative of an exponentially modified Gaussian random variable:

$$P_t(\Delta c_{t+1} = x | N_{t+1} = 1) = \frac{\nu}{2} e^{\frac{\nu}{2}(\nu\sigma_c^2 + 2c - 2\mu_c)} \operatorname{erfc} \left( \frac{\nu\sigma_c^2 + c - \mu_c}{\sqrt{2}\sigma_c} \right).$$

In the case where  $n$  is larger than or equal to two, the integrand contains terms  $(u^m e^{-u^2})$  for  $1 \leq m \leq n-1$ . By applying integration by parts  $\left[ \int u^m e^{-u^2} = -\frac{1}{2}u^{m-1}e^{-u^2} + \int \frac{m-1}{2}u^{m-2}e^{-u^2} \right]$  multiple times, the integral of such terms can also be expressed as the error function.

Under our calibration, the probability of observing more than, say, three jumps within a one-month period is essentially zero even under the depression regime. Therefore, simply considering the terms in equation (B.1) up to  $n = 3$  still provides an accurate approximation for the conditional density of log consumption growth  $P_t(\Delta c_{t+1})$ .

## C The Model with Perfect Information

In this section, we consider a special case of our model in which the representative investor has perfect information about the current economic state  $s_t$ . We assume that the agent's

preferences as well as the underlying dynamics of consumption and the jump intensity are identical to those of the benchmark model. The only distinction is that  $s_t$  is observable, which means that the investor's learning and the resulting belief  $\pi_t$  are no longer relevant. Therefore, under this perfect information setup, the wealth-consumption ratio and the price-dividend ratio become functions of  $\lambda_t$  and  $s_t$  (instead of  $\pi_t$ ). We are able to numerically solve for these two valuation ratios based on the recursive relations in equations (8) and (9).

Note that under perfect information, the value of  $s_t$  is a part of the investor's time- $t$  information set. As a result, it is possible to calculate moments, conditional on  $s_t$ . Thus, the time- $t$  conditional expectation of a general function  $F$  in equation (10) is now expressed as:

$$\mathbb{E}_t [F_{t+1} | s_t = s] = \sum_{s' \in \{0,1\}} p_{ss'} \mathbb{E} [F_{t+1} | s_{t+1} = s', \lambda_t, s_t = s].$$

Conditional on  $(s_{t+1}, \lambda_t, s_t)$ , the distribution of  $F_{t+1}$  is determined by the distributions of  $\epsilon_{t+1}^\lambda$  and  $\Delta c_{t+1}$ . Hence, the value of  $\mathbb{E} [F_{t+1} | s_{t+1} = s', \lambda_t, s_t = s]$  in the above equation can be calculated using a double integral with respect to  $\epsilon_{t+1}^\lambda$  and  $\Delta c_{t+1}$ , similar to the case for the benchmark model. That is, we are able to calculate conditional moments and option prices by implementing the above equation using a double integral.

We repeat the same simulation exercise in Section 4.2 for the model with perfect information and report the results in Table C.1. In Panel A, we present the mean and standard deviation of implied variance together with the mean and standard deviation of the variance risk premium, all of which are expressed in monthly percentage squared terms. Panel B provides the average annual difference between the full equity premium and the premium on the put-protected portfolio with moneyness  $k$ . Also reported is the ratio between the premium on the put-protected portfolio and the full equity premium. Following Section 4.2.4, we consider a wide range of moneyness  $k$  values ranging from 75% to 90%.

