Optimal Risk Sharing with Time Inconsistency and Long-Run Risk

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Abstract

I examine the role of time inconsistency, modeled by hyperbolic discounting, for the dynamics of asset prices and the wealth distribution between agents. Naive time-inconsistent investors with recursive preferences overconsume and have a lower effective elasticity of intertemporal substitution (EIS) than otherwise similar investors who are time-consistent. In both survival and overlapping-generations economies with i.i.d. consumption growth, I show that the suboptimal consumption and saving decisions of the naive time-inconsistent investors endogenously generate long-run risks in the consumption dynamics of the time-consistent agents. As a result, the presence of naive shortsighted investors increases the risk-free rate, volatility, and risk premium in the economy.

Keywords: Time inconsistency, hyperbolic discounting, long-run risk, risk sharing

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1. Introduction

A vast body of experimental and field economics literature shows that a large proportion of people prefer to experience immediate rewards, avoid immediate costs, and revise their plans to postpone unpleasant activities (Loewenstein and Thaler, 1989; Ainslie, 1992). For instance, people may prefer a 7-day holiday now rather than a longer 10-day holiday a month from now, but at the same time they are willing to be more patient in the future and choose a 10-day vacation in 13 months rather than a 7-day vacation in 12 months. Importantly, this short-termism is also prevalent when making financial decisions. As a result of their time inconsistency people may plan to start an aggressive savings plan next year, but when next year comes, they tend to overconsume and save too little, which can have important implications for their long-term wealth. Programs such as Save More Tomorrow (SMarT) raise awareness and aim to alleviate this bias, inducing people to commit to higher savings rates (Thaler and Benartzi, 2004). Economic theory has proposed time-inconsistent preferences captured by hyperbolic discounting as a possible way to model and study such shortsightedness, represented by higher impatience for near-term trade-offs compared to future trade-offs (Phelps and Pollak, 1968; Laibson, 1997).

Even though empirical evidence shows that approximately 50% of people are time-inconsistent (Halevy, 2015), theoretical models usually include a representative investor with time-consistent preferences. In addition, existing models rely on the assumption that investors give a much larger weight to the distant future compared to the near future, which is at odds with the short-termism documented in experimental literature. For instance, the long-run risk model by Bansal and Yaron (2004) that manages to match stylized asset pricing features reasonably well, relies on the assumption that persistent long-run risks exist and investors are highly averse to them, which is captured by a high level of elasticity of intertemporal substitution (EIS). As a result, investors require a higher compensation to invest in risky assets, which increases the price of risk, volatility, and risk premium in the model to levels observed in the real data. It is therefore important to understand how time-inconsistent, shortsighted investors who are less concerned about long-run risks affect asset prices and whether they reduce the importance of such risks for matching asset pricing data. This paper aims to fill in this gap and show how the wealth share of time-inconsistent agents affects equilibrium asset prices in the economy.

To study these effects I construct a general equilibrium model with two agents who are endowed with recursive Epstein and Zin (1989) and Weil (1989) preferences. One of the agents is time-consistent using standard exponential discounting of the future streams of consumption while the other one is time-inconsistent and shortsighted, so she uses quasi-hyperbolic discounting, characterized by higher discount rates in the short-run compared to the long-run. The quasi-hyperbolic discounting gives rise to a disagreement between temporal selves as in Laibson (1997) and Harris and Laibson (2001). Self $t+1$ has a different plan for valuing future consumption than self t and the agent plays an intrapersonal sequential game between her selves to optimize her utility. In addition, I distinguish between two types of time-inconsistent agents based on whether they are aware of their bias, as in O'Donoghue and Rabin

(1999). The sophisticated time-inconsistent agent realizes her bias and corrects for it, while the naive time-inconsistent agent is not aware of it. I consider the cases when consumption growth rate shocks are independent and identically distributed (i.i.d.) and when there are long-run risks to consumption growth, as in Bansal and Yaron (2004). I also show the effect of time inconsistency in two economic settings: a survival economy where investors have infinite investment horizon and one of the agents can be crowded out completely in the long run, and an overlapping-generations economy as in Blanchard (1985) and Garleanu and Panageas (2015) which ensures long-run stationarity. Since the consumption evolution of each agent in the model cannot be expressed analytically, I employ the numerical method of Collin-Dufresne, et al. (2019) using backwards recursion in order to find a solution.

I find that both sophisticated and naive time-inconsistent agents have a lower effective time discount factor than a time-consistent agent and as a result, keeping all else equal, their relative wealth share in the economy decreases over time. Since time-inconsistent agents are relatively more impatient, they require a higher compensation in order to postpone consumption and therefore the risk-free rate increases with their wealth share in the economy. In addition, I show that the naive time-inconsistent investor who is not aware of her bias consumes a steady proportion of her wealth regardless of her wealth share in the economy and the level of the risk-free rate. This consumption behavior reveals an interaction between her time inconsistency bias and the Epstein-Zin preferences she is endowed with, which is not present if she has standard CRRA preferences. Intuitively, the willingness to substitute consumption across time caused by the Epstein-Zin preferences induces a substitution effect, driving the agent to decrease her consumption and save more as she gets wealthier. The time inconsistency bias of the naive agent, however, induces a wealth effect which counteracts this mechanism: she believes she will be time-consistent in the future with the same preferences as the true time-consistent agent and that the two agents will share the aggregate consumption proportionally to their initial endowments. Consequently, since aggregate consumption growth is i.i.d., the naive agent believes that her consumption-wealth ratio and expected consumption growth rate will remain constant as her wealth share in the economy increases. As a result, she consumes a steady proportion of wealth regardless of her wealth share and she has a lower effective elasticity of intertemporal substitution (EIS) than the time-consistent agent, even if both agents are endowed with the same EIS.

This result has two important economic implications. First, since the expected consumption growth rate of naive agents remains constant across different levels of her wealth share in the economy, it generates a persistent component in the expected consumption growth of the time-consistent agent. This represents an endogenous source of long-run risk, even if aggregate consumption growth is i.i.d. Under the preference for early resolution of uncertainty such long-run risks are positively priced. An economy with i.i.d. consumption growth dominated completely by naive agents has a 3-5% larger risk premium than an economy without naive agents. In an overlapping-generations (OLG) setting the effect is even more pronounced. The reason is that the individual consumption growth may contain a persistent component, in case one of the agents has a lower EIS (Garleanu and Panageas, 2015). Since the naive agent has a lower effective EIS than the time-consistent agent, this gives rise to long-run risks. Even though such risks manage to account for many asset pricing features, Bansal and Yaron (2004) assume that they are given exogenously, while their existence remains debatable (Cochrane, 2007). Given that naive agents are shortsighted and less averse to long-run risks, it is striking that their presence gives rise to such risks. Thus, instead of decreasing the importance of long-run risks in the economy, the presence of time-inconsistent agents provides evidence for their existence and new potential sources of them.

Second, the fact that naive agents have lower effective EIS helps to alleviate the concern that long-run risk models require a very high level of EIS to match the data. According to Epstein, Farhi, and Strzalecki (2014), the EIS parameter used in the model is too high, inducing a very strong preference for early resolution of uncertainty. This implies that agents would be prepared to give up more than 30% of lifetime wealth to resolve upcoming uncertainty, which they argue puts a large weight to the distant future and is too extreme to be realistic. However, I argue that even when naive agents are endowed with high EIS, their effective EIS is low. In the presence of long-run risks time-inconsistent agents in the economy lead to an even larger increase in the risk premium as their wealth share increases. Thus, instead of having a negative impact on the importance of long-run risks in the economy, naive agents with lower effective EIS lead to a higher equilibrium risk premium.

To further understand how time-inconsistent agents affect the importance of long-run risks for asset prices, I show the dynamics of the model when such risks are exogenously assumed, as in Bansal and Yaron (2004). In this setting hedging demands appear as a result of the different preferences of investors. The time-inconsistent agent who is less concerned about long-run risk is ready to bear a larger proportion of it and sell insurance to the time-consistent agent against it. Thus, in states with low consumption growth the time-inconsistent investor loses a larger share of her wealth since she pays out insurance to the time-consistent investor. The time-consistent agent in turn gains relative share. Since she is more averse to long-run risk, she requires higher compensation for it, driving the risk premium up in recessions. The opposite effect prevails in states with high consumption growth, which generates countercyclicality implications for the risk premium in the economy. The effect of time inconsistency preferences, however, counteracts this mechanism. Both sophisticated and naive agents lose relative share to the time-consistent agent in expansions as well as recessions, and as a result the model generates a procyclical risk premium.

Taking into account heterogeneity in terms of time inconsistency sheds light on the wealth distribution between time-consistent and time-inconsistent agents in the economy, and their economic importance in the long-run. I find that in a survival economy sophisticated agents are crowded out of the market only after more than 500 years. In a stationary OLG setting they share the total wealth in the economy almost equally with time-consistent agents. Calibrating the time inconsistency parameter to estimates found in experiments ($\delta = 0.8$ annually) leads to extreme outcomes if the time-inconsistent agents are naive, however: they are crowded out of the market in 35 years even if their initial share is 99.99%. Since experimental estimates might pick up the risk and uncertainty related to receiving rewards in the distant future or unexpected liquidity constraints that agents may face, this parameter can be an overestimate of the true shortsightedness of agents (Halevy, 2015). To alleviate this concern I use more conservative time inconsistency levels, $\delta = 0.9$ or $\delta = 0.98$, and find that in the long-run naive agents hold 10% or 37% of the total wealth respectively, depending on their time inconsistency level. Even when their bias is strong and they hold a relatively small portion of the total wealth (10%), naive agents can still affect asset prices, leading to a 10-15% increase in the risk premium upon the baseline model with a representative time-consistent agent.

The novelty of this study is exploring a new line of heterogeneity in terms of time inconsistency in a class of asset pricing models featuring recursive preferences that have shown potential in resolving asset pricing puzzles. Even though accounting for heterogeneity is plausible and can lead to interesting interactions among agents, these models typically analyze a representative agent. The reason, as emphasized by Collin-Dufresne, et al. (2019), is the complexity in solving such models and so far there are only a few studies incorporating agents of different types (Borovička, 2019; Collin-Dufresne, Johannes, and Lochstoer, 2017; Garleanu and Panageas, 2015; Pohl, Schmedders, and Wilms, 2018, among others). These papers have mainly focused on studying investors who differ in their beliefs and preferences. However, the effect of time inconsistency in such models has not been investigated yet.

In addition, this paper contributes to the literature that justifies the existence of long-run risks, which are difficult to show empirically (Hansen, Heaton, and Lee, 2005). In the model by Bansal and Yaron (2004) long-run consumption risks can account for many features of the asset pricing data, but they are modeled using an exogenous process. Previous literature has documented potential sources of such risks. Kaltenbrunner and Lochstoer (2010) find that optimal consumption smoothing induces long-run risks, even if technology growth rate is i.i.d., while Garleanu and Panageas (2015) find that a persistent component in individual agent's consumption growth arises in an overlapping-generations setting, even if aggregate consumption growth is i.i.d. This paper gives an additional source of such risks and provides more evidence for their existence in the presence of naive shortsighted agents.

The rest of the paper is organized as follows. Section 2 presents the model setup and the economy in which I study the effect of time inconsistency on risk sharing and asset prices. Section 3 presents the numerical method used to find the optimal consumption allocation between the agents. Section 4 outlines analytically the equilibrium wealth sharing dynamics. Section 5 presents the asset pricing implications of the model. Section 6 discusses the long-run survival and stationary outcomes, and section 7 concludes the paper.

2. The model

2.1 Agents' preferences and demographics

I consider a discrete time general equilibrium model with heterogeneous agents who have recursive preferences. To study the effect of time inconsistency on wealth distribution and asset prices, I model two types of agents $i = \{C, I\}$, time-consistent (C) and time-inconsistent (I), who can also have different risk aversion, elasticity of intertemporal substitution (EIS), and subjective time discount factors. In addition, I distinguish between two types of time-inconsistent agents: sophisticated (S) and naive (N) as in O'Donoghue and Rabin (1999). The sophisticated agent is fully rational and realizes her time inconsistency bias as in (Strotz, 1956), while the naive one is not aware of it. In the next subsections I describe the individual preferences of each of these agents and derive their intertemporal marginal rates of substitution which I use in section 5. to study the asset pricing implications of the model. In section **2.3** I extend the setting to an overlapping-generations (OLG) model based on Blanchard (1985) and Garleanu and Panageas (2015), which brings long-run stationarity.

2.1.1 Time-consistent agent

The time-consistent agent, denoted by $i = C$, has Epstein-Zin preferences suggested by Koopmans (1960), Kreps-Porteus (1978), Epstein and Zin (1989), and Weil (1989), and she solves the following optimization problem:

$$V_{C,t} = \max_{\{C_{C,t}, W_{C,t+1}\}} \left[(1-\beta_C) C_{C,t}^{\rho_C} + \beta_C E_t[(V_{C,t+1})^{\alpha_C}]^{\frac{\rho_C}{\alpha_C}} \right]^{\frac{1}{\rho_C}} \tag{1}$$

$$\text{s.t. } \Pi_{C,t} C_{C,t} + E_t[\Pi_{C,t+1} W_{C,t+1}] \leq \Pi_{C,t} W_{C,t}, \tag{2}$$

where $\Pi_{C,t}$ is the marginal rate of substitution of consumption at time t. $W_{C,t,s}$ is the total wealth of agent C, defined as the sum of current and discounted future consumption. Epstein and Zin (1989) show that the value of wealth can be expressed as follows:

$$W_{C,t} = \frac{V_{C,t}^{\rho_C}}{(1-\beta_C) C_{C,t}^{\rho_C - 1}}. \tag{3}$$

The investor optimizes dynamically by maximizing her value function $V_{C,t}$ that depends on current consumption $C_{C,t}$ and the certainty equivalent $E_t(V_{C,t+1}^{\alpha_C})^{\frac{1}{\alpha_C}}$ of her continuation value function $V_{C,t+1}$. The time-consistent investor uses standard exponential discounting, where the time discount factor is β_C, ranging from 0 to 1. Under Epstein-Zin preferences the elasticity of intertemporal substitution (EIS), denoted by ψ_C in the literature, and the risk aversion level, denoted by γ_C, can be separated. Throughout the paper I use the following simplified notation: $\rho_C = 1 - 1/\psi_C$ and $\alpha_C = 1 - \gamma_C$. I assume a concave form of the utility functions and thus the parameters γ_C and ψ_C take only positive values. Hence, $\rho_C \leq 1$ and $\alpha_C \leq 1$. Solving the maximization problem, we can find the stochastic discount factor for agent C:

$$\frac{\Pi_{C,t+1}}{\Pi_{C,t}} = \beta_C \left(\frac{C_{C,t+1}}{C_{C,t}}\right)^{\rho_C - 1} \left(\frac{V_{C,t+1}}{E_t[V_{C,t+1}^{\alpha_C}]^{\frac{1}{\alpha_C}}}\right)^{\alpha_C - \rho_C} \tag{4}$$

Note that equation (4) holds for every state of the world. The first part of the stochastic discount factor $\beta_C \left(\frac{C_{C,t+1}}{C_{C,t}}\right)^{\rho_C - 1}$ captures the intertemporal substitution between time t and $t+1$ which depends on the time discount factor β_C. The second part $\left(\frac{V_{C,t+1}}{E_t[V_{C,t+1}^{\alpha_C}]^{\frac{1}{\alpha_C}}}\right)^{\alpha_C - \rho_C}$ compares the ex-post utility $V_{C,t+1}$ (that depends on the shocks to aggregate consumption ε_{t+1}) to the ex-ante expected future utility $E_t[V_{C,t+1}^{\alpha_C}]^{\frac{1}{\alpha_C}}$.

2.1.2 Time-inconsistent sophisticated agent

The time-inconsistent agent, denoted by I, is short-sighted and gives a stronger relative weight to consumption the closer it is in the future. In contrast to the time-consistent agent who uses exponential discounting, both the sophisticated and naive time-inconsistent agents use quasi-hyperbolic discounting (Phelps and Pollak, 1968). As a result, their short-run discount rates are larger than their long-term ones.

The sophisticated time-inconsistent agent, denoted by S, maximizes her utility by playing a sequential intrapersonal game with her future selves where the solution yields the subgame-perfect equilibrium. I use a recursive formulation of her value function and optimization problem, similar to the one with horizon-dependent risk aversion defined by Andries, Eisenbach, and Schmalz (2018):

$$V_{S,t,t} = \max_{\{C_{S,t,t}, W_{S,t,t+1}\}} \left[(1-\beta_S) C_{S,t,t}^{\rho_S} + \beta_S \delta_S E_t[(V_{S,t,t+1})^{\alpha_S}]^{\frac{\rho_S}{\alpha_S}}\right]^{\frac{1}{\rho_S}} \tag{5}$$

$$\text{s.t.} \quad \Pi_{S,t,t} C_{S,t,t} + E_t[\Pi_{S,t,t+1} W_{S,t,t+1}] \leq \Pi_{S,t,t} W_{S,t,t}, \tag{6}$$

$$V_{S,t,t+1} = \left[(1-\beta_S) C_{S,t,t+1}^{\rho_S} + \beta_S E_{t+1}[(V_{S,t,t+2})^{\alpha_S}]^{\frac{\rho_S}{\alpha_S}}\right]^{\frac{1}{\rho_S}}, \tag{7}$$

where δ_S is the discount factor that the agent uses in the short run in addition to the time discount factor β_S. In comparison, the time-consistent investor uses a time discount factor β_C for both the short and the long run. The range of the parameter δ_S is between 0 and 1, where 1 corresponds to time consistency and lower values correspond to higher levels of time inconsistency. The continuation value function $V_{S,t,t+1}$ represents the value self t gives to future consumption, that differs from the actual value function that self $t+1$ will optimize $V_{S,t+1,t+1}$. The first subindex of the value functions and consumption denotes the time at which the agent makes a plan about a given variable and the second subindex shows the time at which the value takes place. For instance, $V_{S,0,1}$ denotes the value function at time 1 according to the plan that self 0 makes at time 0. Equation (7) shows that the time discount factor that self t plans to use in the future is greater than her current discount factor ($\beta_S > \delta_S\beta_S$). In other words her short-run discount rate ($-\ln\delta_S\beta_S$) is larger than her long-run discount rate ($-\ln\beta_S$). This means that self t gives larger relative weight to immediate consumption $C_{S,t,t}$ at time t than she plans to give to immediate consumption $C_{S,t,t+1}$ at time $t+1$.