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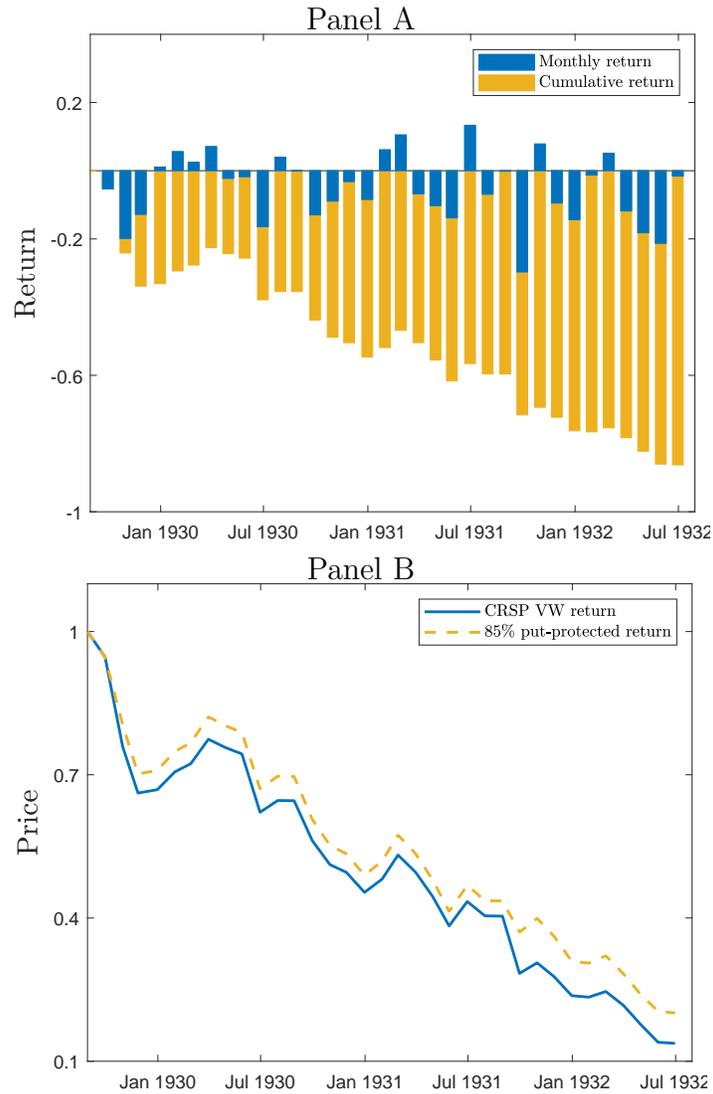
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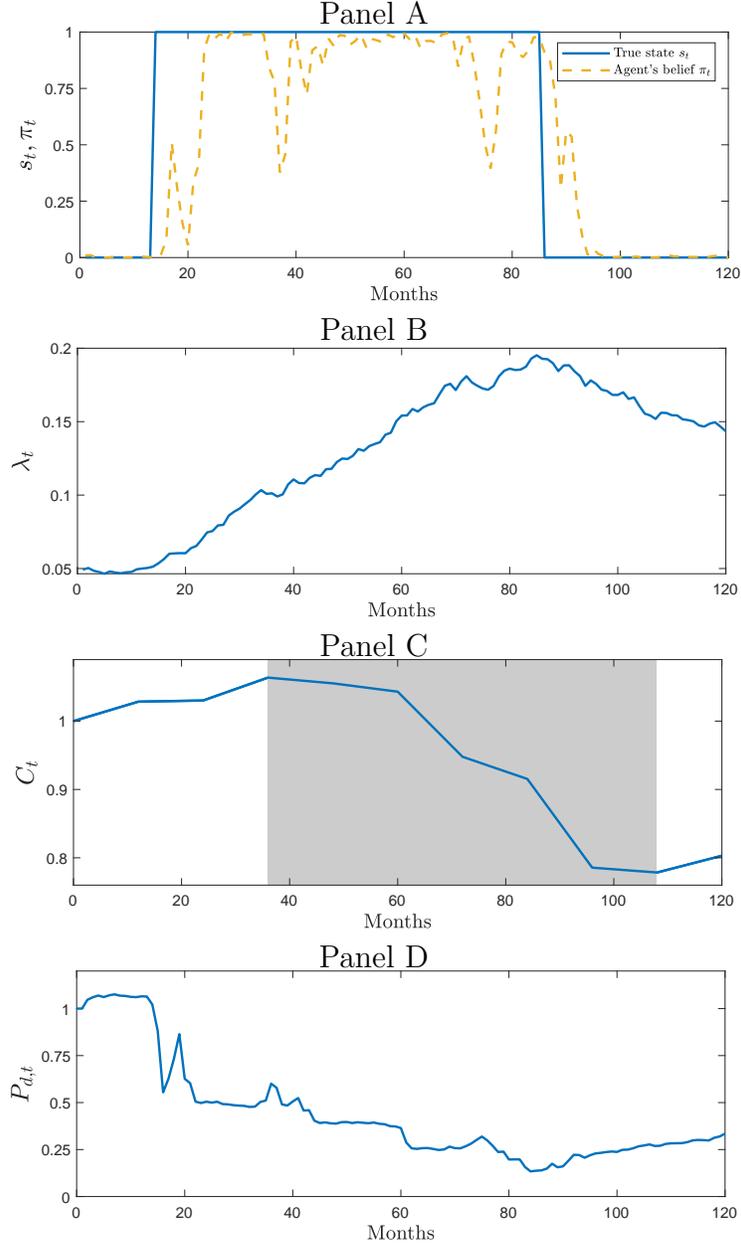
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Figure 1: Monthly Returns During the Great Depression