Deriving the first-order conditions of the optimization problem (5) – (7) in Appendix B, I show that the stochastic discount factor of the time-inconsistent sophisticated investor is given by:

$$\frac{\Pi_{S,t+1}}{\Pi_{S,t}} \quad = \quad \delta_S\beta_S \frac{\frac{1}{\rho_S}\frac{\partial V^{\rho_S}_{S,t,t+1}}{\partial W_{S,t,t+1}}}{C^{\rho_S-1}_{S,t,t}} \left(\frac{V_{S,t,t+1}}{E_t[V^{\alpha_S}_{S,t,t+1}]^{\frac{1}{\alpha_S}}}\right)^{\alpha_S-\rho_S} \tag{8}$$

Note that for a time-consistent agent the planned value function equals the actual one $V_{S,t,t+1} = V_{S,t+1,t+1}$ and the optimal consumption $C_{t+1,t+1}$ maximizes both the planned and the actual value functions. Since Envelope theorem holds in this case we have:

$$\frac{1}{\rho_C}\frac{\partial V^{\rho_C}_{C,t,t+1}}{\partial W_{C,t,t+1}} = \frac{1}{\rho_C}\frac{\partial V^{\rho_C}_{C,t+1,t+1}}{\partial W_{C,t+1,t+1}} = C^{\rho_C-1}_{C,t+1,t+1} \tag{9}$$

The sophisticated time-inconsistent agent, however, realizes that the continuation function planned by self t, $V_{S,t,t+1}$, differs from the actual function that self $t+1$ will optimize, $V_{S,t+1,t+1}$. Unable to change her type and realizing she will overconsume in the future periods, she incorporates this in her plan for time $t+1$. Hence, the planned consumption and wealth will be equal to the optimal consumption and wealth that self $t+1$ will solve for ($C_{S,t,t+1} = C_{S,t+1,t+1}$). Therefore, in her optimization she takes into account the optimal consumption $C_{S,t+1,t+1}$ that maximizes the actual value function $V_{S,t+1,t+1}$, instead of the consumption planned by self t, $C_{S,t,t+1}$, that maximizes the continuation value function $V_{S,t,t+1}$. This means, however, that the optimal $C_{S,t+1,t+1}$ does not maximize the continuation function $V_{S,t,t+1}$ and the Envelope theorem cannot be applied to the following term of the stochastic discount factor:

$$\frac{1}{\rho_S}\frac{\partial V^{\rho_S}_{S,t,t+1}}{\partial W_{S,t,t+1}} \neq C^{\rho_S-1}_{S,t+1,t+1} \tag{10}$$

In order to express the stochastic discount factor in terms of the optimal consumption $C_{S,t+1,t+1}$ of agent S at time $t+1$ I explicitly show how the agent adjusts her stochastic discount factor to account for her time inconsistency bias:

$$\frac{\Pi_{S,t+1}}{\Pi_{S,t}} = \delta_S\beta_S\left(\frac{C_{S,t+1,t+1}}{C_{S,t,t}}\right)^{\rho_S-1}\left(\frac{V_{S,t,t+1}}{E_t[(V_{S,t,t+1})^{\alpha_S}]^{\frac{1}{\alpha_S}}}\right)^{\alpha_S-\rho_S}\left(\frac{V_{S,t,t+1}}{V_{S,t+1,t+1}}\right)^{\rho_S} \tag{11}$$

The last term $\left(\frac{V_{S,t,t+1}}{V_{S,t+1,t+1}}\right)^{\rho_S}$ represents the adjustment term which ensures that the agent allocates sufficient resources for her overconsumption at time $t+1$. Since she accounts for the fact that her marginal utility of current wealth at time $t+1$ will be lower than the one self t expects, she requires a higher compensation for any unit of consumption she postpones from time t to time $t+1$.

2.1.3 Time-inconsistent naive agent

The naive agent, denoted by N, has the same specification of preferences as the sophisticated agent (equations (5) – (7)) and believes that she will be time-consistent in valuing her future strings of consumption. Unlike the sophisticated agent, she is not aware of her time inconsistency and she wrongly assumes that her future self $t+1$ will optimize the continuation value function $V_{N,t,t+1}$ according to the plan of self t. Thus, instead of realizing that at time $t+1$ her self $t+1$ will optimize the following actual value function:

$$V_{N,t+1,t+1} = \left[(1-\beta_N)C_{N,t+1,t+1}^{\rho_N} + \beta_N\delta_N E_{t+1}\left[(V_{N,t,t+2})^{\alpha_N}\right]^{\frac{\rho_N}{\alpha_N}}\right]^{\frac{1}{\rho_N}}, \tag{12}$$

she optimizes the value function planned by self t:

$$V_{N,t,t+1} = \left[(1-\beta_N)C_{N,t,t+1}^{\rho_N} + \beta_N E_{t+1}\left[(V_{N,t,t+2})^{\alpha_N}\right]^{\frac{\rho_N}{\alpha_N}}\right]^{\frac{1}{\rho_N}}. \tag{13}$$

Thus, in contrast to the sophisticated agent whose planned consumption equals the optimal consumption ($C_{S,t,t+1} = C_{S,t+1,t+1}$), the planned consumption of the naive agent $C_{N,t,t+1}$ will be different from her optimal consumption $C_{N,t+1,t+1}$. In Appendix B I show that the stochastic discount factor of the naive agent is given as follows for each state of the economy:

$$\frac{\Pi_{N,t+1}}{\Pi_{N,t}} = \delta_N\beta_N\left(\frac{C_{N,t,t+1}}{C_{N,t,t}}\right)^{\rho_N-1}\left(\frac{V_{N,t,t+1}}{E_t[(V_{N,t,t+1})^{\alpha_N}]^{\frac{1}{\alpha_N}}}\right)^{\alpha_N-\rho_N}, \tag{14}$$

where $C_{N,t,t+1}$ is the consumption that the naive agent believes she will have at time $t+1$. This is the optimal consumption the agent would have had, had she been time-consistent. Note that $C_{N,t,t+1}$ maximizes the continuation value function since the agent does not adjust for her time inconsistency bias and thus the Envelope theorem in this case holds.

2.2 Pareto problem with recursive preferences

The two-agent Pareto problem can be represented as the optimization of a social planner who maximizes the weighted sum of utilities of the investors of both types at time t subject to the market clearing condition that the sum of consumption equals the exogenous aggregate consumption C_t. As shown by Lucas and Stokey (1984), Kan (1995), and Backus, Routledge, and Zin (2009) a recursive formulation of this problem exists. Applying Theorem 3 from Lucas and Stockey (1984) it follows that the Pareto optimal consumption allocation between the agents is given by the following Bellman equation:

$$\begin{aligned} V_{I,t,t}(C_{I,t,t}, V_{C,t}) &= \max_{\{C_{I,t,t}, V^*_{C,t+1}\}} \left[C^{\rho_I}_{I,t,t} + \delta_I \beta_I E_t [V_{I,t,t+1}(C_{I,t,t+1}, V_{C,t+1})^{\alpha_I}]^{\frac{\rho_I}{\alpha_I}} \right]^{\frac{1}{\rho_I}} \quad (15) \\ &\text{s.t.} \quad V_{C,t}(C_{C,t}, V_{C,t+1}) \geq V^*_{C,t} \\ &\quad V_{I,t,t+1} = \left[C^{\rho_I}_{I,t,t+1} + \beta_I E_{t+1} [V_{I,t,t+2}(C_{I,t,t+2}, V_{I,t,t+2})^{\alpha_I}]^{\frac{\rho_I}{\alpha_I}} \right]^{\frac{1}{\rho_I}} \\ &\quad C_{I,t,t} + C_{C,t} = C_t, \end{aligned}$$

where $V^*_{C,t}$ is the so-called promised utility to agent C at time t and $V_{I,t,t}(C_{I,t,t}, V_{C,t})$ is the value function of agent I. Since there is monotonicity in preferences, the utility-promise constraint is binding and hence, the constraint can be replaced by $V_{C,t}(C_{C,t}, V_{C,t+1}) = V^*_{C,t}$.

The outcome of the optimization is not determined by the social planner's preferences, so either of them can act as a social planner. In that sense the problem is equivalent to finding the competitive equilibrium that is Pareto optimal for both agents. We can view this as maximizing the utility of agent I at time t subject to her own consumption $C_{I,t,t}$ and the promised utility to agent C at time $t+1$, $V^*_{C,t+1}$, that is the aggregate utility over the remaining horizon that agent I promises to agent C. Agent I can increase her consumption at time t up to the point where the utility of agent C at time t does not fall below the promised utility $V^*_{C,t}$. Thus, agent I can either choose to have higher consumption $C_{I,t,t}$ at time t and lower utility $V_{I,t,t+1}$ at time $t+1$ by promising higher utility to agent C at time $t+1$, or alternatively agent I can choose to have lower consumption $C_{I,t,t}$ at time t and higher utility $V_{I,t,t+1}$ at time $t+1$ by promising lower utility $V^*_{C,t+1}$ to agent C at time $t+1$. The promised utility $V^*_{C,t+1}$ that was chosen at time t will serve as a constraint for the minimum utility agent C will receive at time $t+1$, and so on until the terminal date T. This means that the promised utility at time 0 determines the initial endowments of the agents and binds the consumption allocation of the agents between the current and following periods, and between themselves.

I solve a normalized version of the model with the value functions and consumption of each agent divided by aggregate consumption. I denote $v_{i,t} = V_{i,t}/C_t$ and $c_{i,t} = C_{i,t}/C_t$ and hence, the value functions can be written as:

$$v_{i,t} = \left[(1-\beta_i) c^{\rho_i}_{i,t} + \beta_i E_t \left[v^{\alpha_i}_{i,t+1} (C_{t+1}/C_t)^{\alpha_i} \right]^{\frac{\rho_i}{\alpha_i}} \right]^{\frac{1}{\rho_i}}, \quad (16)$$

where the market clearing condition is $c_{I,t} + c_{C,t} = 1$.

Under the assumption of frictionless complete markets the equilibrium requirement (presented here in a normalized form) for solving the maximization problem is given by the first-order condition – the marginal intertemporal rates of substitution of the two agents must be equal for each state and over each time period. The equilibrium condition for a time-consistent and a sophisticated time-inconsistent agent is:

$$\delta_S\beta_S\left(\frac{c_{S,t+1,t+1}}{c_{S,t,t}}\right)^{\rho_S-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_S-1}\left(\frac{v_{S,t,t+1}}{E_t\left[v_{S,t,t+1}^{\alpha_S}(C_{t+1}/C_t)^{\alpha_S}\right]^{\frac{1}{\alpha_S}}}\right)^{\alpha_S-\rho_S}\left(\frac{v_{S,t,t+1}}{v_{S,t+1,t+1}}\right)^{\rho_S}=$$

$$=\beta_C\left(\frac{c_{C,t+1}}{c_{C,t}}\right)^{\rho_C-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_C-1}\left(\frac{v_{C,t+1}}{E_t\left[v_{C,t+1}^{\alpha_C}(C_{t+1}/C_t)^{\alpha_C}\right]^{\frac{1}{\alpha_C}}}\right)^{\alpha_C-\rho_C}. \tag{17}$$

Detailed derivation is provided in Appendix C. Analogously, the equilibrium condition for a time-consistent and a naive time-inconsistent agent is given by:

$$\delta_N\beta_N\left(\frac{c_{N,t,t+1}}{c_{N,t,t}}\right)^{\rho_N-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_N-1}\left(\frac{v_{N,t,t+1}}{E_t\left[v_{N,t,t+1}^{\alpha_N}(C_{t+1}/C_t)^{\alpha_N}\right]^{\frac{1}{\alpha_N}}}\right)^{\alpha_N-\rho_N}=$$

$$=\beta_C\left(\frac{c_{C,t+1}}{c_{C,t}}\right)^{\rho_C-1}\left(\frac{C_{t+1}}{C_t}\right)^{\alpha_C-1}\left(\frac{v_{C,t+1}}{E_t\left[v_{C,t+1}^{\alpha_C}(C_{t+1}/C_t)^{\alpha_C}\right]^{\frac{1}{\alpha_C}}}\right)^{\alpha_C-\rho_C}. \tag{18}$$

Given a Pareto-optimal allocation, we can find the equilibrium prices and hence, all competitive equilibria can be determined. However, we can only estimate the initial endowments if we are given an equilibrium condition, but we cannot find the equilibrium given the initial endowments. The reason is that the current period value functions (in particular the utility that agent I promises to agent C, V_C^*) that implicitly serve as initial endowments depend on the future value functions and consumption allocations that are unknown. This poses a challenge for finding a closed-form solution of the model since the consumption sharing rule I solve for depends on the future value functions that are unknown. Therefore, I employ the numerical method of Collin-Dufresne, et al. (2019) using backwards recursion in order to find a solution.

2.3 OLG setting

In this section I extend the setting to a discrete time overlapping-generations (OLG) model based on Blanchard (1985) and Garleanu and Panageas (2015). At each point in time a mass λ of agents is born and a randomly chosen mass λ of all agents die, such that the agents face a hazard rate $\lambda \in [0,1]$ of dying and the population remains constant. The size of the population is normalized to 1, so newborn agents start with an initial share of the total population λ and a $(1-\lambda)$ share of each generation survives every period. Thus, agents born at time s represent a $\lambda(1-\lambda)^{t-s}$ part of the population at time t. Summing up the shares of agents born at each period from $-\infty$ to t equals the total population size of 1 at time t:

$$\sum_{s=-\infty}^{t}\lambda(1-\lambda)^{t-s}=\lambda\frac{1}{\lambda}=1, \tag{19}$$

where $\sum_{s=-\infty}^{t}(1-\lambda)^{t-s}$ is a geometric series which sums up to $1/\lambda$. At each point of time there are two types of agents in the economy, time-consistent and time-inconsistent, whose individual preferences when they have infinite investment horizon are described in the previous subsections.

The total wealth at time t of each agent born at time s is denoted by $W_{i,t,s}$ and is comprised of the sum of her financial wealth $F_{i,t,s}$ and labor income $H_{i,t,s}$. At every period t every agent born at time s has the same human capital and receives an equal amount of earnings $y_{i,t,s} = \frac{1}{2}\omega Y_t$ per capita, where Y_t is aggregate output, $\omega \in (0,1)$ is the fraction of output allocated to labor income, and $1-\omega$ is paid out as dividends. Labor and goods markets clear and in equilibrium aggregate consumption equals total output $C_t = Y_t$. The total labor income of agent i can thus be represented as the discounted sum of all future earnings:

$$H_{i,t,s} = E_s \sum_{t=s}^{\infty} \frac{\Pi_t}{\Pi_s}(1-\lambda)^{(t-s)} y_{i,t,s} = E_s \sum_{t=s}^{\infty} \frac{\Pi_t}{\Pi_s}(1-\lambda)^{(t-s)} \frac{1}{2}\omega C_t \tag{20}$$

The financial wealth is the difference between total wealth and labor income:

$$F_{i,t,s} = W_{i,t,s} - H_{i,t,s} = W_{i,t,s} - E_s \sum_{t=s}^{\infty} \frac{\Pi_t}{\Pi_s}(1-\lambda)^{(t-s)} \frac{1}{2}\omega C_t \tag{21}$$

I assume that there exists a zero-profit insurance company that provides agents who survive with annuity payments $\lambda F_{t,s}$ (a fraction λ of their financial wealth $F_{t,s}$) in exchange for receiving the agents' terminal financial wealth at the time of death.

As in Blanchard (1985), at birth newborn agents do not have any financial wealth, but only earnings $y_{s,s} = \frac{1}{2}\omega Y_s$ per capita, representing their total wealth $W_{i,s,s}$. Since the newborn agents of each type are equally skilled, they share the earnings allocated to them equally and the wealth of newborn agents of type i is given as half of the total labor income ωW_t per capita.

$$\begin{aligned} W_{i,t,t} &= \frac{1}{2}\omega W_t, \\ \text{where } W_t &= W_{I,t} + W_{C,t}. \end{aligned} \tag{22}$$

Then the ratio of the total wealth of newborn time-consistent agents, for instance, relative to the total wealth of all existing time-consistent agents at time t, $W_{C,t}$, is:

$$\frac{W_{C,t+1,t+1}}{W_{C,t+1}} = \frac{\frac{1}{2}\omega W_{t+1}}{W_{C,t+1}} = \frac{1}{2}\omega \frac{W_{I,t+1} + W_{C,t+1}}{W_{C,t+1}} = \frac{1}{2}\omega \left(1 + \frac{W_{I,t+1}}{W_{C,t+1}}\right), \tag{23}$$

where $W_{I,t+1}$ and $W_{C,t+1}$ can be expressed in terms of the value functions and consumption of the agents. Note that since there are no intra-generational differences among agents of the same type, their consumption-wealth ratios are equal $\left(\frac{C_{C,t+1,t+1}}{W_{C,t+1,t+1}} = \frac{C_{C,t+1}}{W_{C,t+1}}\right)$. Hence, the ratio of consumption of newborn agents of type i to the total consumption of agents of type i equals the ratio of wealth of newborn agents of type i to the total wealth of agents of type i:

$$\frac{C_{C,t+1,t+1}}{C_{C,t+1}} = \frac{W_{C,t+1,t+1}}{W_{C,t+1}} = \frac{1}{2}\omega\left(1 + \frac{W_{I,t+1}}{W_{C,t+1}}\right) \tag{24}$$

In order to find the equilibrium condition in a competitive economy, both individual and aggregate optimality must hold. Hence, I start by solving the optimization problem of each individual investor of a certain type and generation, and then aggregate within type over generations. I show the aggregation for the time-consistent type of agents. The case of time-inconsistent agents is analogous. An individual time-consistent agent born at time s optimizes the following problem:

$$V_{C,s,s} = \max_{C_{C,s,s},W_{C,t,s}} \left[(1-\beta_C)C_{C,s,s}^{\rho_C} + (1-\lambda)\beta_C E_s\left[V_{C,s+1,s}^{\alpha_C}\right]^{\frac{\rho_C}{\alpha_C}}\right]^{\frac{1}{\rho_C}} \tag{25}$$
$$\text{s.t.}\quad C_{C,s,s} + E_s\left[\frac{\Pi_t}{\Pi_s}(1-\lambda)^{(t-s)}W_{C,t,s}\right] \leq W_{C,s,s},$$

where $(1-\lambda)$ represents the survival probability of the agent. The probability that the agent will survive until time t, $t-s$ periods ahead of her birth s, is thus $(1-\lambda)^{t-s}$ and she discounts future utility taking this into account. Once we solve for the first-order conditions of this agent we get the following stochastic discount factor from time s to time t:

$$\frac{\Pi_t}{\Pi_s} = \beta_C^{t-s}\left(\frac{C_{C,t,s}}{C_{C,s,s}}\right)^{\rho_C-1}\prod_{j=s+1}^{t}\left(\frac{V_{C,j,s}}{E_{j-1}[V_{C,j,s}^{\alpha_C}]^{\frac{1}{\alpha_C}}}\right)^{\alpha_C-\rho_C} \tag{26}$$

I rewrite this equation to express consumption at time t with consumption at the time of birth s and sum up the consumption of each agent of type C who is alive at time t, weighted by her population share in order to find aggregate consumption of time-consistent agents in that period $C_{C,t}$:

$$C_{C,t} = \sum_{s=-\infty}^{t}\lambda(1-\lambda)^{(t-s)}C_{C,s,s}\left(\beta_C^{-(t-s)}\frac{\Pi_t}{\Pi_s}\right)^{\frac{1}{\rho_C-1}}\prod_{j=s+1}^{t}\left(\frac{V_{C,j,s}}{E_{j-1}[V_{C,j,s}^{\alpha_C}]^{\frac{1}{\alpha_C}}}\right)^{\frac{\rho_C-\alpha_C}{\rho_C-1}}. \tag{27}$$

Similarly we can write consumption at time $t+1$ and combine with equation (27) to get the total consumption of agents of type C at time $t+1$:

$$C_{C,t+1} = (1-\lambda)C_{C,t}\left(\beta_C^{-1}\frac{\Pi_{t+1}}{\Pi_t}\right)^{\frac{1}{\rho_C-1}}\left(\frac{V_{C,t+1}}{E_t[V_{C,t+1}^{\alpha_C}]^{\frac{1}{\alpha_C}}}\right)^{\frac{\rho_C-\alpha_C}{\rho_C-1}} + \lambda C_{C,t+1,t+1}. \tag{28}$$

Intuitively the first term gives the consumption of the already existing agents at time $t+1$ and the second term represents the consumption of the newborn agents. Dividing both sides of equation (28) by C_t we can solve for the intertemporal marginal rate of substitution:

$$\frac{\Pi_{C,t+1}}{\Pi_{C,t}} = \beta_C \left(\frac{C_{C,t+1}}{C_{C,t}}\right)^{\rho_C - 1} \left(\frac{V_{C,t+1}}{E_t[V_{C,t+1}^{\alpha_C}]^{\frac{1}{\alpha_C}}}\right)^{\alpha_C - \rho_C} \left[\frac{1}{1-\lambda}\left(1 - \lambda \frac{C_{C,t+1,t+1}}{C_{C,t+1}}\right)\right]^{\rho_C - 1}. \tag{29}$$

The consumption share of the newborn agents of type C relative to the aggregate consumption share of agents of type C can be expressed in terms of the wealth shares as in expression (24): $\frac{C_{C,t+1,t+1}}{C_{C,t+1}} = \frac{W_{C,t+1,t+1}}{W_{C,t+1}}$.

Note that the last term of the stochastic discount factor (in square brackets) can take negative values, turning the whole expression negative. Negative stochastic discount factors, however, imply arbitrage opportunities and cannot be a solution to the problem. Thus, I express $\frac{W_{C,t+1,t+1}}{W_{C,t+1}}$ in terms of wealth shares and require it to take positive values. Hence,

$$w_{C,t+1} > \frac{\omega\lambda}{2} \tag{30}$$

This requirement puts a natural bound on the wealth share of the agents, that has an intuitive economic interpretation: the minimum wealth of all agents of a certain type must be larger than the wealth share of newborn agents who are endowed with positive and equal initial labor income ($\frac{\omega\lambda}{2}$) by definition.

Similarly we can express the stochastic discount factors of the sophisticate and naive time-inconsistent agents as follows:

$$\frac{\Pi_{S,t+1}}{\Pi_{S,t}} = \delta_S \beta_S \left(\frac{C_{S,t+1,t+1}}{C_{S,t,t}}\right)^{\rho_S - 1} \left(\frac{V_{S,t,t+1}}{E_t[(V_{S,t,t+1})^{\alpha_S}]^{\frac{1}{\alpha_S}}}\right)^{\alpha_S - \rho_S} \left(\frac{V_{S,t,t+1}}{V_{S,t+1,t+1}}\right)^{\rho_S} \left[\frac{1}{1-\lambda}\left(1 - \frac{1}{2}\lambda\omega\left(1 + \frac{W_{S,t+1,t+1}}{W_{S,t+1}}\right)\right)\right]^{\rho_C - 1} \tag{31}$$

$$\frac{\Pi_{N,t+1}}{\Pi_{N,t}} = \delta_N \beta_N \left(\frac{C_{N,t,t+1}}{C_{N,t,t}}\right)^{\rho_N - 1} \left(\frac{V_{N,t,t+1}}{E_{t+1}[(V_{N,t,t+1})^{\alpha_N}]^{\frac{1}{\alpha_N}}}\right)^{\alpha_N - \rho_N} \left[\frac{1}{1-\lambda}\left(1 - \frac{1}{2}\lambda\omega\left(1 + \frac{W_{N,t+1,t+1}}{W_{N,t+1}}\right)\right)\right]^{\rho_N - 1} \tag{32}$$

2.4 The Economy

I assume that markets are complete and aggregate consumption is exogenous and given by C_t. In the baseline results I consider an economy where the only shock to consumption growth rate is an i.i.d. shock ε_{t+1}:

$$g_{t+1} = \ln\left(\frac{C_{t+1}}{C_t}\right) = \mu + \sigma\varepsilon_{t+1} \tag{33}$$

I also consider a Bansal and Yaron (2004) economy where the dynamics of the log-dividend $g_{d,t+1}$ and log-consumption g_{t+1} growth rates contain a persistent and predictable component x_t and are determined as follows:

$$g_{t+1} = \ln\left(\frac{C_{t+1}}{C_t}\right) = \mu + x_t + \sigma\varepsilon_{t+1} \tag{34}$$
$$x_{t+1} = \rho_x x_t + \varphi_e \sigma e_{t+1}$$
$$g_{d,t+1} = \mu_d + \phi_x x_t + \varphi_d \sigma u_{t+1}.$$

I study the case when economic uncertainty σ is constant and the unconditional mean consumption and dividend growth rates are equal ($\mu = \mu_d$). The shocks ε_{t+1}, e_{t+1}, and u_{t+1} are i.i.d., mutually independent and standard normally distributed. The parameters $\varphi_d > 1$, and $\phi_x > 1$ allow for calibration of the dividend volatility and its correlation with consumption. As in Abel (1999) ϕ_x represents the leverage ratio on expected consumption growth. I calibrate $\varphi_d = 4.5$ and $\phi_x = 2.5$. For the purpose of numerical convenience I use a Markov two-state regime switching model and I represent the log aggregate consumption growth and log dividend growth rate as follows:

$$g_{t+1} = \mu_{s_t} + \sigma\varepsilon_{t+1} \tag{35}$$
$$g_{d,t+1} = \mu_d + \phi_x (g_{t+1} - \mu) + \varphi_d \sigma u_{t+1}. \tag{36}$$

There are two possible states of the economy at time t, recession ($s_t = 1$) and expansion ($s_t = 2$). The average consumption growth rate depends on the state s_t, where $\mu_1 < \mu_2$:

$$\mu_{s_t} = \begin{cases} \mu_1 & \text{if } s_t = 1 \\ \mu_2 & \text{if } s_t = 2 \end{cases}, \tag{37}$$

To match the consumption growth moments from the Markov two-state regime switching model with the moments implied by the Bansal and Yaron (2004) parametrization I first assume that the states s_t have a constant, symmetric transition matrix:

$$P = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix}, \tag{38}$$

where $\Pr\{s_{t+1} = j | s_t = i\} = P(i, j)$. The states switch independently of the shocks ε_t according to this matrix. In addition, the state is conditionally known:

$$E_t [g_{t+1}] = E [g_{t+1} | s_t = i] = \mu_i. \tag{39}$$

The unconditional probabilities of state 1 and state 2 are:

$$\pi_1 = \frac{1-p_{22}}{2-p_{11}-p_{22}} \tag{40}$$

$$\pi_2 = \frac{1-p_{11}}{2-p_{11}-p_{22}}. \tag{41}$$

Assuming that $p_{11} = p_{22} = p$, as in Bansal and Yaron (2004) we get:

$$\pi_1 = \pi_2 = \frac{1}{2}. \tag{42}$$

I find the parameters p, μ_1, and μ_2 by matching the unconditional mean, variance, and autocorrelation of consumption growth implied by this Markov switching process with the moments estimated based on the following model (Bansal and Yaron, 2004) :

$$g_{t+1} = x_t^* + \sigma\varepsilon_{t+1}, \tag{43}$$

where

$$x_t^* - \mu = \rho_x(x_{t-1}^* - \mu) + \varphi\sigma e_t. \tag{44}$$

2.5 Asset prices

In section 5. I study the asset pricing implications of the model. The return R_{t+1} on every asset in the economy must satisfy:

$$E_t\left[\frac{\Pi_{t+1}}{\Pi_t}R_{t+1}\right] = 1, \tag{45}$$

where $\frac{\Pi_{t+1}}{\Pi_t}$ is the intertemporal marginal rate of substitution (stochastic discount factor), which I derived in the previous subsections.

The aggregate market is presented as a claim on the future dividend stream D_t. From equation (45) the ex-dividend price, denoted by $P_{d,t}$ must satisfy:

$$E_t\left[\frac{\Pi_{t+1}}{\Pi_t}\frac{P_{d,t+1}+D_{t+1}}{P_{d,t}}\right] = 1. \tag{46}$$

I rewrite this in terms of the $\frac{P_{d,t}}{D_t}$ ratio:

$$\frac{P_{d,t}}{D_t} = E_t\left[\frac{\Pi_{t+1}}{\Pi_t}\frac{D_{t+1}}{D_t}\left(1+\frac{P_{d,t+1}}{D_{t+1}}\right)\right], \tag{47}$$

where $\ln(\frac{D_{t+1}}{D_t}) = g_{d,t+1}$. I determine the dynamics of the $\frac{P_{d,t}}{D_t}$ ratio over time as a function of the endogenous state variable, the consumption share $c_{A,t}$. Since in autarky the $\frac{P_{d,t}}{D_t}$ ratio is constant we can find its terminal closed-form value and use backwards recursion to find its dynamics at the initial time period, given the optimal investment strategy of the agents. Equity is a claim on dividends, so we can write the return on equity as:

$$E_t\left[R_{d,t+1}\right] = E_t\left[\frac{P_{d,t+1}+D_{t+1}}{P_{d,t}}\right] = E_t\left[\frac{\frac{D_{t+1}}{D_t}\left(1+\frac{P_{d,t+1}}{D_{t+1}}\right)}{\frac{P_{d,t}}{D_t}}\right]. \tag{48}$$

The equity premium is the difference between the return on equity and the risk-free rate, where the risk-free rate is given by the inverse of the expected stochastic discount factor:

$$R_{ft+1} = \frac{1}{E_t\left[\frac{\Pi_{t+1}}{\Pi_t}\right]}. \tag{49}$$

3. Numerical Method

I solve the optimization problem (15) numerically using backwards recursion, starting from time $t = T$ (Section **3.1**) and subsequently iterating (Section **3.2**). The solution gives the consumption share distribution between a time-inconsistent agent I who has hyperbolic preferences and a time-consistent agent C who has Epstein-Zin-Weil preferences.

3.1 Time $t = T$

I assume that at time T the agents enter autarky and thus they cannot trade and share risk any longer. The value functions at time T then correspond to the representative agent value functions. At time $t = T$, the value functions are given analytically by:

$$v_{I,T,T} = c_{I,T,T}\tilde{v}_{I,T,T} \tag{50}$$

$$v_{C,T} = (1-c_{I,T,T})\tilde{v}_{C,T}, \tag{51}$$

where the continuation value functions if the agents consume aggregate consumption from time T on are:

$$\tilde{v}_{I,T,T} = \left[1-\beta_I+\beta_I\delta_I\mathbb{E}_T\left[(\tilde{v}_{I,T,T+1})^{\alpha_I}e^{(\mu+\sigma\varepsilon_{T+1})\alpha_I}\right]^{\frac{\rho_I}{\alpha_I}}\right]^{\frac{1}{\rho_I}} \tag{52}$$

$$\tilde{v}_{I,T,T+1} = \left[1-\beta_I+\beta_I\mathbb{E}_{T+1}\left[(\tilde{v}_{I,T+1,T+2})^{\alpha_I}e^{(\mu+\sigma\varepsilon_{T+2})\alpha_I}\right]^{\frac{\rho_I}{\alpha_I}}\right]^{\frac{1}{\rho_I}} \tag{53}$$

$$\tilde{v}_{C,T} = \left[1-\beta_C+\beta_C\mathbb{E}_T\left[(\tilde{v}_{C,T+1})^{\alpha_C}e^{(\mu+\sigma\varepsilon_{T+1})\alpha_C}\right]^{\frac{\rho_C}{\alpha_C}}\right]^{\frac{1}{\rho_C}} \tag{54}$$

Since the normalized function $\tilde{v}_{I,T+1,T+2}$ is independent of the shock ε_{T+2} and the horizon is infinite, it follows that $\tilde{v}_{I,T,T+1} = \tilde{v}_{I,T+1,T+2}$. Thus, we can rewrite (53) and find a closed-form expression for

$\tilde{v}_{I,T,T+1}$:

$$
\begin{aligned}
(\tilde{v}_{I,T,T+1})^{\rho_I} &= 1-\beta_I+\beta_I(\tilde{v}_{I,T,T+1})^{\rho_I}e^{\rho_I\mu+\frac{1}{2}\rho_I\alpha_I\sigma^2} && (55)\\
\Rightarrow \tilde{v}_{I,T,T+1} &= (1-\beta_I)^{\frac{1}{\rho_I}}\left(1-\beta_I e^{\rho_I\mu+\frac{1}{2}\rho_I\alpha_I\sigma^2}\right)^{-\frac{1}{\rho_I}} && (56)
\end{aligned}
$$

We substitute (56) in (52) to find a closed-form expression for the autarky value function $\tilde{v}_{I,T,T}$:

$$
\tilde{v}_{I,T,T}=\left[1-\beta_I+\beta_I\delta_I(1-\beta_I)e^{\rho_I\mu+\frac{1}{2}\rho_I\alpha_I\sigma^2}\left(1-\beta_I e^{\rho_I\mu+\frac{1}{2}\rho_I\alpha_I\sigma^2}\right)^{-1}\right]^{\frac{1}{\rho_I}} \quad (57)
$$

Analogously, since the normalized function $\tilde{v}_{C,T+1}$ is independent of the shock ε_{T+1} and the horizon is infinite, we have $\tilde{v}_{C,T}=\tilde{v}_{C,T+1}$. Rewriting (54) gives a closed-form expression for $\tilde{v}_{C,T}$:

$$
\begin{aligned}
(\tilde{v}_{C,T})^{\rho_C} &= 1-\beta_C+\beta_C(\tilde{v}_{C,T})^{\rho_C}e^{\rho_C\mu+\frac{1}{2}\rho_C\alpha_C\sigma^2} && (58)\\
\Rightarrow \tilde{v}_{C,T} &= (1-\beta_C)^{\frac{1}{\rho_C}}\left(1-\beta_C e^{\rho_C\mu+\frac{1}{2}\rho_C\alpha_C\sigma^2}\right)^{-\frac{1}{\rho_C}} && (59)
\end{aligned}
$$

We will use these functions when we are solving for the optimal $c_{I,T,T}$ at time $t=T-1$.

3.2 Recursion at time t

In this step, it is convenient to use $c_{I,t,t}$ as the endogenous state variable and solve for the optimal consumption share $c^*_{I,t,t+1}$ on a grid for $c_{I,t,t}$. Thus, we can write the value function of agent i at time t as a function of the state variable $c_{I,t,t}$ at time t: $v_{i,t}=v_{i,t}(c_{i,t,t},\varepsilon_{t+1})$. Since we are solving the problem backwards, we may assume that it is already solved at time $t+1$ and we are given the optimal value functions (denoted by $v^*_{i,t+1}$), normalized by aggregate consumption. In the numerical implementation, the functions will not be known for any possible value of the state variables they depend on, but only at a number of (bivariate) grid points of $c_{I,t,t+1}$. Intermediate values have to be calculated by interpolation:

$$
\begin{aligned}
v^*_{I,t,t+1} &= \frac{V^*_{I,t,t+1}}{C_{t+1}} = v^*_{I,t,t+1}(c_{I,t,t+1},\varepsilon_{t+2}), && (60)\\
v^*_{I,t+1,t+1} &= \frac{V^*_{I,t+1,t+1}}{C_{t+1}} = v^*_{I,t+1,t+1}(c_{I,t+1,t+1},\varepsilon_{t+2}), && (61)\\
v^*_{C,t+1} &= \frac{V^*_{C,t+1}}{C_{t+1}} = v^*_{C,t+1}(c_{I,t,t+1},\varepsilon_{t+2}). && (62)
\end{aligned}
$$

At time t the value function $v_{I,t,t}(v_{C,t+1}|c_{I,t,t+1})$ depends on the consumption share of agent I, $c_{I,t,t}$. I also replace $C_{t+1}/C_t=e^{\mu+\sigma\varepsilon_{t+1}}$ that follows from equation (34). The problem to solve now reads as follows:

$$
\begin{aligned}
v_{I,t,t} = & \max_{\{c_{I,t,t}, v_{C,t+1}\}} \left[(1-\beta_I)c_{I,t,t}^{\rho_I} + \delta_I \beta_I E_t\left[v_{I,t,t+1}^{\alpha_I} e^{(\mu+\sigma\varepsilon_{t+1})\alpha_I}\right]^{\frac{\rho_I}{\alpha_I}}\right]^{\frac{1}{\rho_I}} \\
\text{s.t. } v_{C,t} = & \left[(1-\beta_C)c_{C,t}^{\rho_C} + \beta_C E_t\left[v_{C,t+1}^{\alpha_C}\left(e^{(\mu+\sigma\varepsilon_{t+1})\alpha_C}\right)\right]^{\frac{\rho_C}{\alpha_C}}\right]^{\frac{1}{\rho_C}} \\
v_{I,t,t+1} = & \left[(1-\beta_I)c_{I,t,t+1}^{\rho_I} + \beta_I E_{t+1}\left[v_{I,t,t+2}^{\alpha_I} e^{(\mu+\sigma\varepsilon_{t+2})\alpha_I}\right]^{\frac{\rho_I}{\alpha_I}}\right]^{\frac{1}{\rho_I}} \qquad (63) \\
& c_{I,t,t} + c_{C,t} = 1. \qquad (64)
\end{aligned}
$$

As mentioned before, we actually solve numerically the induced first-order conditions:

$$
\begin{aligned}
& \beta_I\left(\frac{c_{I,t,t+1}}{c_{I,t,t}}\right)^{\rho_I-1}\left(e^{\mu+\sigma\varepsilon_{t+1}}\right)^{\alpha_I-1}\left(\frac{v_{I,t,t+1}}{E_t[v_{I,t,t+1}^{\alpha_I}e^{(\mu+\sigma\varepsilon_{t+1})\alpha_I}]^{\frac{1}{\alpha_I}}}\right)^{\alpha_I-\rho_I}\left(\frac{v_{I,t,t+1}}{v_{I,t+1,t+1}}\right)^{\rho_I} - \\
& \beta_C\left(\frac{1-c_{I,t,t+1}}{1-c_{I,t,t}}\right)^{\rho_C-1}\left(e^{\mu+\sigma\varepsilon_{t+1}}\right)^{\alpha_C-1}\left(\frac{v_{C,t+1}}{E_t[v_{C,t+1}^{\alpha_C}e^{(\mu+\sigma\varepsilon_{t+1})\alpha_C}]^{\frac{1}{\alpha_C}}}\right)^{\alpha_C-\rho_C} = 0 \qquad (65) \\
& v_{C,t} = \left[(1-\beta_C)(1-c_{I,t,t}^{\rho_C} + \beta_C E_t\left[v_{C,t+1}^{\alpha_C}(e^{(\mu+\sigma\varepsilon_{t+1})\alpha_C})\right]^{\frac{\rho_C}{\alpha_C}}\right]^{\frac{1}{\rho_C}} \qquad (66) \\
& c_{I,t,t} + c_{C,t} = 1. \qquad (67)
\end{aligned}
$$

Instead of solving the problem for the decision variables $c_{I,t,t}$, $c_{B,t}$, and $v_{B,t+1}$ implied by the system of equations (65), (66), and (67) it is convenient to use a grid for $c_{I,t,t}$ and determine the optimal $c^*_{I,t,t+1}$ and the utilities $v_{I,t,t+1}$ and $v_{C,t+1}$ for all combinations of the grid points. In order to do this we first rewrite equation (65) and solve it as a fixed point problem:

$$
\underbrace{\frac{c_{I,t,t+1}^{\rho_I-1}}{(1-c_{I,t,t+1})^{\rho_C-1}}\frac{v_{I,t,t+1}^{\alpha_I-\rho_I}}{v_{C,t+1}^{\alpha_C-\rho_C}}\left(\frac{v_{I,t,t+1}}{v_{I,t+1,t+1}}\right)^{\rho_I}}_{\Phi_{t,t+1}} = k_t\frac{\beta_C}{\beta_I}\frac{c_{I,t,t}^{\rho_I-1}}{(1-c_{I,t,t})^{\rho_C-1}}\left(e^{\mu+\sigma\varepsilon_{t+1}}\right)^{\alpha_C-\alpha_I} \qquad (68)
$$

$$
\text{where } k_t = \frac{E_t\left[v_{C,t+1}^{\alpha_C}e^{(\mu+\sigma\varepsilon_{t+1})\alpha_C}\right]^{\frac{\rho_C-\alpha_C}{\alpha_C}}}{E_t\left[v_{I,t,t+1}^{\alpha_I}e^{(\mu+\sigma\varepsilon_{t+1})\alpha_I}\right]^{\frac{\rho_I-\alpha_I}{\alpha_I}}} \qquad (69)
$$

It is evident that the evolution of the consumption share from time t to $t+1$ depends on k_t. As Collin-Dufresne, et al. (2019) show, k_t uniquely determines $c_{I,t,t+1}$ since $c_{I,t,t+1} \in (0,1)$ is decreasing in k_t (when $\alpha_i - 1 < 0$, which is the case we consider). Thus, we can solve equations (68) and (69) jointly for k_t as a fixed point problem. From equation (69) we only know k_t as a function of $c_{I,t,t+1}$. The relation between k_t and $c_{I,t,t+1}$, however, is nonlinear and an algorithm which calculates the corresponding

$c_{I,t,t+1}$ for every guess of k_t is computationally costly. Therefore, we estimate k_t as a function of a monotonic transformation of $c_{I,t,t+1}$, $\Phi_{t,t+1}$ which is linear in k_t and more efficient. Once we know the optimal solution for k_t we can find the corresponding $c_{I,t,t+1}$ for each combination of the grid points of the endogenous state variables.