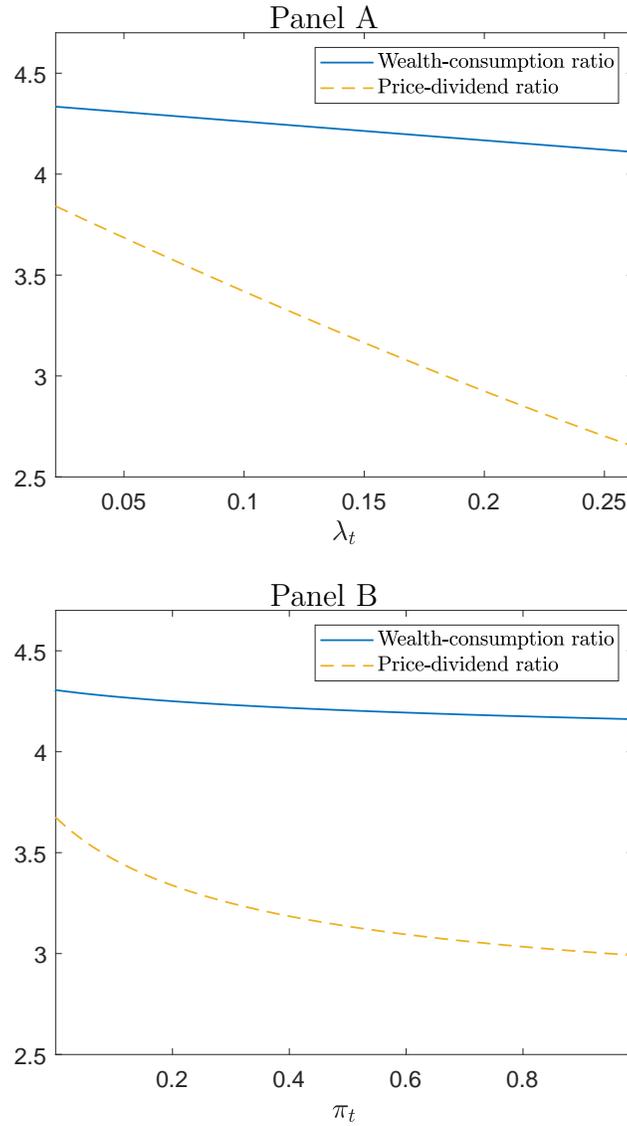
Notes: Panel A depicts the monthly returns on the CRSP value-weighted index excluding dividends, from September 1929 to June 1932. The blue bars represent month-over-month returns and the yellow bars represent cumulative returns. In Panel B, the solid blue line represents the path of the equity index while the dashed yellow line represents the value of the put-protected portfolio with 85% moneyness. The value of the put-protected portfolio is calculated assuming a zero cost for acquiring put options.