As already mentioned, due to the backwards recursion method I use in order to solve the problem numerically, the optimal $v^*_{I,t,t+1}$, $v^*_{I,t+1,t+1}$, and $v^*_{C,t+1}$ will be known from the previous recursion on a grid of different possible values of the state variables $c_{I,t,t+1}$. Thus, at time t I interpolate both $v^*_{I,t,t+1}$, $v^*_{I,t+1,t+1}$, and $v^*_{C,t+1}$ for the values of the optimal $c^*_{I,t,t+1}$ that I solve for. I use 96 grid points for the endogenous state variable c_I at any point of time:

$$c_I = [0.0001, 0.9999]. \tag{70}$$

It is important to point out that $c_{I,t,t+1}$ and $v_{C,t+1}$, and hence $\Phi_{t,t+1}$, will depend on the exogenous evolution of C_{t+1} and thus on the shock to consumption growth rate ε_{t+1}. Therefore, I choose 7 different grid points for ε_{t+1}, and for each combination of state variables I solve equations (68) and (69) for each of the values of this shock. The expectation of the shock ε is then approximated using Gaussian quadrature. The terminal time T is set far in the future at 500 years. I use the model parameters estimated by Bansal and Yaron (2004) with a quarterly calibration and set $\mu = 0.0045$, $\rho_x = 0.979$, $\varphi_e = 0.044$, $\sigma = 0.0135$. I assume that the risk aversion level is $\gamma_i = 10$ and thus the risk aversion parameter equals $\alpha_i = -9$. The elasticity of intertemporal substitution is set to $\psi_i = 1.5$ and the EIS parameter is $\rho_i = 1/3$. The time discount factor is set to $\beta_i = 0.994$. I also allow the preference parameters of the two investors to differ and provide sensitivity analysis in order to understand the role of the agents' preferences for the relation between investment horizon and risk premia.

At the end of time t the problem is solved and we know the optimal $c^*_{I,t,t+1}$ and the utilities $v^*_{I,t,t}$ and $v^*_{C,t}$ on a grid for the state variable $c_{I,t,t}$ and the shock ε_{t+1}:

$$v^*_{I,t,t} = \frac{V^*_{I,t,t}}{C_t} = v^*_{I,t,t}(c_{I,t,t}, \varepsilon_{t+1}), \tag{71}$$

$$v^*_{I,t,t+1} = \frac{V^*_{I,t,t+1}}{C_t} = v^*_{I,t,t+1}(c_{I,t,t+1}, \varepsilon_{t+1}), \tag{72}$$

$$v^*_{C,t} = \frac{V^*_{C,t}}{C_t} = v^*_{C,t}(c_{I,t}, \varepsilon_{t+1}). \tag{73}$$

For the following recursion at time $t-1$ we interpolate their values corresponding to the grids of the monotonic transformation of the state variable $c_{I,t-1,t-1}$, $\Phi_{t,t+1}$, and the shock ε_t and we solve the problem for the decision variable $c_{I,t-1,t}$.

3.3 Time $t = 0$

At time 0 the equilibrium condition can be written as follows:

$$\beta_I \left(\frac{c_{I,0,1}}{c_{I,0,0}}\right)^{\rho_I - 1} \left(e^{\mu+\sigma\varepsilon_1}\right)^{\alpha_I - 1} \left(\frac{v_{I,0,1}}{E_1[v_{I,0,1}^{\alpha_I} e^{(\mu+\sigma\varepsilon_1)\alpha_I}]^{\frac{1}{\alpha_I}}}\right)^{\alpha_I - \rho_I} \left(\frac{v_{I,0,1}}{v_{I,1,1}}\right)^{\rho_I} =$$

$$= \beta_C \left(\frac{1 - c_{I,0,1}}{1 - c_{I,0}}\right)^{\rho_C - 1} \left(e^{\mu+\sigma\varepsilon_1}\right)^{\alpha_C - 1} \left(\frac{v_{C,1}}{E_0[v_{C,1}^{\alpha_C} e^{(\mu+\sigma\varepsilon_1)\alpha_C}]^{\frac{1}{\alpha_C}}}\right)^{\alpha_C - \rho_C} \tag{74}$$

In this period I solve for the optimal $c^*_{I,0,1}$ (and $c^*_{C,1} = 1 - c^*_{I,0,1}$) and I determine the optimal utilities $v^*_{I,0}$ and $v^*_{C,0}$ of the two agents on a grid of the monotonic transformation $\Phi_{0,1}$ of the state variable $c_{I,0,0}$. Due to the backwards recursion we assume that we know the optimal value functions at time $t+1$, $v^*_{I,0,1}$ and $v^*_{C,1}$, and thus, the initial utilities of the two agents will depend on the promised utility by agent I to agent C, $v_{C,1}$. It is important to note that we can determine these initial utilities and the initial endowments of the two agents, $c_{I,0,1}$ and $c_{C,0}$, given the equilirbia for all the periods ahead until the terminal date T, but we cannot find the possible equilibria knowing only the initial endowments.

The estimations using the described numerical method will show the consumption distribution of the two agents over time and will determine the optimal investment strategies of the two agents. Analyzing the conditions under which one of the two agents is allocated a larger consumption share and dominates the economy will shed light on the market interaction between the investors and the wealth distribution between them.

4. Equilibrium wealth distribution in a survival economy

4.1 Sophisticated time-inconsistent and time-consistent agents

This section discusses the wealth distribution between a time-consistent agent who discounts future consumption exponentially and a sophisticated time-inconsistent agent who uses hyperbolic discounting. The sophisticated agent is aware of her time inconsistency and therefore realizes that today she values future strings of consumption differently from the way she will value them the next periods. As shown in Section **2.2** the equilibrium condition for risk sharing is given by:

$$\delta_S \beta_S \left(\frac{C_{S,t+1,t+1}}{C_{S,t,t}}\right)^{\rho_S - 1} \left(\frac{V_{S,t,t+1}}{E_t[(V_{S,t,t+1})^{\alpha_S}]^{\frac{\rho_S}{\alpha_S}}}\right)^{\alpha_S - \rho_S} \left(\frac{V_{S,t,t+1}}{V_{S,t+1,t+1}}\right)^{\rho_S}$$

$$= \beta_C \left(\frac{C_{C,t+1}}{C_{C,t}}\right)^{\rho_C - 1} \left(\frac{V_{C,t+1}}{E_t[(V_{C,t+1})^{\alpha_C}]^{\frac{\rho_C}{\alpha_C}}}\right)^{\alpha_C - \rho_C} \tag{75}$$

Theorem 1. *The effective discount factor of the time-inconsistent investor is smaller than the one of the time-consistent agent. As a result, keeping the rest of the preference parameters equal for both agents, the time-inconsistent investor is more impatient and her relative wealth share decreases over time.*

Proof: There are two channels at play that determine the dynamics of the relative wealth share of the time-consistent and sophisticated time-inconsistent agents. On the one hand, the time-inconsistency channel captures the higher degree of impatience the time-inconsistent agent has in the short run. On the other hand, the sophistication channel captures the behavior adjustment of the time-inconsistent agent who realizes her bias and corrects for her future overconsumption. Both of these channels affect the effective discount factor of the sophisticated time-inconsistent agent ($\delta_S \beta_S \frac{V_{S,t,t+1}}{V_{S,t+1,t+1}}$) compared to the one of the time-consistent agent (β_C) and determine which of the two agents is overall more impatient.

First, the time inconsistency channel (the extra discount rate δ_S) has a negative effect on the effective discount factor, because it takes values smaller than 1 and decreases the time discount factor of the time-inconsistent agent relative to the one of the time-consistent agent. Second, I consider the effect of the sophistication channel, represented by the term $\frac{V_{S,t,t+1}}{V_{S,t+1,t+1}}$. Note that since $\beta_S > \delta_S \beta_S$ and since the sophisticated agent evaluates both the continuation value function $V_{S,t,t+1}$ and the actual value function she optimizes $V_{S,t+1,t+1}$ at the optimal consumption $C_{S,t+1,t+1}$, the sophistication channel has a positive effect on the effective discount factor:

$$\left(\frac{V_{S,t,t+1}}{V_{S,t+1,t+1}}\right)^{\rho_S} = \frac{C_{S,t+1,t+1}^{\rho_S} + \beta_S E_{t+1}\left[(V_{S,t,t+2})^{\alpha_S}\right]^{\frac{\rho_S}{\alpha_S}}}{C_{S,t+1,t+1}^{\rho_S} + \delta_S \beta_S E_{t+1}\left[(V_{S,t,t+2})^{\alpha_S}\right]^{\frac{\rho_S}{\alpha_S}}} > 1. \tag{76}$$

Intuitively, this term of the sophisticated agent S reflects the adjustment that she makes due the time inconsistency between the periods that she is aware of. Self $t+1$ will overconsume compared to what self t plans to consume. In other words, the marginal utility that self t plans to receive from a unit of consumption at time $t+1$ is larger than the actual marginal utility she will get. This is why in the risk sharing process the time-inconsistent agent will require lower than optimal compensation from agent C to shift consumption from time t to time $t+1$. However, since the sophisticated agent anticipates this at time t, she adjusts her stochastic discount factor to take this difference into account.

Thus, on the one hand the time-inconsistent agent is more impatient in the short-run compared to the long-run and has an additional discount rate δ_S. On the other hand, however, the agent is sophisticated and realizes this inconsistency, so she adjusts her stochastic discount factor accordingly. Which of these two channels, the sophistication or time inconsistency channel, has a stronger effect determines whether the agent's relative wealth share will decrease compared to the one of the time-consistent agent in a competitive equilibrium between the two. To answer this question I take into account the total effect of both the additional discount factor δ_S and the adjustment term of the stochastic discount factor of the agent and find that:

$$\delta_S \left(\frac{V_{S,t,t+1}}{V_{S,t+1,t+1}} \right)^{\rho_S} = \frac{\delta_S C^{\rho_S}_{S,t+1,t+1} + \delta_S \beta_S E_{t+1} \left[(V_{S,t,t+2})^{\alpha_S} \right]^{\frac{\rho_S}{\alpha_S}}}{C^{\rho_S}_{S,t+1,t+1} + \delta_S \beta_S E_{t+1} \left[(V_{S,t,t+2})^{\alpha_S} \right]^{\frac{\rho_S}{\alpha_S}}} < 1 \tag{77}$$

Thus, the effective discount rate of the sophisticated time-inconsistent agent will always be smaller than the one of te time-consistent agent:

$$\delta_S \beta_S \left(\frac{V_{S,t,t+1}}{V_{S,t+1,t+1}} \right)^{\rho_S} < \beta_C, \tag{78}$$

where I set $\beta_S = \beta_C$. Hence, this condition shows that, keeping the risk aversion, EIS and time discounting parameters equal, and consumption growth rate i.i.d., the effective discount factor of the time-inconsistent sophisticated agent is always smaller than the one of the time-consistent agent, so the first is overall more impatient than the latter. Hence, the relative wealth share of the time-consistent agent will increase over time compared to the one of the time-inconsistent agent, all else equal. The parameter δ_S governs the degree of time inconsistency of agent S. This results shows that even though the investor is sophisticated and predicts how much extra consumption she needs to allocate to her future self in order to account for her overconsumption, she still suffers from the current period time inconsistency bias and gives higher weight to her current consumption than to saving. This is the reason why despite her sophistication the agent can never fully resolve her bias and her relative wealth share shrinks over time.

4.2 Naive time-inconsistent and time-consistent agents

The naive agent is not aware of her time inconsistency bias, so self t expects that from time period $t+1$ on she will be fully rational and time-consistent. The rest of the preference parameters of the naive agent are the same as the ones of the time-consistent agent, so she anticipates that consumption in the future will be shared based on the initial endowment. However, she does not realize that self $t+1$ will be more impatient and overconsume compared to what self t plans to consume. This means that self t does not allocate sufficient immediate consumption to satisfy the consumption needs of self $t+1$. That is, the marginal utility of consumption at time $t+1$ from the point of view of self t will be larger than the actual marginal utility of consumption at time $t+1$. Hence, in the risk sharing process the naive agent will require a lower compensation for postponing consumption from time t to time $t+1$ than is optimal for her and as a consequence her share of aggregate consumption will shrink over time. Since the naive agent is not correcting for this error, the deviations from optimal consumption will accumulate, her effective time discount factor will be smaller than the one of the time-consistent agent and her relative wealth will shrink over time. Alternatively, we can see that in the following equation:

$$\delta_N \beta_N \left(\frac{C_{N,t,t+1}}{C_{N,t,t}} \right)^{\rho_N - 1} \left(\frac{V_{N,t,t+1}}{E_t \left[V_{N,t,t+1}^{\alpha_N} \right]^{\frac{1}{\alpha_N}}} \right)^{\alpha_N - \rho_N} =$$

$$= \beta_C \left(\frac{C_{C,t+1}}{C_{C,t}} \right)^{\rho_C - 1} \left(\frac{V_{C,t+1}}{E_t \left[V_{C,t+1}^{\alpha_C} \right]^{\frac{1}{\alpha_C}}} \right)^{\alpha_C - \rho_C} . \tag{79}$$

The consumption that the naive agent plans to have $C_{N,t,t+1}$ is the one that optimizes the continuation value function $V_{N,t,t+1}$ according to which the agent is time-consistent. Thus, in an optimization between two time-consistent agents the equation above holds without the δ_N additional discount factor of the naive agent. With the additional discounting the naive agent becomes more impatient, so the only way the equation above would hold is when she receives a smaller consumption share in the future relative to today.

The sophisticated agent performs better than the naive agent since unlike self t of the naive agent, self t of the sophisticated agent knows that self $t+1$ will overconsume compared to what self t plans to consume and self $t+1$ will have a lower marginal utility of wealth than the one predicated by self t. Therefore, self t of the sophisticated agent incorporates this in her optimization and requires a higher compensation from agent C for shifting consumption from time t to time $t+1$. Thus, the lower the marginal utility that self $t+1$ requires compared to what self t has predicted, the more additional consumption she will require and the closer she will resemble the behavior of a time-consistent agent. Since the naive agent does not correct for the overconsumption of self $t+1$ compared to what self t predicts, her wealth share decreases faster than the one of the sophisticated agent.

5. Asset pricing with time inconsistency

I solve the model using the numerical procedure described in Section 3. I consider the cases when consumption growth rate is i.i.d. and when there is exogenous long-run risk in the economy. The equilibrium condition is that the intertemporal marginal rates of substitution of both agents need to be equal. In this section I show how the relative wealth share of the time-inconsistent agent in the economy affects the equilibrium conditional risk-free rate and the risk premium.

5.1 Survival economy with i.i.d. consumption growth

I first consider a survival economy with i.i.d. consumption growth, where the only risk that agents share is the aggregate consumption growth rate risk. In this economy there is no stationarity constraint, so one type of agents gets crowded out of the market and the other type survives in the long run, turning into the representative investor. The risk aversion parameter of agents γ determines their appetite for risk. To focus on the effect of time inconsistency, I assume that both agents have the same level of risk aversion γ. Thus, their portfolio allocation between the risk-free and risky assets and their return on wealth are the same. I also assume that their subjective time discount factors (β) and elasticity of intertemporal substitution parameters (ψ) are equal. In the baseline case I follow the parametrization of Bansal and Yaron (2004): EIS $\psi = 1.5$, risk aversion $\gamma = 10$, and time discount factor $\beta = 0.998$.

5.1.1 Risk-free rate

Under these conditions Theorem 1 given in Section 4. holds and provides a direct implication for the risk-free rate in the economy for different levels of wealth share of the time-inconsistent agent. Keeping all else fixed the effective time discount factor of the time-inconsistent agent is lower than the one of the time-consistent agent, meaning the former she is more impatient. The greater the share of the shortsighted, time-inconsistent agent is, the larger the overall impatience level in the economy becomes. Intuitively, more impatient agents require a higher risk-free rate in order to be induced to postpone consumption and save. Thus, as the share of time-inconsistent agents increases, the risk-free rate becomes higher.

Figure 1 quantifies this result in a model with a time-consistent and a sophisticated time-inconsistent agents, where the time inconsistency parameter is $\delta_S = 0.9$. As the wealth share of the time-inconsistent agent increases from 0.1% to 99.99%, the annualized risk-free rate increases from 3.03% to 3.22%. Since the sophisticated time-inconsistent agent is aware of her bias and adjusts her stochastic discount factor accordingly, her overall impatience level is not much lower than the one of the time-consistent agent. Therefore, as the share of the sophisticated time-inconsistent agent in the economy grows, the risk-free rate increases only slightly.

The presence of naive time-inconsistent agents on the market has a much more substantial effect on the risk-free rate, as shown in Figure 2. Using the same time inconsistency parameter ($\delta_N = 0.9$) I find that the risk-free rate increases from about 3% when naive agent's share of total wealth is 0.1% to over 14% if her share is 99.99%. The reason for the larger increase in the risk-free rate when a naive agent dominates, compared to the case when a sophisticated agent dominates can be traced to the fact that the naive agent is not aware of her bias and future overconsumption. Since the marginal utility of consumption at time $t+1$ that her self t plans to have is much larger than the actual marginal utility self $t+1$ will get, the agent requires a lower than optimal compensation to postpone consumption to the future. Over time these errors accumulate and the intertemporal marginal rate of substitution (the stochastic discount factor) becomes much smaller than the one of the time-consistent agent. Thus, when the naive agent dominates, the risk-free rate which is the inverse of the expected stochastic discount factor becomes higher.

The consumption-wealth (C/W) ratios of the agents (Figure 3) shed more light on the change in their consumption behavior as a response to the increase in the risk-free rate as the share of the time-inconsistent agent in the economy becomes larger. Two possible effects can be at play when the risk-free rate increases: a substitution effect, which drives agents to save at a higher rate as a result of the higher interest rate they can earn, and a wealth effect which drives agents to consume more since they feel richer. For agents with Epstein-Zin preferences and sufficiently large EIS parameter the substitution effect dominates since they have a weak preference for smoothing consumption. This means that the agents are willing to postpone consumption to future periods as an insurance against uncertainty in their consumption. While for the sophisticated time-consistent agent the substitution effect dominates and she decreases her consumption level when the risk-free rate in the economy increases, the naive agent keeps consuming at a constant rate. Thus, for the naive agent the wealth effect is stronger and partially counteracts the substitution effect.

5.1.2 Risk premium

As a next step I consider the effect of time inconsistency on the risk premium (the difference between the return on the dividend claim and the risk-free rate). Figure 1 shows that the risk premium remains constant at 1.84% as the wealth share of the sophisticated time-inconsistent agent (relative to the one of the time-consistent agent) in the economy increases. In addition, in the same figure I plot the annualized conditional volatility of the dividend claim and the market price of risk (defined as the ratio of the conditional volatility of the stochastic discount factor to the expectation of the stochastic discount factor), on which the equity premium depends. As we vary the wealth share of the sophisticated agent these variables also remain constant which is consistent with the constant risk premium. Thus, the larger level of impatience in the economy when the sophisticated shortsighted agent dominates only results in a higher risk-free rate as an extra compensation for postponing consumption.

Although the presence of sophisticated agents does not affect the risk premium in the economy, Figure 2 shows that the wealth share of naive agents has an impact on it. The conditional risk premium increases from 1.84% when the naive agent holds 0.01% of the wealth to 4.6% when she has a wealth share of 99.99%. On the same figure we can see that the volatility of the dividend claim also increases as the share of the naive agent becomes larger (from 13.5% when the time-consistent agent dominates to 14.5%) while the price of risk remains constant. Thus, the increase in risk premium reflects an increase in the volatility of the dividend claim.

To understand whether this effect is the result of an interaction between Epstein-Zin preferences and time inconsistency or is purely the effect of time inconsistency I consider the case when both agents have standard CRRA preferences. In this case the risk aversion equals the inverse of the EIS parameter of the agents, so they do not have a preference for early resolution of uncertainty and a high willingness to substitute consumption across periods. Under these conditions, when the wealth share of the naive agent increases, the risk premium remains constant (Figure 5). Thus, we can conclude that the effect of the wealth share of naive agents on the risk premium is the result of the interaction between time inconsistency and the Epstein-Zin preferences.