Figure 2: Learning and Slowly Unfolding Disasters



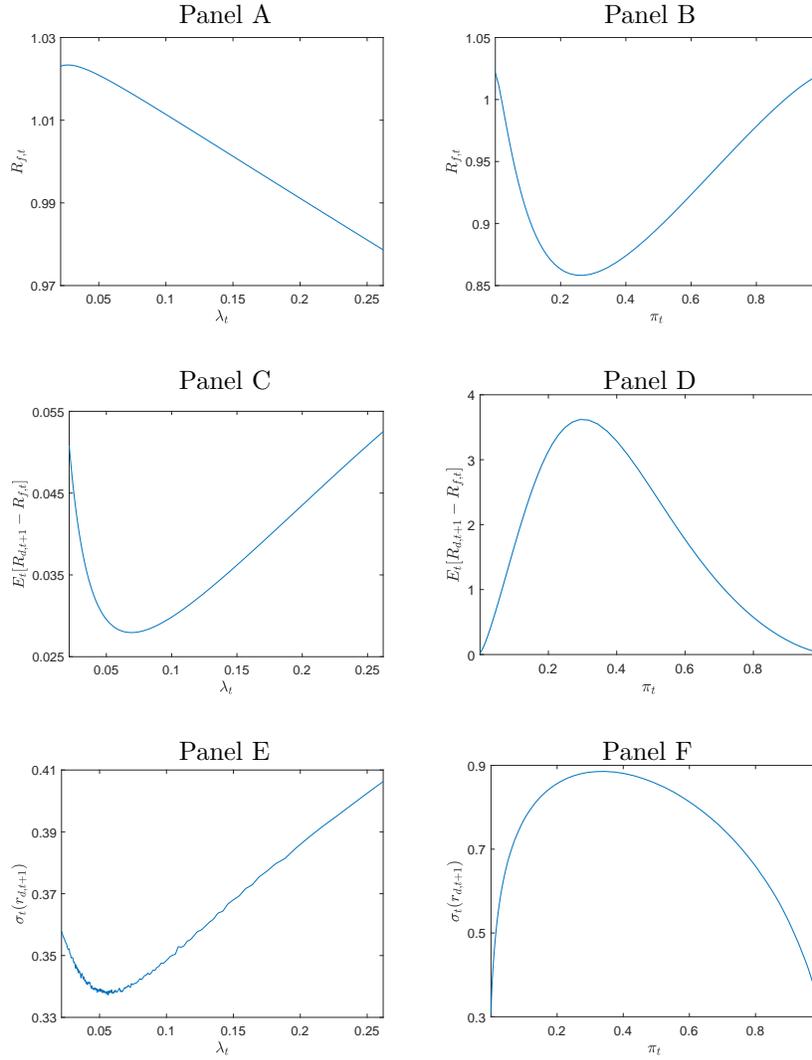
Notes: This figure plots the dynamics of our model in a sample path that includes a consumption disaster. Panel A plots the path of the state  $s_t$  and belief  $\pi_t$ . Panel B plots the path of jump intensity  $\lambda_t$ . Panel C plots the path of annual consumption  $C_t$ , normalized to one at month 0. Panel D plots the path of the equity price  $P_{d,t}$ , normalized to one at month 0.

Figure 3: Valuation Ratios



Notes: This figure plots the annualized log wealth-consumption ratio and the annualized log price-dividend ratio as a function of jump intensity  $\lambda_t$  (Panel A) or belief  $\pi_t$  (Panel B). We set  $\pi_t$  to its median value in Panel A. Likewise, we set  $\lambda_t$  to its median value in Panel B.

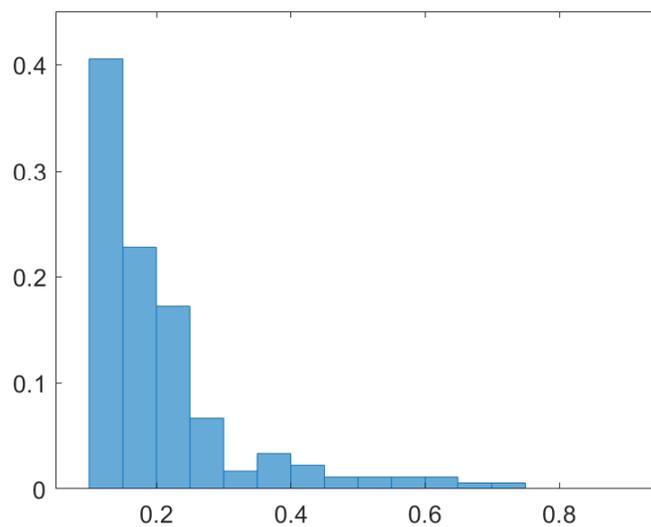
Figure 4: Risk-Free Rate, Equity Premium and Conditional Volatility