Intuitively, the willingness to substitute consumption across periods induced by the Epstein-Zin preferences when the EIS parameter is sufficiently large drives agents to decrease their consumption in the current period and save more for the future, meaning that the substitution effect dominates. However, the naive agent incorrectly believes that she will be time-consistent in the future and have the same preference parameters as the true time-consistent agent. Thus, she believes that aggregate consumption will be split based on initial endowments. This results in a constant consumption-wealth ratio across wealth levels of the naive agent, which is always larger than the one of the time-consistent agent, meaning that the wealth effect for her dominates (Figure 3). As a result, she has a steady expected consumption growth rate which equals the aggregate expected consumption growth, regardless of her wealth share in the economy (Figure 4). Thus, the time inconsistency bias counteracts the effect of the high elasticity of intertemporal substitution and leads the naive agent to a smoother consumption path choice where she overconsumes in the current period and is less willing to postpone consumption to the future. This translates into a lower effective elasticity of intertemporal substitution than the one of a time-consistent agent, even if both are endowed with the same EIS.

The fact that the expected consumption growth rate of the naive agent is constant across wealth levels generates a persistent component, or long-run risks in the expected consumption growth rate of the time-consistent agent. Under preferences for early resolution of uncertainty such risks are is positively priced. Figure 4 shows that while the believed expected consumption growth (labeled E[TI]) of the naive time-inconsistent agent is constant, the one of the time-consistent agent is not, meaning that she faces additional risks in her expected consumption. As a result, the agent requires a higher premium in order to invest in risky assets, driving the risk premium up. Thus, instead of hampering the importance of long-run risks in the economy, shortsighted naive agents give rise to such risks even if aggregate consumption growth rate is i.i.d. and provide evidence of the existence of such risks.

To demonstrate that the behavior of the naive time-inconsistent agent is in line with that of an agent with a lower level of EIS, I solve a model with two time-consistent agents (A and B) who have different EIS parameters ($\psi_A = 0.2, \psi_B = 2$). Lower EIS parameter corresponds to a higher consumption smoothing. In Figure 6 I find that the equity premium increases as the wealth share of the agent with low EIS parameter increases and it follows a similar pattern as the risk premium in a model with a naive time-inconsistent investor and a time-consistent investor. To provide further evidence that the implications of naivete on asset prices is consistent with that of having a lower elasticity of intertemporal substitution, in Figure 7 I vary the EIS levels of both the naive and the time-consistent agents (keeping them equal to each other) and compare the resulting risk premium. When EIS $= 0.1$, the agents have CRRA preferences and there is no effect of the wealth share of naive agent on the risk premium. When I increase the EIS parameter of both agents to EIS $= 2$, compared to the value of 1.5 in the baseline specification, I find that the risk premium increases to about 6.5% if the naive agent fully dominates the economy (compared to 4.6% in the baseline case).

To alleviate the concern that the behavior of a naive time-inconsistent agent could be solely the result of her higher level of impatience and not due to a lower effective EIS caused by her time inconsistency, I solve a model with two time-consistent agents, one of whom is more impatient and has a lower time discount rate ($\beta_A = 0.995, \beta_B = 0.998$). Then I check the effect of the wealth share of the more impatient agent on the risk premium. Figure 8 shows that in contrast to the wealth share of a naive agent, the wealth share of an impatient agent has almost no effect on the risk premium.

5.2 OLG economy with i.i.d. consumption growth

5.2.1 Risk-free rate

As a next step, I study the effect of the wealth share of time-inconsistent agents in an overlapping-generations economy (OLG) where every period newborn agents of each type enter the market with an equal wealth share. The economy is described in detail in section **2.3**. This setting brings stationarity in the long run and none of the agents is completely crowded out. Figure 9 depicts the changes in asset prices with the change in wealth share of the sophisticated time-inconsistent agent. We can see that the level of the risk-free rate is higher: from 3.7% when the sophisticated agent hold 0.01% of the consumption share to about 4% when she dominates the economy. The reason is that in the OLG setting agents take into account their probability to die in the future and discount their consumption stream by a higher rate. Effectively, these agents have a shorter horizon and higher impatience than the infinite-horizon investors I studied in the previous sections. As a result they require a higher risk-free rate to postpone consumption. As in the survival economy case the risk-free rate increases with the share of more shortsighted and impatient time-inconsistent agents. The risk-free rate responds stronger to an increase in the wealth share of the naive agent, increasing from 4% to 12.5% (Figure 10).

5.2.2 Risk premium

As in the survival economy, the risk premium in the economy increases the larger the share of naive agents is (Figure 10). In the previous subsection I showed that the presence of naive agents gives rise to long-run risks and agents require a larger risk premium as a result. The effect in the OLG setting is more pronounced because as Garleanu and Panageas (2015) show, the individual consumption growth of agents may contain a persistent component, even though aggregate consumption growth is i.i.d. The condition for this to occur is that one of the agents has lower EIS and both of them prefer early or late resolution of uncertainty. As I showed previously, the consumption behavior of the naive agent is consistent with that of an agent with low EIS. Thus, in an OLG setting, long-run risks emerge endogenously in the presence of naive agents, even though they are less averse to such risk. Under the preference of early resolution of uncertainty these risks are priced. Figure 11 shows that realized consumption growth of the naive agent decreases with her wealth share (as a result of the entrance of newborn agents with equal share), while her consumption-wealth ratio increases (Figure 12). The consumption growth of the time-consistent agent then increases. Since this agent has a higher EIS and is more averse to long-run risk she requires a higher premium in order to hold a larger share of the risky asset, which drives the risk premium and the market price of risk up.

An interesting pattern in the risk premium as a function of the wealth share of the sophisticated agents emerges in the OLG economy setting. In contrast to the survival economy case, the risk premium, volatility of the dividend claim and the price of risk increase with the increase of the sophisticated agent's wealth share (Figure 9). The reason is that in this setting, the consumption-wealth ratio of the sophisticated agent increases with her share in the economy (Figure 12). This behavior is in line with choosing a smoother consumption path and having a lower effective EIS parameter. The reason for this choice is the decreasing expected consumption growth rate of the sophisticated agent as her wealth share increases. This effect is caused by newborn agents who enter the economy with equal proportion and decrease the relative share of the dominating type of agents. Thus, taking into account the heterogeneity between agents in the economy allows us to understand the effect of their interactions on asset prices. In the presence of short-sighted agents the importance of long-run risks does not decrease, although they are less averse to them. Instead, such risks emerge endogenously and drive the risk premium up.

5.3 Survival economy with exogenous long-run risk

In this section I consider the effect of long-run risk on the risk sharing between a time-inconsistent and a time-consistent agent and the asset prices in the economy. As described in section 2.4. I model the aggregate consumption growth with long-run risk using a Markov regime switching model with two states – expansion (up) and recession (down). Note that the persistent component in the consumption growth rate corresponding to the parametrization of Bansal and Yaron (2004) imposes that there is a 98% probability that the economy will remain in the same state as it was in the previous period.

The time-inconsistent agent puts a larger weight on the short-run compared to the time-consistent agent, so she is less averse to long-run risks and finds it optimal to sell insurance to the time-consistent agent against them. The agent who sells insurance effectively invests a larger share of her portfolio in risky assets, which have a higher expected return in an up state than in a down state. In an up state the time-inconsistent agent gains share because risky assets bring a higher expected return. Thus, the insurance fee that the time-consistent agent pays in an up state is the foregone premium from investing less in risky assets. In a down state the time-inconsistent agent loses share because of the decrease in expected return on risky assets, so this loss is effectively the insurance she pays to the time-consistent agent.

To show this effect, I plot the change in the consumption share of the time-inconsistent agent in the transition from up to down or down to up states. In a shift from down to up state the agent who sells insurance only collects the insurance fees, so her relative share increases, while in a down state her share decreases since she needs to pay out the insurance to the other agent. Figure 13 shows the results when the time-inconsistent agent is sophisticated or naive and holds 50% of the total consumption at the initial period. As the state of the economy changes from up (denoted by 2) to down (denoted by 1), the time-inconsistent investor loses a larger share of her wealth than when the shift is from a recession to an expansion. We know from Theorem 1 (section 4) that all else equal the relative wealth share of the time-inconsistent agent decreases over time compared to that of the time-consistent agent. The fact that she loses a larger share in a shift from down to up state than from an up to a down state means that the time-inconsistent agent finds it optimal to sell insurance to the time-consistent agent who is more averse to long-run risk. Due to her time-inconsistency bias, however, even in an expansion period the time-inconsistent agent cannot gain a larger share than the time-consistent agent. Figure 13 shows that the effect is more pronounced for the naive agent.

The effect of time inconsistency on the risk-free rate and the risk premium in the economy with long-run risk is similar to the one in an economy with i.i.d. shocks. While the risk-free rate increases with the wealth share of the time-inconsistent agent both when she is sophisticated and naive, the risk premium only becomes larger with the increase in the wealth share of the time-inconsistent agent if she is naive (Figures 14 and 15). In an up state as her consumption share increases from 0.1% to 99.99% the premium increases from 2.7% to over 6% and in a down state – from 2.1% to over 6%. Since the agent invests a larger proportion of her portfolio in risky assets she requires a higher compensation for bearing extra risk, which drives the premium in an economy with long-run risk higher than the one in an i.i.d. economy. The volatility of the dividend claim also increases while the price of risk remains flat.

Contrary to what we observe in reality, however, the equity premium implied by the model with time-inconsistent agents is higher in the up state compared to the down state, meaning that it is procyclical instead of countercyclical. The fact that the time-inconsistent agent who is less averse to long-run risk sells insurance to the time-consistent agent induces a countercyclicality mechanism. Since the time-inconsistent agent is less averse to long-run risk she requires a lower premium for it compared to the time-consistent agent. The time-inconsistent agent collects insurance fees in the up state but pays out insurance to the time-consistent agent in the down state, so she has a relatively larger (smaller) wealth share and drives the equity premium down (up) in an expansion (recession). However, this mechanism is counteracted by the relative loss of share of the time-inconsistent agent in every state due to her bias, which results in a net procyclicality instead of countercyclicality in the economy.

The results presented in this section have important implications for the parametrization of long-run risk models. Epstein, Farhi, and Strzalecki (2014) raise the concern that the long-run risk model uses an extreme value for the EIS parameter in order to match asset pricing moments, which induces that agents would be willing to give up more than 30% of their wealth in order to resolve future uncertainty. The fact that naive have a lower effective EIS, but the risk premium increases as their wealth share in the economy increases, alleviates this concern. Instead of reducing the importance of long-run risks and decreasing the risk premium, agents who are less averse to such risks lead to a higher risk premium, even though their effective EIS is lower.

6. Long-run and stationarity implications

In this section I study the long-run implications of the model in a survival economy and in a stationary setting with overlapping generations. This will shed light on the actual share and importance of the time-inconsistent agents for determining the asset prices in the economy.

6.1 Long-run survival

Theorem 1 shows that the effective time discount factor of the sophisticated time-inconsistent agent is smaller than the one of the time-consistent agent meaning that she is more impatient and her relative wealth share decreases over time. The naive agent does not realize her bias and does not predict her overconsumption. This misjudgment of future immediate consumption drives the agent to consume out of her savings and as her mistakes accumulate over time she loses share even faster than the sophisticated agent. To quantify the result, I consider two values of the time inconsistency parameter δ_N, 0.9 and 0.98, annually. The rest of the parameters are equal for both agents and based on the parametrization of Bansal and Yaron (2004): EIS $\psi = 1.5$, risk aversion $\gamma = 10$, and time discount factor $\beta = 0.998$. Table 1 and Figure 16 show the consumption share of the naive time-inconsistent agent with $\delta_N = 0.9$ over time. If her initial share is $c_N = 50\%$, after 10 years, it drops down to 43.86%. After 20 years the agent is left with 37.28% and after 50 years she is crowded out of the market. When the time-inconsistent agent is instead endowed with 95% of total consumption in the beginning, the decrease in consumption share is much slower. After 50 years she loses less than 20% of her wealth and is left with 75.56% of the total wealth. Thus, the smaller share the naive time-inconsistent agent controls, the faster she loses wealth. The effect is weaker when the time inconsistency parameter is higher ($\delta_N = 0.98$) and the agent is less biased (Figure 17). In this case, when her initial share is 50% she is crowded out of the market only after 200 years.

In Table 2 and Figure 16 I present the consumption share evolution of the sophisticated time-inconsistent agent using two time inconsistency parameter values δ_S, 0.9 and 0.98. We see that the sophisticated agent is crowded out of the market by the time-consistent agent much slower than the naive one. The reason is that she corrects for her time inconsistency bias and her errors do not accumulate over time. After 500 years her share decreases to 40.41% if she is highly short-sighted ($\delta_S = 0.9$). If she is not as short-sighted ($\delta_S = 0.98$) her loss of share is less than 2%.

6.2 OLG setting and stationarity implications

Next I consider the wealth distribution between time-consistent and time-inconsistent agents in an OLG setting. Since every quarter newborn agents of each type enter the market and they are endowed with equal wealth shares, both types of agents survive in the long run and their wealth shares converge to a stationary value. Figure 18 shows the consumption share evolution of the sophisticated time-inconsistent agent with $\delta_S = \delta_N = 0.9$. Regardless of her starting share (5%, 50% or 95%) the agent holds 49.5%

in the long run. Since the sophisticated agent corrects for her bias and her overconsumption does not accumulate over time, her share remains very close to that of the time-consistent agent. The naive agent, however, does not realize her bias and keeps consuming out of her savings every period, which shrinks her relative share to the time-consistent agent more than compared to the sophisticated agent. Stationarity is achieved after about 100 years and the naive agent's share of aggregate consumption converges to about 10% in the long-run. Decreasing the level of time inconsistency and calibrating $\delta_S = \delta_N = 0.98$ brings the consumption shares of both the sophisticated and the naive agents closer to the one of the time-consistent agent. Figure 19 shows that the stationary consumption share of the sophisticated agent is nearly 50% while the one of the naive agent is about 37%.

As previously pointed out, the larger the share of naive time-inconsistent agents in the economy is, the larger impact they have on the risk-free rate and the risk premium. In addition, even though a larger level of time inconsistency emphasizes this effect, it is associated with higher level of impatience and overconsumption of the agents which shrinks their stationary wealth share in the economy. Thus, if the time inconsistency level is higher ($\delta_N = 0.9$), the risk premium increases from 1.84% to over 6% as the wealth share of the naive agent increases from 0.01% to 99.99%. Since in the stationary long run she only holds 10% of the total wealth share, the stationary risk premium becomes 2.04%. Hence, even though the naive agent holds a relatively small fraction of the total wealth, she still increases the equilibrium premium by about 20-30bp. This represents a 10-15% increase in the premium compared to the baseline model with a representative time-consistent agent, which is economically sizable. If instead the time inconsistency level is lower ($\delta_N = 0.98$), the effect of the naive agent's wealth share on the premium is smaller, but they hold about 37% of the total wealth in the economy. Thus, the stationary risk premium amounts to about 2.1%, which again represents an increase of 20-30bp in the premium. We can conclude that considering heterogeneity among agents is important and sheds light on the wealth distribution in the economy and the corresponding implications for asset prices.

6.3 Parameter sensitivity

The time inconsistency parameter values I pick (0.9 and 0.98 annually) are higher than the ones experimental literature estimates (0.6 to 0.85 annually). The reason I choose a more conservative level of time inconsistency is twofold. First, Halevy (2015) shows that present-biased preferences are not the only source of time inconsistency. Two alternative sources can be the higher risk of distant future rewards attributed to the inherently larger uncertainty the future holds as well as the higher than expected demand for liquidity created by potential unpredicted constraints the agents may face. In experiments these effects are difficult to isolate and can be picked up by the time inconsistency parameter, leading to a larger short-term discounting than short-termism induces.

Second, solving an equilibrium model with a time-consistent and a naive time-inconsistent investor who has an annual $\delta_N = 0.8$ leads to extreme wealth distribution dynamics. Figure 20 shows that if in the starting period the naive agent holds 50% of total consumption, she will be entirely crowded out of the market in 3 years. When she starts with 95% of initial consumption share, she still loses her entire wealth in less than 35 years. Even considering an OLG setting in which every quarter newborn naive agents enter the market and are endowed with equal share to the time-consistent newborn agents still brings an extreme outcome. Regardless of the initial consumption share, the naive agents end up with a 0.5% consumption share when the economy comes to a stationary point (Figure 21). Thus, in an equilibrium setting naive agents with a time inconsistency parameter $\delta_N = 0.8$ consume almost their entire wealth within a very short period of time and hold a negligibly small fraction of the total wealth in the economy. This is an extreme and unlikely outcome, which would contradict the vast experimental findings showing that investors with such time inconsistency biases do exist.

7. Conclusion

This paper examines the role of time inconsistency for the risk sharing and wealth distribution among agents and the corresponding effect on asset prices. I find that a naive time-inconsistent agent over-consumes and has a lower effective elasticity of intertemporal substitution (EIS) than a time-consistent agent, even if both agents are endowed with the same EIS. This result has two important economic implications. First, in both survival and overlapping-generations settings, the suboptimal consumption and savings decisions of naive agents give rise to long-run risks, even though they are shortsighted and more averse to such risks. Under the preference for early resolution of uncertainty such risks are positively priced. As a result, in a general equilibrium model I show that when the wealth share of time-inconsistent agents increases, so do the risk-free rate, volatility, and equity premium in the economy. Second, this outcome alleviates the concern that long-run risk models use an extreme level of EIS to match asset pricing moments. Instead of decreasing the risk premium, naive agents who have a lower effective EIS drive the premium up. In an economy with exogenous long-run risk both sophisticated and naive time-inconsistent agents who are less averse to these risks find it optimal to sell insurance to the time-consistent agents. This generates a countercyclical mechanism in the risk premium, which is, however, counteracted by the fact that both in recessions and expansions the relative wealth share of time-inconsistent agents decreases over time compared to that of a time-consistent agents. Accounting for the plausible heterogeneity in terms of time inconsistency sheds light on the long-run and stationary wealth distribution among agents and its effect on asset prices. Even if time-inconsistent agents hold a small fraction of total wealth (10%), they still contribute to a sizable 10-15% increase in the risk premium above the one in an economy with a representative time-consistent agent.

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A Tables and figures

Figure 1: Conditional moments of the dividend claim: TC and Sophisticated TI agents

The figure plots the conditional annualized risk-free rate, risk premium, volatility, and price of risk against the consumption share of the time-inconsistent (TI) agent in a survival economy with i.i.d. consumption growth rate and two types of agents. One of the agents is time-consistent (TC) and the other one is sophisticated time-inconsistent (TI) with time inconsistency parameter $\delta_S = 0.9$. Both agents are endowed with Epstein-Zin-Weil preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

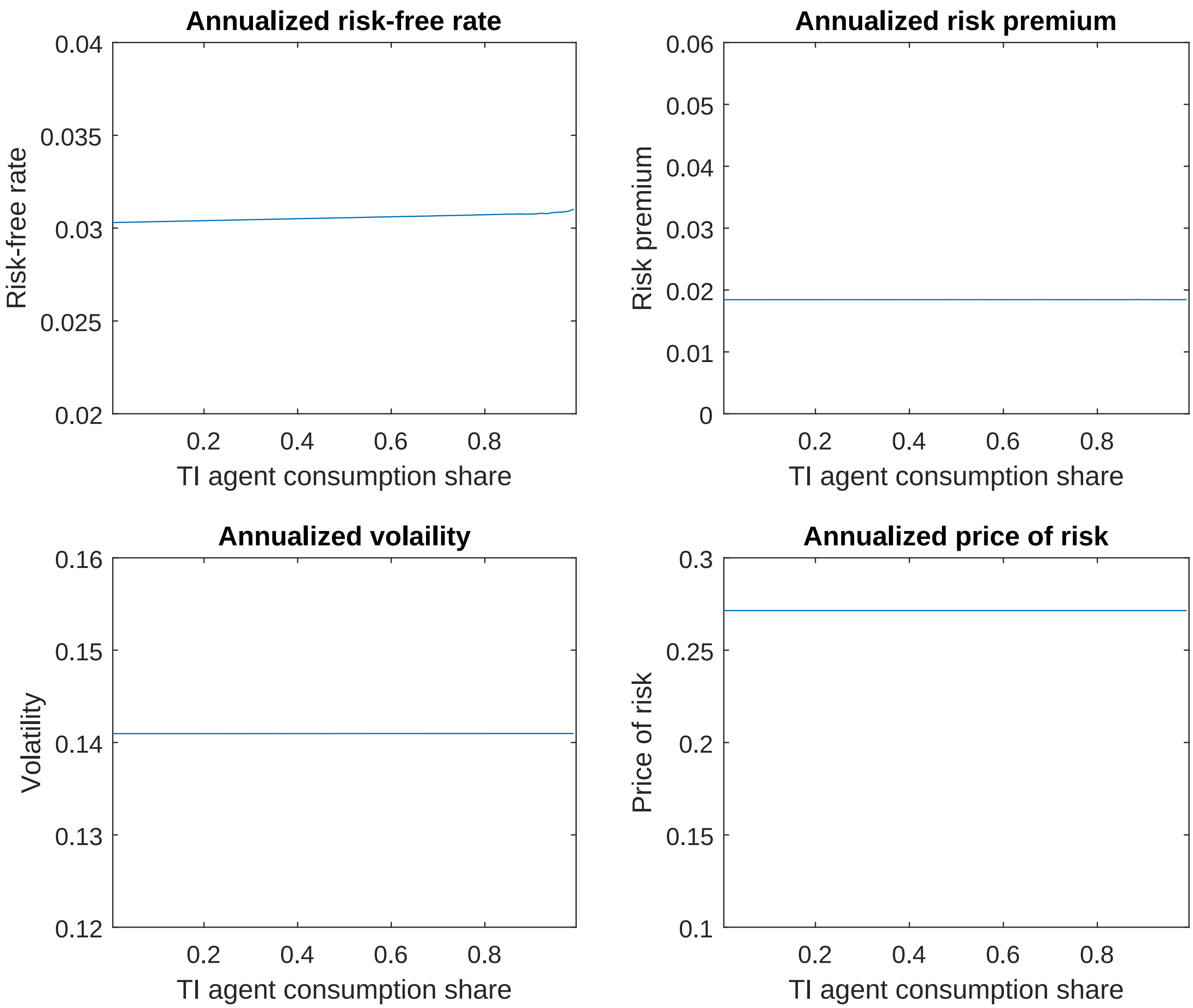


Figure 2: Conditional moments of the dividend claim: TC and Naive TI agents

The figure plots the conditional annualized risk-free rate, risk premium, volatility, and price of risk against the consumption share of the time-inconsistent (TI) agent in a survival economy with i.i.d. consumption growth rate and two types of agents. One of the agents is time-consistent (TC) and the other one is naive time-inconsistent (TI) with time inconsistency parameter $\delta_S = 0.9$. Both agents are endowed with Epstein-Zin-Weil preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

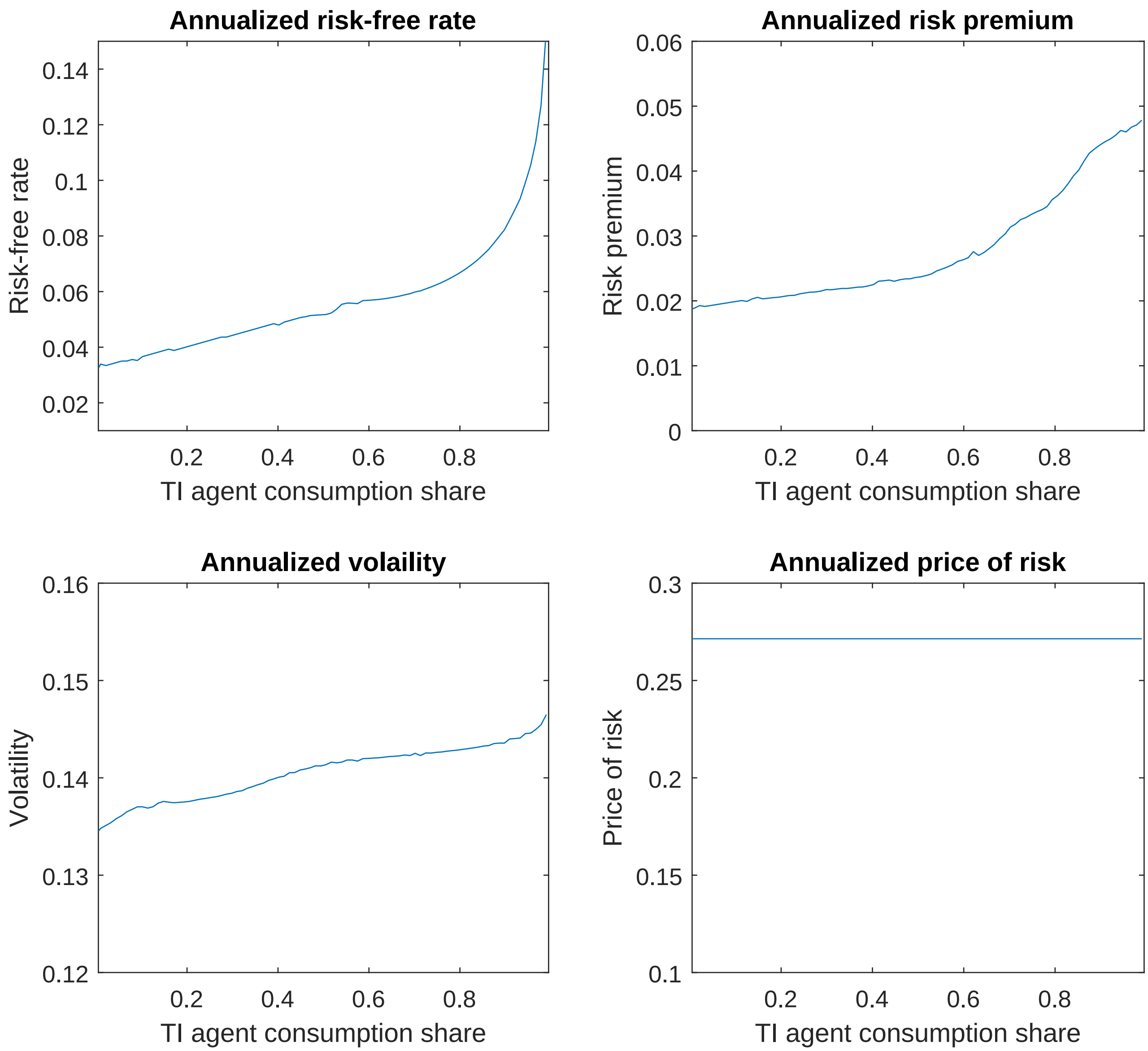


Figure 3: Consumption-wealth ratios: Survival economy

This figure shows the consumption-wealth ratios (C/W) of the sophisticated and naive time-inconsistent (TI) agents and the time-consistent (TC) agent. The setting is a survival economy with i.i.d. consumption growth rate and two types of agents. The time inconsistency parameter is $\delta_S = 0.9$. Both agents are endowed with Epstein-Zin-Weil preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

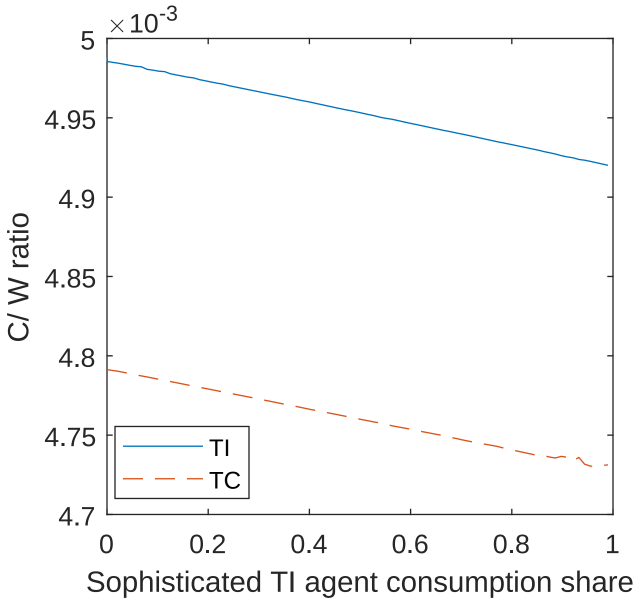


Figure 4: Individual consumption growth rate: Survival economy

The figure plots the expected consumption growth rate of the time-consistent (TC), and the time-inconsistent (TI) sophisticated and naive agents against the consumption share of the time-inconsistent agent in the economy. E[TI] denotes the believed expected consumption growth of the naive agent.

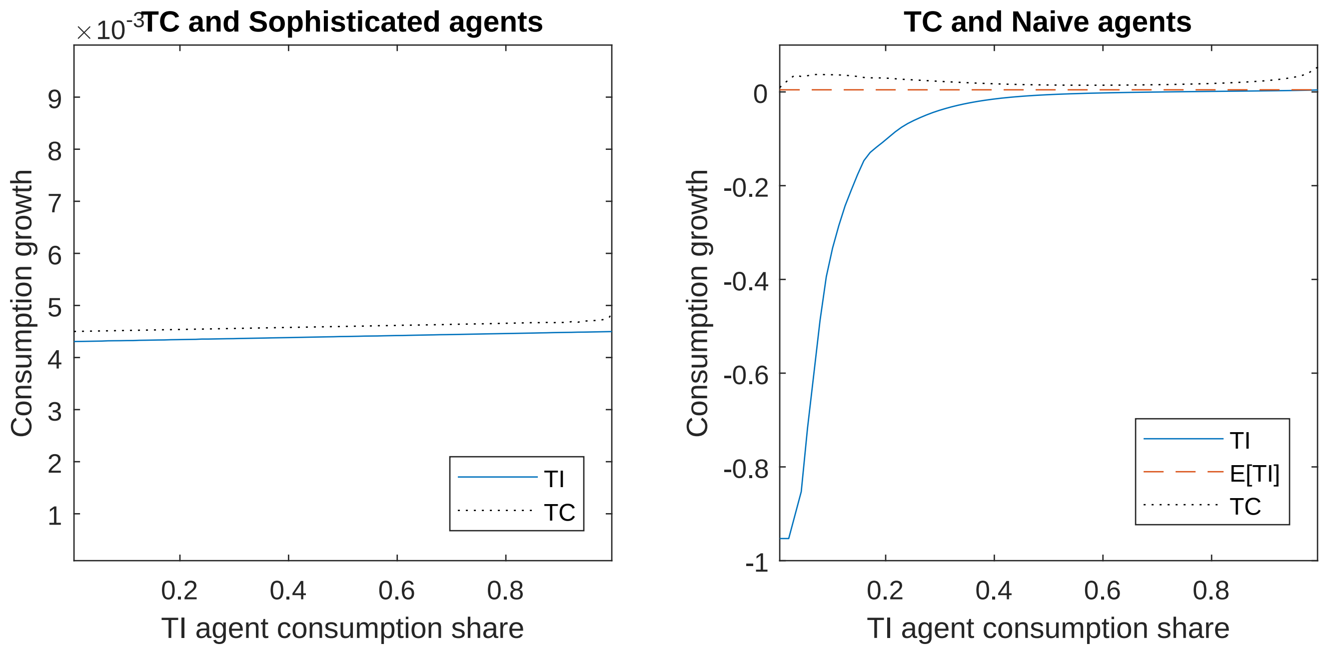


Figure 5: Conditional moments of the dividend claim: TC and naive TI agents with CRRA

The figure plots the conditional annualized risk-free rate, risk premium, volatility, and price of risk against the consumption share of the time-inconsistent (TI) agent in a survival economy with i.i.d. consumption growth rate. One of the agents is time-consistent (TC) and the other one is naive time-inconsistent (TI) with time inconsistency parameter $\delta_S = 0.9$. Both agents are endowed with CRRA preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 0.1$.

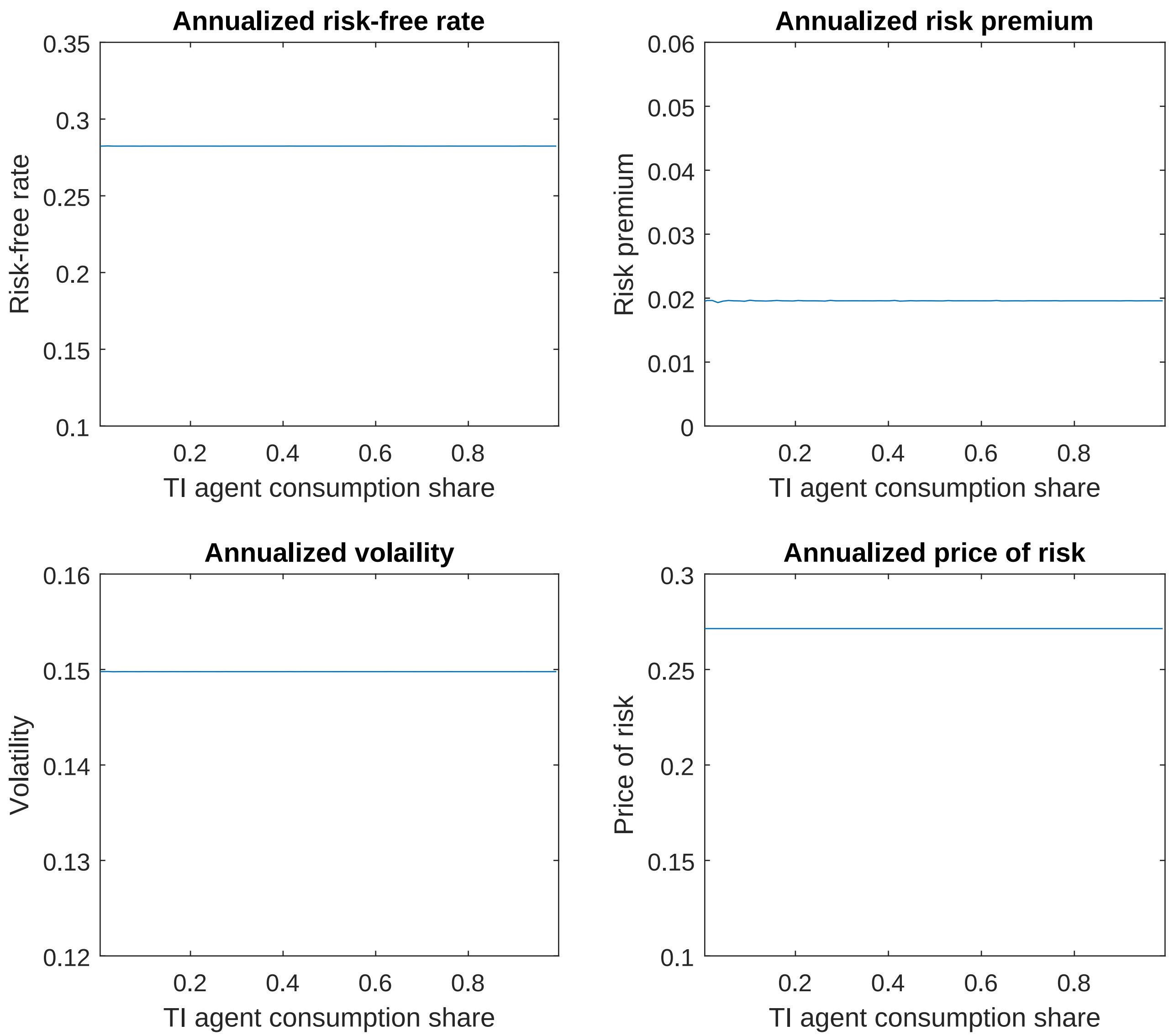


Figure 6: Conditional moments of the dividend claim: TC agents with different EIS

The figure plots the conditional annualized risk-free rate, risk premium, volatility, and price of risk against the consumption share of the time-consistent (TC) agent with lower elasticity of intertermporal substitution (EIS) in a survival economy with i.i.d. consumption growth rate and two time-consistent agents with different EIS. One of the agents has and $\psi_A = 0.2$ and the other one $\psi_B = 2$. Both agents are endowed with Epstein-Zin preferences with $\beta = 0.998, \gamma = 10$.

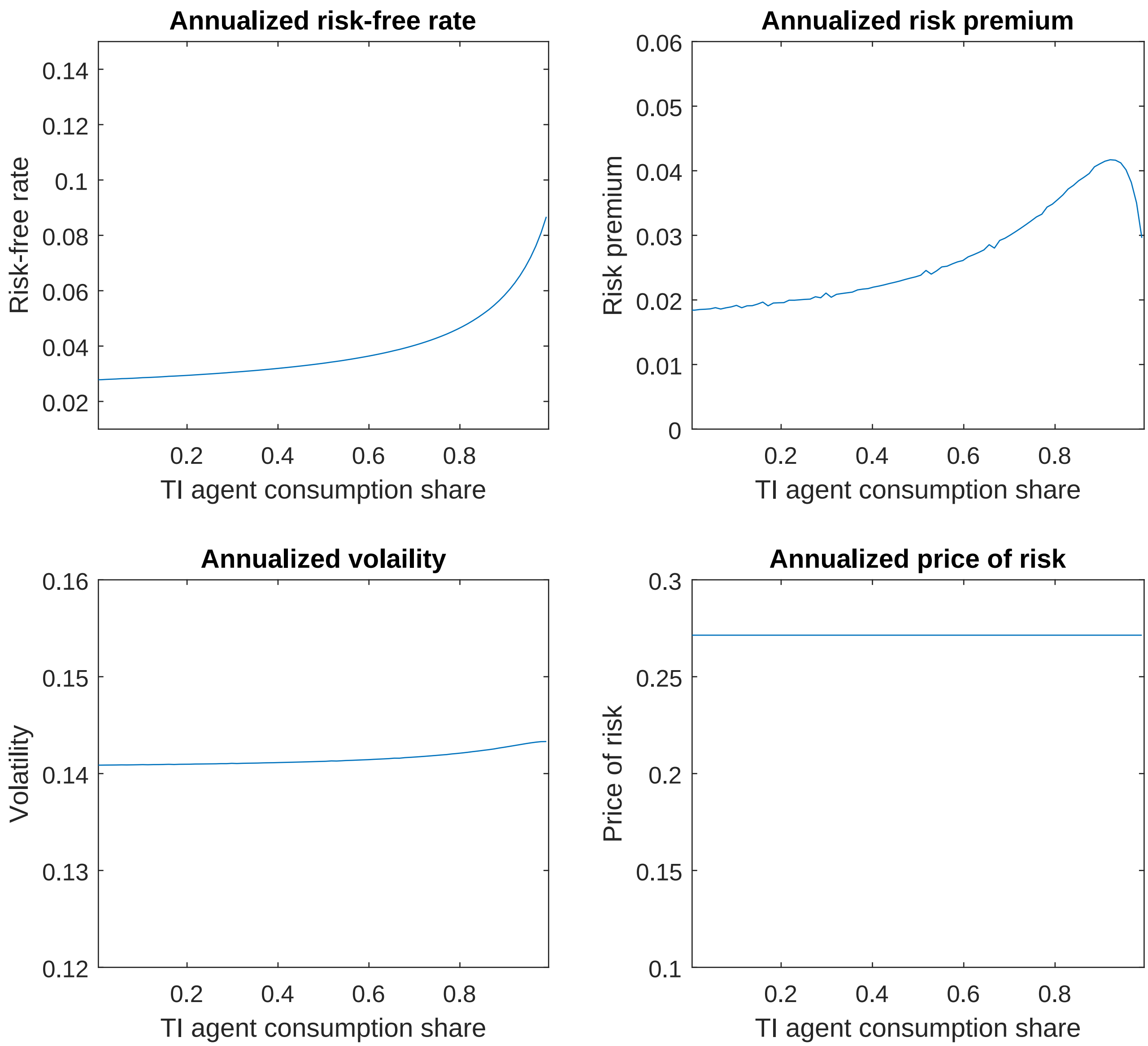


Figure 7: Conditional risk premium and volatility across different EIS levels

The figure plots the conditional annualized risk premium and volatility against the consumption share of the time-inconsistent naive (TI) in a survival economy with i.i.d. consumption growth rate. One of the agents is time-consistent (TC) and the other is naive time-inconsistent (TI). The elasticity of intertemporal susbstitution (EIS) varies between 0.1, 1.5, and 2. Both agents are endowed with Epstein-Zin preferences with $\beta = 0.998, \gamma = 10$.