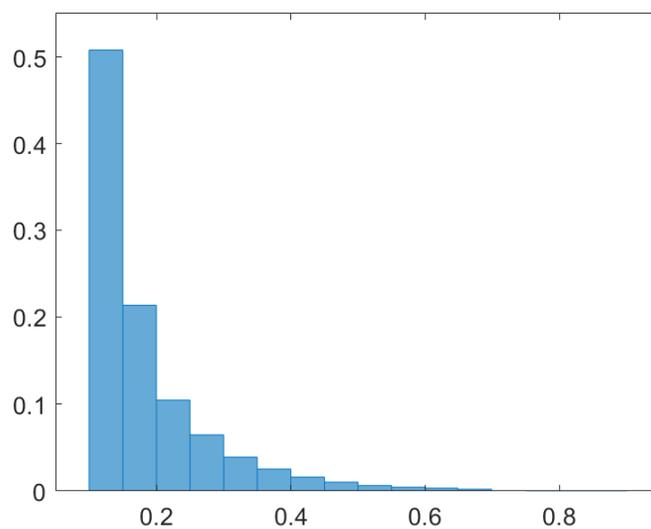
Notes: This figure plots the gross risk-free rate  $R_{f,t}$ , the conditional equity premium  $\mathbb{E}_t[R_{d,t+1} - R_{f,t}]$ , and the conditional volatility of log equity return  $\sigma_t(r_{d,t+1})$  as a function of jump intensity  $\lambda_t$  or belief  $\pi_t$ . We set  $\pi_t$  to its median value in Panels A, C, and E. Likewise, we set  $\lambda_t$  to its median value in Panels B, D, and F.

Figure 5: Disaster Size Distribution

Panel A: Data



Panel B: Model



Notes: This figure plots the distribution of cumulative consumption declines during disasters in the data (Panel A) and in the model (Panel B). Disasters are identified as peak-to-trough consumption declines that are larger than 10%, consistent with Barro and Ursúa (2008).

Table 1: Model Parameters

Risk aversion, $\gamma$	5
Elasticity of intertemporal substitution, $\psi$	1.5
Time discount, $\delta$	0.9990
Mean consumption growth in the absence of jumps, $\mu_c$	0.0026
Volatility of consumption growth in the absence of jumps, $\sigma_c$	0.0020
Leverage parameter, $\phi$	3
Persistence of jump intensity, $\rho_\lambda$	0.9933
Conditional volatility of shocks to jump intensity, $\sigma_\lambda$	0.0083
Average jump intensity in normal times, $\lambda_L$	0.0417
Average jump intensity in depression times, $\lambda_H$	0.4167
Mean jump size, $\mu_Z$	-0.0200
Transition probability from the normal state to depression state, $p_{01}$	0.0017
Transition probability from the depression state to normal state, $p_{10}$	0.0208

Notes: This table reports the parameters for the benchmark model, calibrated at a monthly frequency.

Table 2: Consumption and Asset Pricing Moments

	$\sigma(\Delta c)$	$\sigma(\Delta d)$	$\mathbb{E}[r_d - r_f]$	$\sigma(r_d)$	$\mathbb{E}[r_f]$	$\sigma(r_f)$
Data	1.45	6.46	7.25	17.80	1.25	2.75
Panel A: Benchmark model						
Median	1.80	5.40	7.13	17.06	1.51	0.61
5%	1.23	3.71	5.48	12.45	0.67	0.42
95%	2.38	7.14	8.47	21.36	1.97	0.77
Population	2.50	7.50	6.51	18.17	1.06	0.68
Panel B: Model with instantaneous disaster risk						
Median	1.62	4.21	6.15	14.58	2.12	0.32
5%	1.38	3.58	4.31	11.09	0.84	0.17
95%	1.87	4.87	8.61	19.37	2.82	0.60
Population	5.47	14.36	5.38	24.30	0.85	4.10

Notes: This table reports the annual consumption and equity market statistics in the data and in the model.  $\sigma(\Delta c)$  is the standard deviation of log consumption growth.  $\sigma(\Delta d)$  is the standard deviation of log dividend growth.  $\mathbb{E}[r_d - r_f]$  is the average excess log return on the market.  $\sigma(r_d)$  is the standard deviation of the log market return.  $\mathbb{E}[r_f]$  is the average and  $\sigma(r_f)$  is the standard deviation of the risk-free rate. Data values are for the period from 1947 to 2017. In the model, we simulate 10,000 70-year-long samples and report the 50th, 5th, and 95th percentiles of the model statistics from no-disaster samples. The population statistics are obtained from a long simulation path of 1,000,000 years.

Table 3: Return Predictability

	$\beta_{1y}$	$\beta_{3y}$	$\beta_{5y}$	$R_{1y}^2$	$R_{3y}^2$	$R_{5y}^2$
Data	-0.09	-0.09	-0.08	4.4	17.0	26.9
Panel A: Benchmark model						
Median	-0.24	-0.15	-0.11	9.27	15.32	17.29
5%	-0.69	-0.30	-0.21	0.30	0.33	0.27
95%	-0.02	0.01	0.02	30.63	40.82	46.71
Population	-0.05	-0.04	-0.03	1.58	2.99	3.62
Panel B: Model with instantaneous disaster risk						
Median	-0.23	-0.20	-0.17	13.66	34.16	46.41
5%	-0.39	-0.31	-0.24	6.05	14.11	17.76
95%	-0.12	-0.11	-0.10	22.38	50.92	66.70
Population	-0.10	-0.09	-0.08	3.68	9.80	14.51

Notes: This table reports the statistics from the predictability regressions of the following form:

$$\frac{1}{h} \sum_{j=1}^h r_{d,t+j} - r_{f,t+j-1} = \beta_0 + \beta_{hy} pd_t + \epsilon_{t+h},$$

where  $r_{d,t+j} - r_{f,t+j-1}$  is the excess log return on the market from year  $t + j - 1$  to year  $t + j$ , and  $pd_t$  is the log price-dividend ratio of the aggregate equity. The results are reported for  $h = 1, 3$ , and 5 years and the regressions are run at an annual frequency from 1947 to 2017. In the model, we simulate 10,000 70-year-long samples and report the 50th, 5th, and 95th percentiles of the model statistics from no-disaster samples. The population statistics are obtained from a long simulation path of 1,000,000 years.

Table 4: Consumption Growth Predictability

	$\beta_{1y}$	$\beta_{3y}$	$\beta_{5y}$	$R_{1y}^2$	$R_{3y}^2$	$R_{5y}^2$
Data	0.01	0.00	0.00	6.0	1.3	0.0
Panel A: Benchmark model						
Median	0.02	0.02	0.01	1.74	4.37	6.15
5%	-0.02	-0.02	-0.02	0.01	0.03	0.05
95%	0.06	0.05	0.04	14.66	33.13	43.12
Population	0.03	0.03	0.03	14.50	33.10	43.06
Panel B: Model with instantaneous disaster risk						
Median	0.00	0.00	0.00	0.66	1.77	2.70
5%	-0.02	-0.02	-0.02	0.01	0.02	0.02
95%	0.02	0.02	0.02	5.65	15.40	22.40
Population	0.02	0.02	0.01	1.37	3.47	4.84

Notes: This table reports the statistics from the predictability regressions of the following form:

$$\frac{1}{h} \sum_{j=1}^h \Delta c_{t+j} = \beta_0 + \beta_{hy} pd_t + \epsilon_{t+h},$$

where  $\Delta c_{t+j}$  is log consumption growth from year  $t + j - 1$  to year  $t + j$ , and  $pd_t$  is the log price-dividend ratio of the aggregate equity. The results are reported for  $h = 1, 3,$  and  $5$  years and the regressions are run at an annual frequency from 1947 to 2017. In the model, we simulate 10,000 70-year-long samples and report the 50th, 5th, and 95th percentiles of the model statistics from no-disaster samples. The population statistics are obtained from a long simulation path of 1,000,000 years.

Table 5: Implied Variance and the Variance Risk Premium

	$\mathbb{E}[IV]$	$\sigma(IV)$	$\mathbb{E}[VRP]$	$\sigma(VRP)$
Data	36.05	33.52	16.20	20.57
Panel A: Benchmark model				
Median	40.16	39.41	16.38	17.89
5%	34.70	26.32	8.57	12.66
95%	47.95	52.42	21.28	23.87
Population	40.71	39.68	14.49	18.47
Panel B: Model with instantaneous disaster risk				
Median	165.96	103.86	134.96	86.36
5%	65.21	46.61	51.55	38.87
95%	413.99	227.57	339.48	188.05
Population	270.43	239.28	221.13	197.96

Notes: This table reports the statistics for implied variance and the variance risk premium.  $\mathbb{E}[IV]$  is the average implied variance, and  $\mathbb{E}[VRP]$  is the average variance risk premium. Both quantities are expressed in monthly percentage squared terms. Data values are for the period from 1990 to 2017. In the model, we simulate 10,000 28-year-long samples and report the 50th, 5th, and 95th percentiles of the model statistics from no-disaster samples. The population statistics are obtained from a long simulation path of 1,000,000 years.