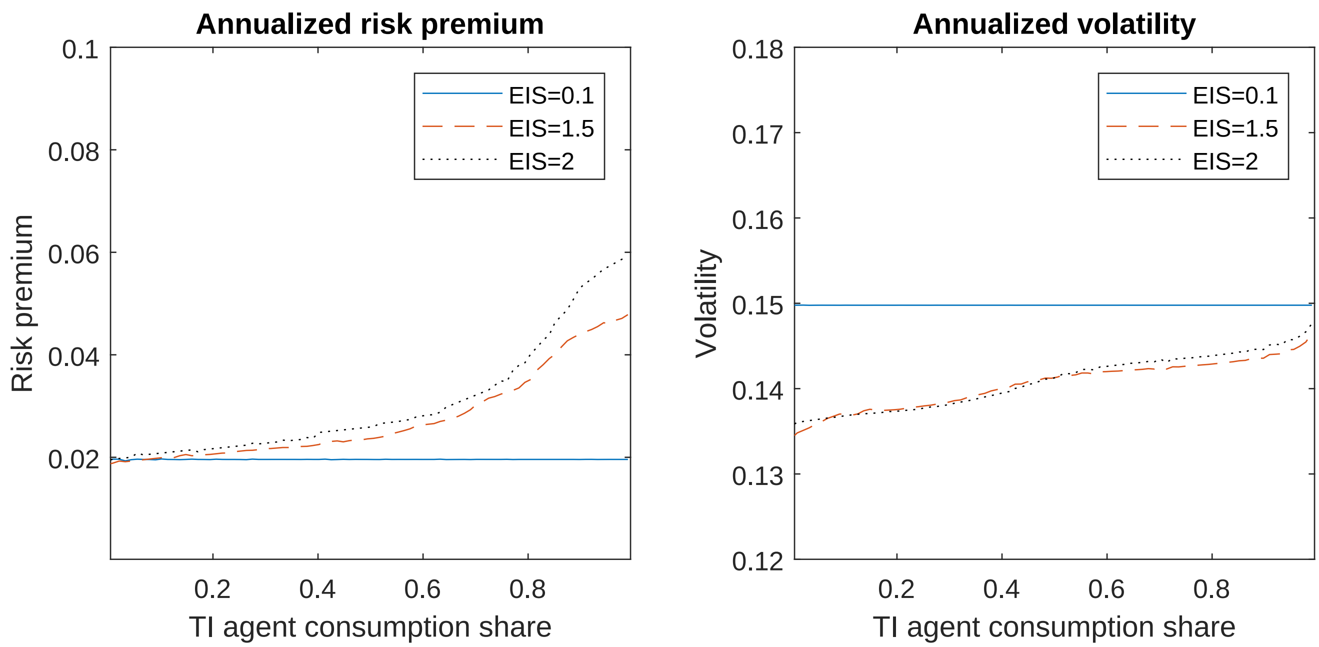


Figure 8: Conditional moments of the dividend claim: TC agents with different time discount rates

The figure plots the conditional annualized risk-free rate, risk premium, volatility, and price of risk against the consumption share of the time-consistent (TC) agent with lower time discount rate in a survival economy with i.i.d. consumption growth rate and two time-consistent agents with different time discount factors. One of the agents has $\beta_A = 0.995$ and the other one $\beta_B = 0.998$. Both agents are endowed with Epstein-Zin preferences with $\psi = 1.5, \gamma = 10$.

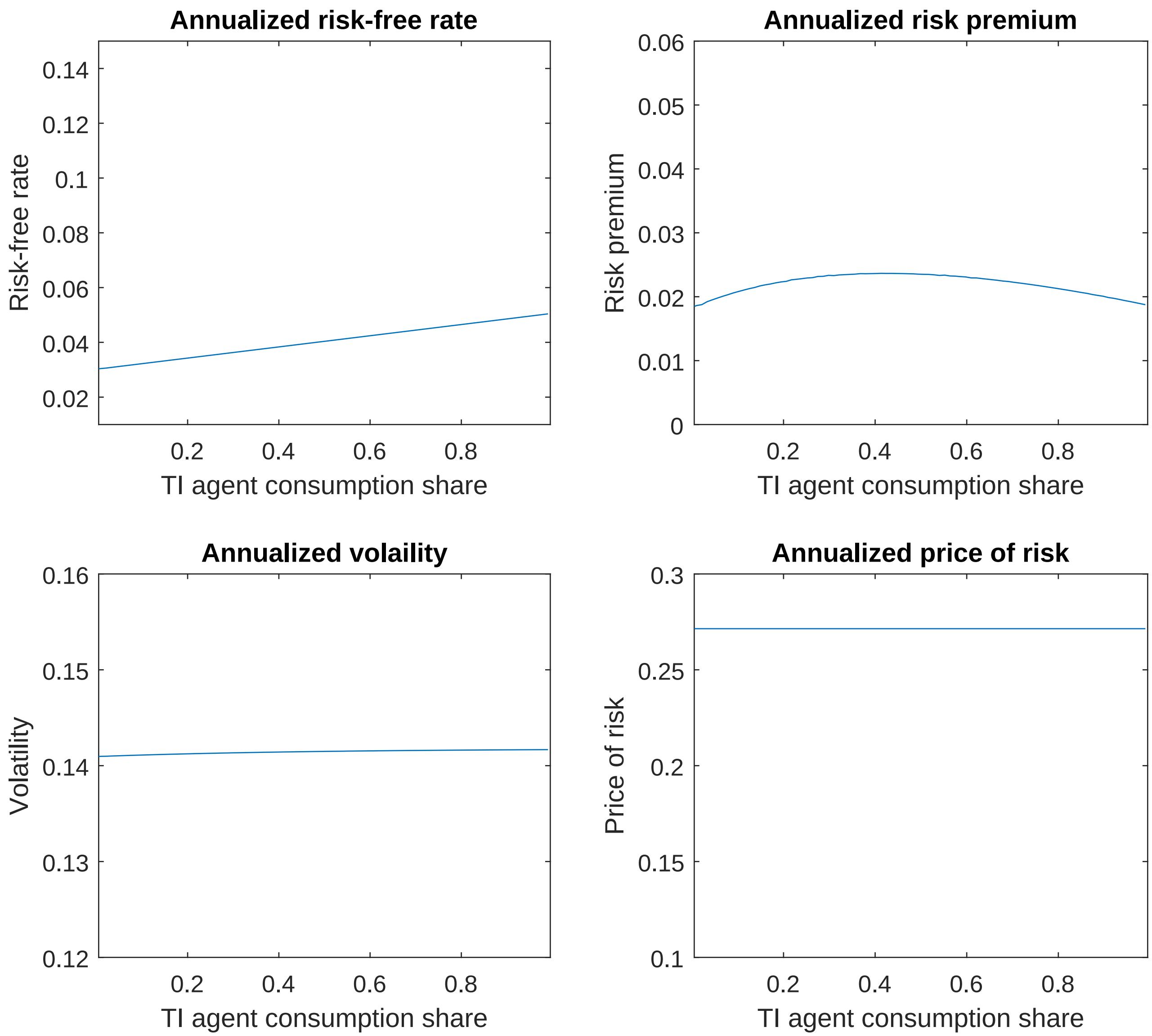


Figure 9: Conditional moments of the dividend claim in OLG economy: TC and Sophisticated TI agents

The figure plots the conditional annualized risk-free rate, risk premium, volatility, and price of risk against the consumption share of the time-inconsistent (TI) agent in an OLG economy with i.i.d. consumption growth rate and two types of agents. One of the agents is time-consistent (TC) and the other one is sophisticated time-inconsistent (TI) with time inconsistency parameter $\delta_S = 0.9$. Both agents are endowed with Epstein-Zin-Weil preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

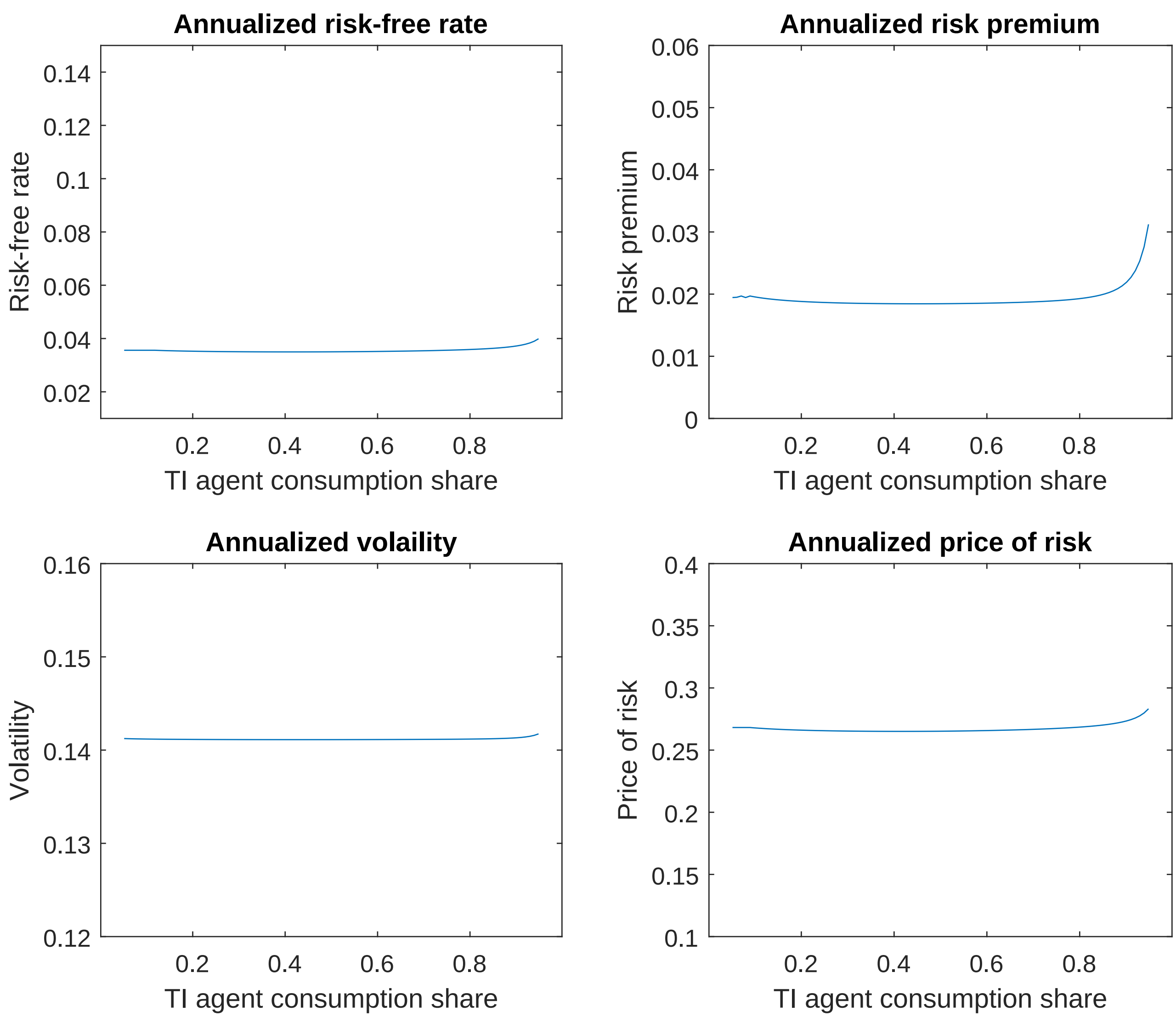


Figure 10: Conditional moments of the dividend claim in OLG economy: TC and Naive TI agents

The figure plots the conditional annualized risk-free rate, risk premium, volatility, and price of risk against the consumption share of the time-inconsistent (TI) agent in an OLG economy with i.i.d. consumption growth rate and two types of agents. One of the agents is time-consistent (TC) and the other one is naive time-inconsistent (TI) with time inconsistency parameter $\delta_S = 0.9$. Both agents are endowed with Epstein-Zin-Weil preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

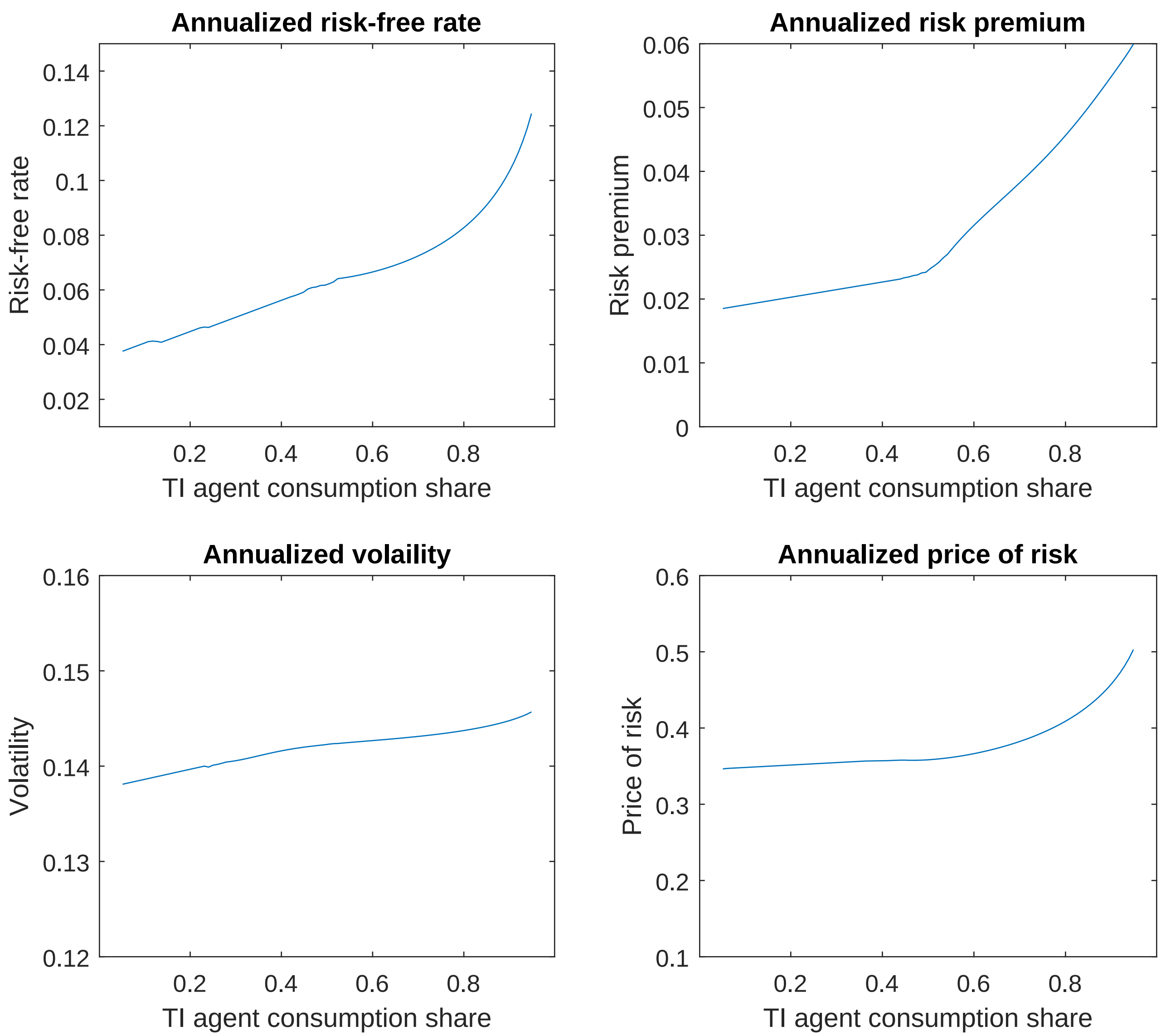


Figure 11: Individual consumption growth rate: OLG economy

The figure plots the expected consumption growth rate of time-consistent (TC), sophisticated and naive time-inconsistent (TI) agents against the consumption share in the economy of the time-inconsistent agent. E[TI] denotes the believed expected consumption growth of the naive agent.

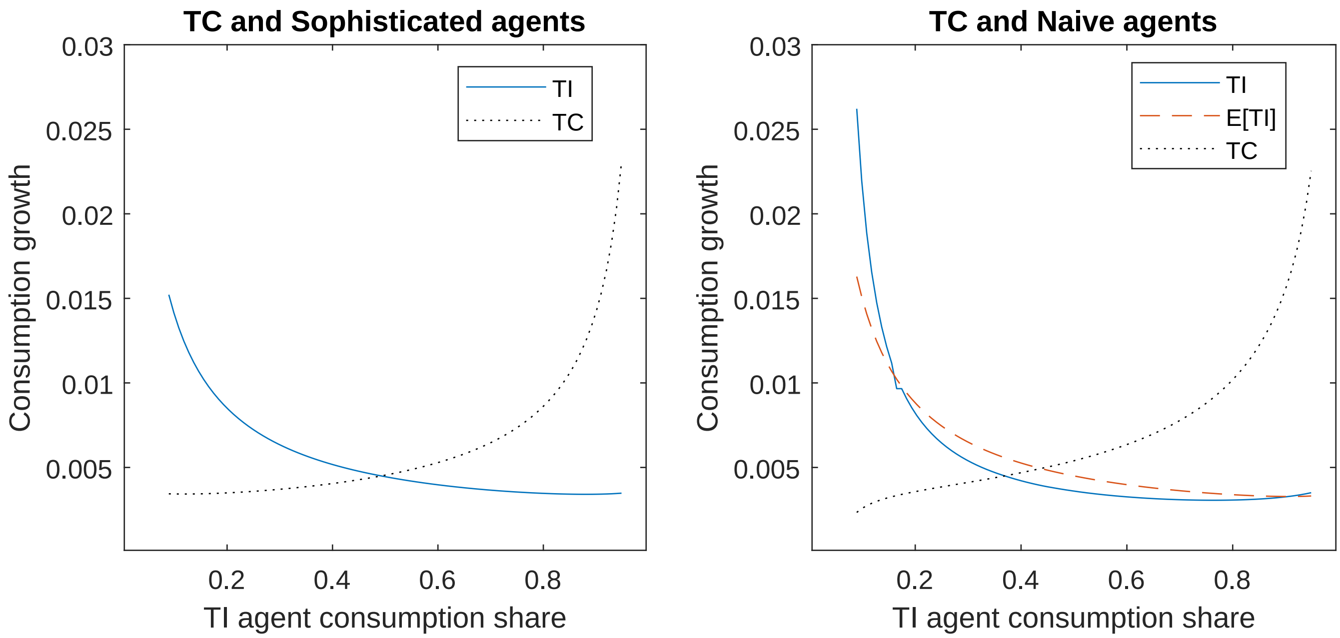


Figure 12: Consumption-wealth ratios: OLG economy

This figure shows the consumption-wealth ratios (C/W) of the sophisticated and naive time-inconsistent (TI) agents and the time-consistent (TC) agent. The setting is an OLG economy with i.i.d. consumption growth rate and two types of agents. One of the agents is time-consistent (TC) and the other one is naive or sophisticated time-inconsistent (TI) with time inconsistency parameter $\delta_S = 0.9$. Both agents are endowed with Epstein-Zin-Weil preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