Table 6: Put-Protected Portfolio Moments

Moneyness	Premium difference $\mathbb{E}[r_d - r_d^k]$				Premium ratio $\frac{\mathbb{E}[r_d^k - r_f]}{\mathbb{E}[r_d - r_f]}$			
	75%	80%	85%	90%	75%	80%	85%	90%
Data	0.75	1.19	1.98	3.43	0.89	0.83	0.71	0.50
Panel A: Benchmark model								
Median	0.73	1.17	1.78	2.56	0.90	0.83	0.75	0.64
5%	-0.38	-0.32	-0.13	0.25	0.78	0.67	0.53	0.39
95%	1.38	2.11	3.00	4.07	1.06	1.06	1.02	0.96
Population	0.53	0.88	1.40	2.07	0.92	0.87	0.79	0.68
Panel B: Model with instantaneous disaster risk								
Median	2.48	2.80	3.14	3.52	0.59	0.54	0.48	0.42
5%	0.97	1.09	1.22	1.34	0.09	-0.02	-0.14	-0.26
95%	6.24	7.03	7.81	8.70	0.81	0.79	0.77	0.74
Population	2.66	2.96	3.29	3.64	0.51	0.45	0.39	0.32

Notes: This table reports the statistics for the returns on put-protected portfolios with various moneyness values  $k = 0.75, 0.80, 0.85,$  and  $0.90$ . The premium difference,  $\mathbb{E}[r_d - r_d^k]$ , is the difference between the average equity premium and the average put-protected portfolio premium. The premium ratio,  $\frac{\mathbb{E}[r_d^k - r_f]}{\mathbb{E}[r_d - r_f]}$ , is the ratio of the average put-protected portfolio premium to the average equity premium. Data values are for the period from 1990 to 2017. In the model, we simulate 10,000 28-year-long samples and report the 50th, 5th, and 95th percentiles of the model statistics from no-disaster samples. The population statistics are obtained from a long simulation path of 1,000,000 years.

Table C.1: The Results from the Model with Perfect Information

Panel A: Implied variance and the variance risk premium				
	$\mathbb{E}[IV]$	$\sigma(IV)$	$\mathbb{E}[VRP]$	$\sigma(VRP)$
Data	36.05	33.52	16.20	20.57
Median	76.69	2.26	65.05	2.51
5%	65.65	1.78	24.05	1.76
95%	79.01	20.07	68.76	75.10
Population	73.09	12.94	51.47	46.88

Panel B: Put-protected portfolio moments									
Moneyness	Premium difference $\mathbb{E}[r_d - r_d^k]$				Premium ratio $\frac{\mathbb{E}[r_d^k - r_f]}{\mathbb{E}[r_d - r_f]}$				
	75%	80%	85%	90%	75%	80%	85%	90%	
Data	0.75	1.19	1.98	3.43	0.89	0.83	0.71	0.50	
Median	3.66	4.23	4.70	5.19	0.59	0.53	0.46	0.38	
5%	0.65	0.73	0.94	1.27	0.39	0.30	0.22	0.14	
95%	3.81	4.40	5.06	5.94	0.90	0.89	0.86	0.82	
Population	2.67	3.06	3.51	3.99	0.64	0.59	0.53	0.46	

Notes: This table reports the results from a special case of our model with perfect information. Panel A reports the statistics for implied variance and the variance risk premium.  $\mathbb{E}[IV]$  is the average implied variance, and  $\mathbb{E}[VRP]$  is the average variance risk premium. Both quantities are expressed in monthly percentage squared terms. Panel B reports the statistics for the returns on put-protected portfolios with various moneyness values  $k = 0.75, 0.80, 0.85, \text{ and } 0.90$ . The premium difference,  $\mathbb{E}[r_d - r_d^k]$ , is the difference between the average equity premium and the average put-protected portfolio premium. The premium ratio,  $\frac{\mathbb{E}[r_d^k - r_f]}{\mathbb{E}[r_d - r_f]}$ , is the ratio of the average put-protected portfolio premium to the average equity premium. In both panels, data values are for the period from 1990 to 2017. In the model, we simulate 10,000 28-year-long samples and report the 50th, 5th, and 95th percentiles of the model statistics from no-disaster samples. The population statistics are obtained from a long simulation path of 1,000,000 years.