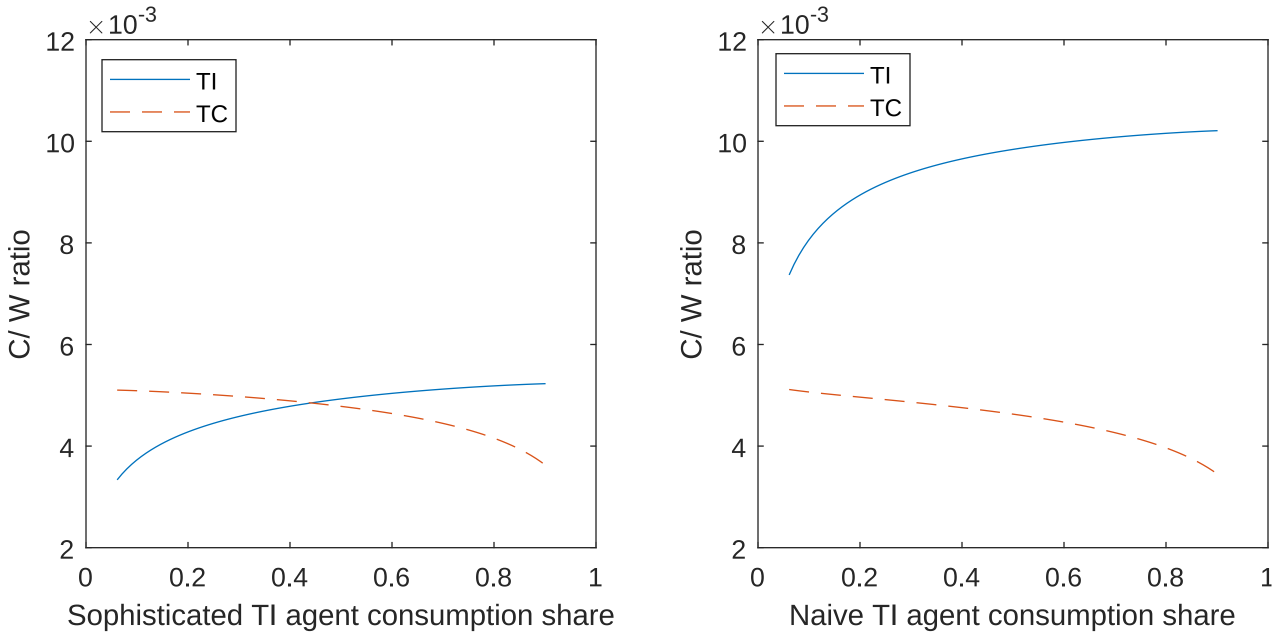


Figure 13: Consumption sharing rule: Survival economy with LRR

The figure plots the response of the consumption share of the sophsticated and naive time-inconsistent agents in down (1) and up (2) states. Both agents are endowed with an initial consumption share of $c_0 = 0.5$ of aggregate consumption and with Epstein-Zin-Weil preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

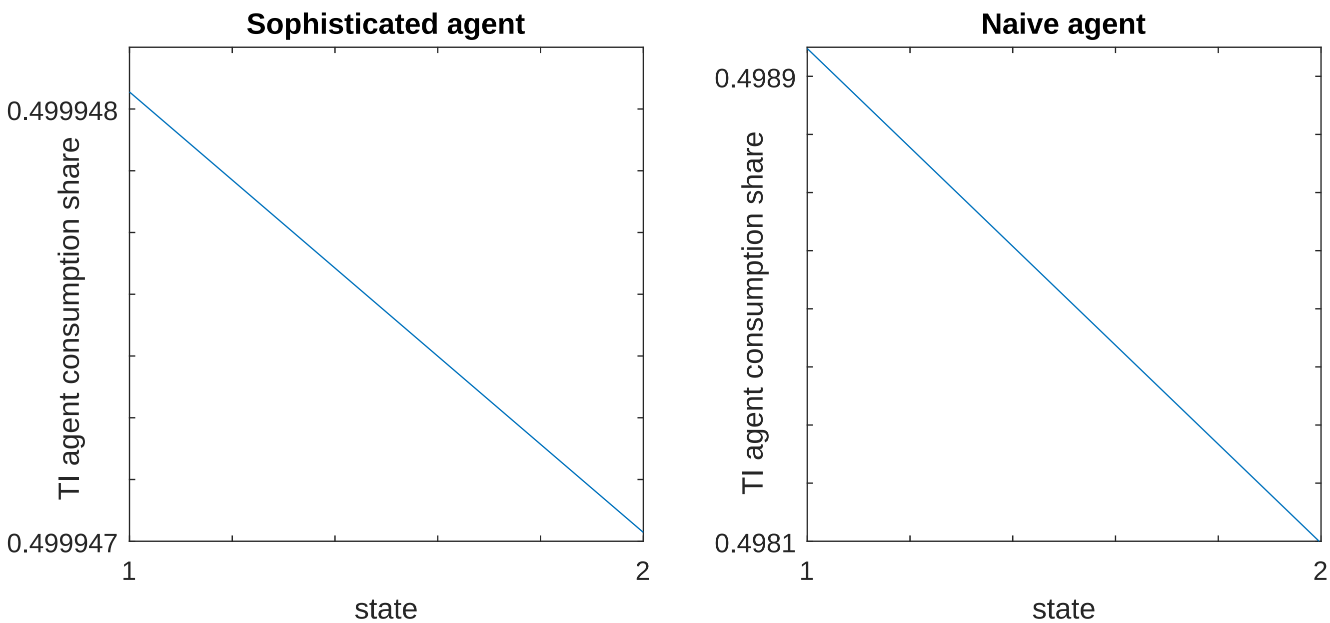


Figure 14: Conditional moments of the dividend claim with LRR: TC and Sophisticated TI agents

The figure plots the conditional annualized risk-free rate, risk premium, volatility, and price of risk against the consumption share of the time-inconsistent (TI) agent in an economy with long-run risks in the aggregate consumption growth and two types of agents. One of the agents is time-consistent (TC) and the other one is sophisticated time-inconsistent (TI) with time inconsistency parameter $\delta_S = 0.9$. Both agents are endowed with Epstein-Zin-Weil preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

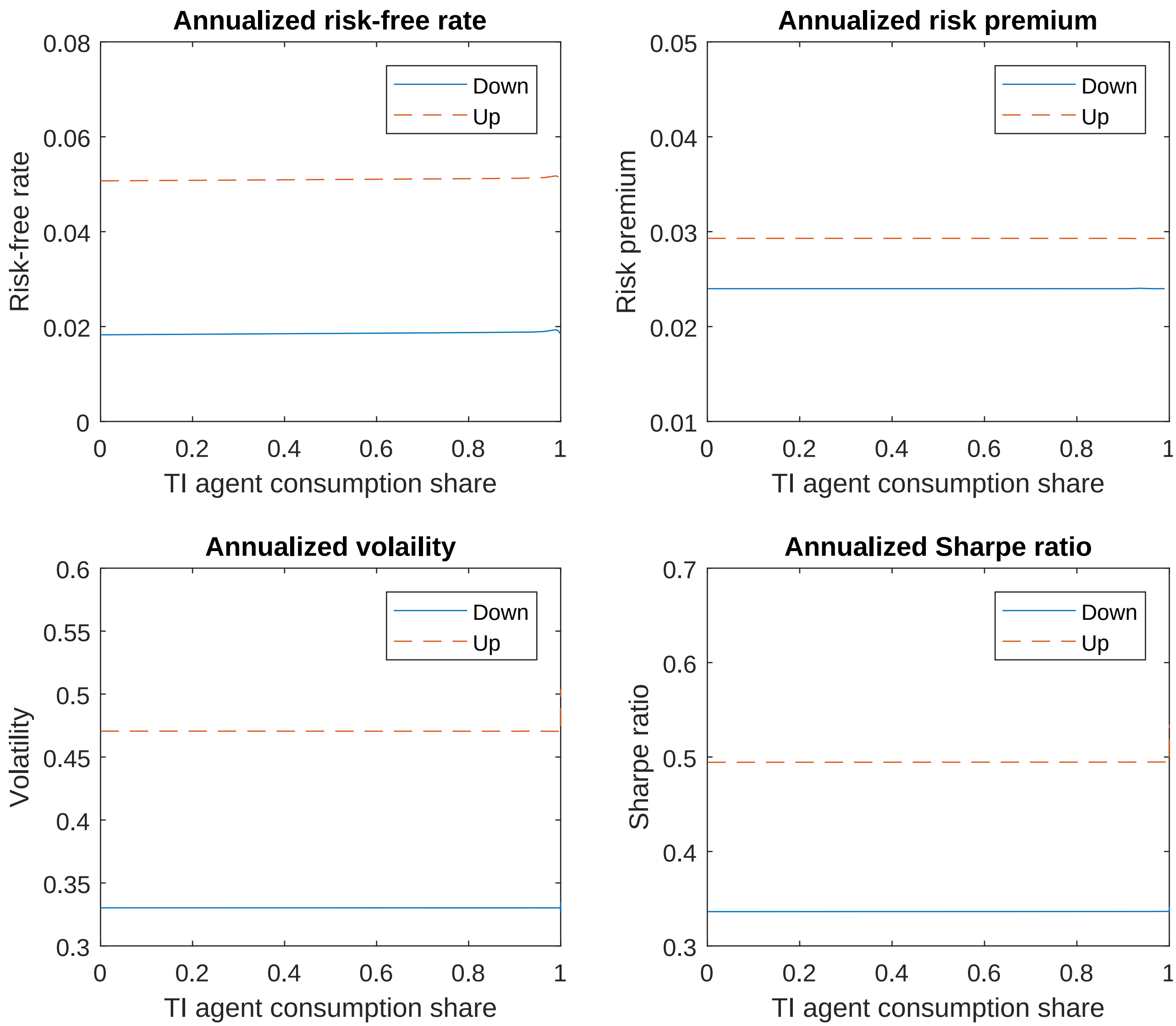


Figure 15: Conditional moments of the dividend claim with LRR: TC and Naive TI agents

The figure plots the conditional annualized risk-free rate, risk premium, volatility, and price of risk against the consumption share of the time-inconsistent (TI) agent in an economy with long-run risks in the aggregate consumption growth and two types of agents. One of the agents is time-consistent (TC) and the other one is naive time-inconsistent (TI) with time inconsistency parameter $\delta_S = 0.9$. Both agents are endowed with Epstein-Zin-Weil preferences with equal preference parameters: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

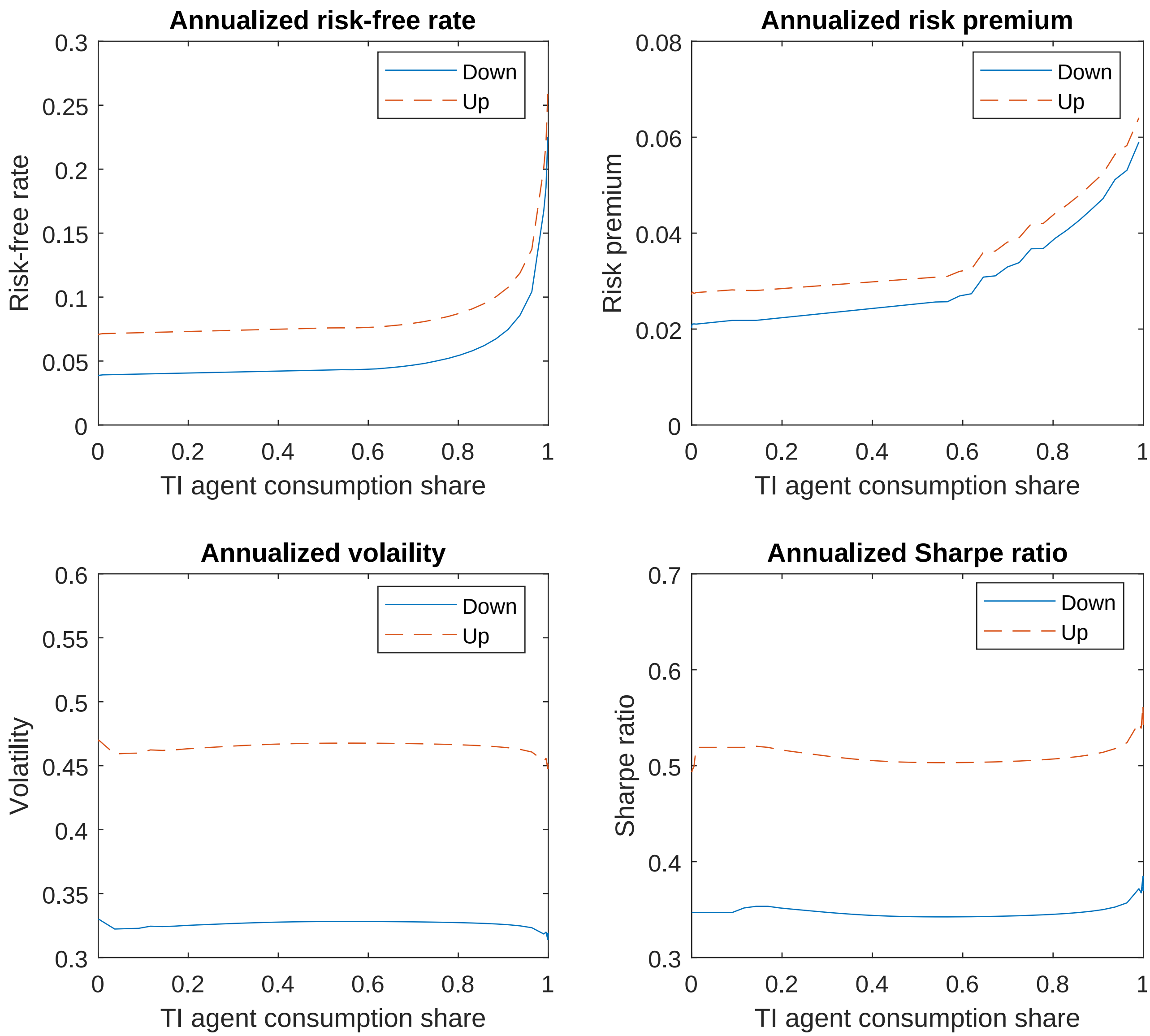


Table 1: Equilibrium consumption share evolution: naive agent

The table shows the consumption share evolution of the naive time-inconsistent agent over time. The median consumption share over 1000 simulations is reported. Two initial endowment levels of the naive agent are considered: $c_N = 50.00\%$ and $c_N = 95.00\%$. The time inconsistency parameter is either $\delta_N = 0.9$ or $\delta_N = 0.98$. The rest of the preference parameters are equal for the two investors: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

	$\delta_N = 0.9$			$\delta_N = 0.98$		
	10 years	20 years	50 years	50 years	100 years	200 years
$c_N = 50.00\%$	43.86%	37.28%	0.00%	39.17%	29.14%	0.00%
$c_N = 95.00\%$	92.16%	88.17%	75.56%	89.15%	81.59%	63.61%

Table 2: Equilibrium wealth distribution: sophisticated agent

The table shows the consumption share of the sophisticated time-inconsistent agent after 50, 100, and 500 years of trading with the time-consistent agent. The median consumption share over 1000 simulations is reported. Two initial endowment levels of the time-inconsistent agent are considered: $c_S = 50.00\%$ and $c_S = 95.00\%$. The time inconsistency parameter is either $\delta_S = 0.9$ or $\delta_S = 0.98$. The rest of the preference parameters are equal for the two investors: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

	$\delta_S = 0.9$			$\delta_S = 0.98$		
	50 year	100 years	500 years	50 year	100 years	500 year
$c_S = 50.00\%$	49.04%	48.07%	40.41%	49.82%	49.63%	48.11%
$c_S = 95.00\%$	94.86%	94.52%	94.11%	94.99%	94.96%	94.53%

Table 3: Equilibrium wealth distribution: naive agent

The table shows the consumption share of the naive time-inconsistent agent after 1, 3, and 35 years of trading with the time-consistent agent. The median consumption share over 1000 simulations is reported. Two initial endowment levels of the time-inconsistent agent are considered: $c_N = 50.00\%$ and $c_N = 99.99\%$. The time inconsistency parameter is either $\delta_N = 0.6$ or $\delta_N = 0.8$. The rest of the preference parameters are equal for the two investors: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

	$\delta_N = 0.6$			$\delta_N = 0.8$		
	1 year	3 years	35 years	1 year	3 years	35 year
$c_N = 50.00\%$	3.04%	0.00%	0.00%	38.83%	0.00%	0.00%
$c_N = 95.00\%$	99.98%	99.84%	0.00%	99.98%	99.96%	0.00%

Figure 16: Consumption share evolution in a survival economy ($\delta = 0.9$)

The figure shows the consumption share evolution of the sophisticated and naive time-inconsistent agents in a survival economy with i.i.d. consumption growth over 2000 quarters (500 years) ahead. The starting endowment is 50% or 95% of the total aggregate consumption. The time inconsistency parameter is $\delta_A = 0.9$. The rest of the preference parameters are equal for the two investors: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

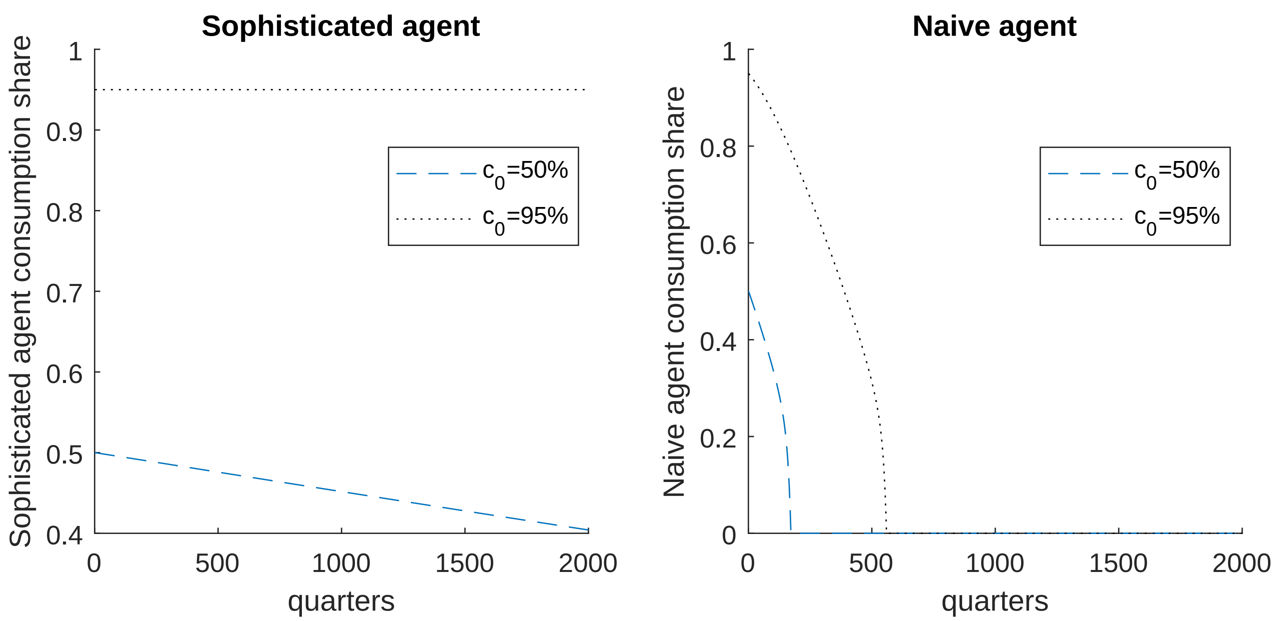


Figure 17: Consumption share evolution in a survival economy ($\delta = 0.98$)

The figure shows the consumption share evolution of the sophisticated and naive time-inconsistent agents in a survival economy with i.i.d. consumption growth over 2000 quarters (500 years) ahead. The starting endowment is 50% or 95% of the total aggregate consumption. The time inconsistency parameter is $\delta_A = 0.98$. The rest of the preference parameters are equal for the two investors: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

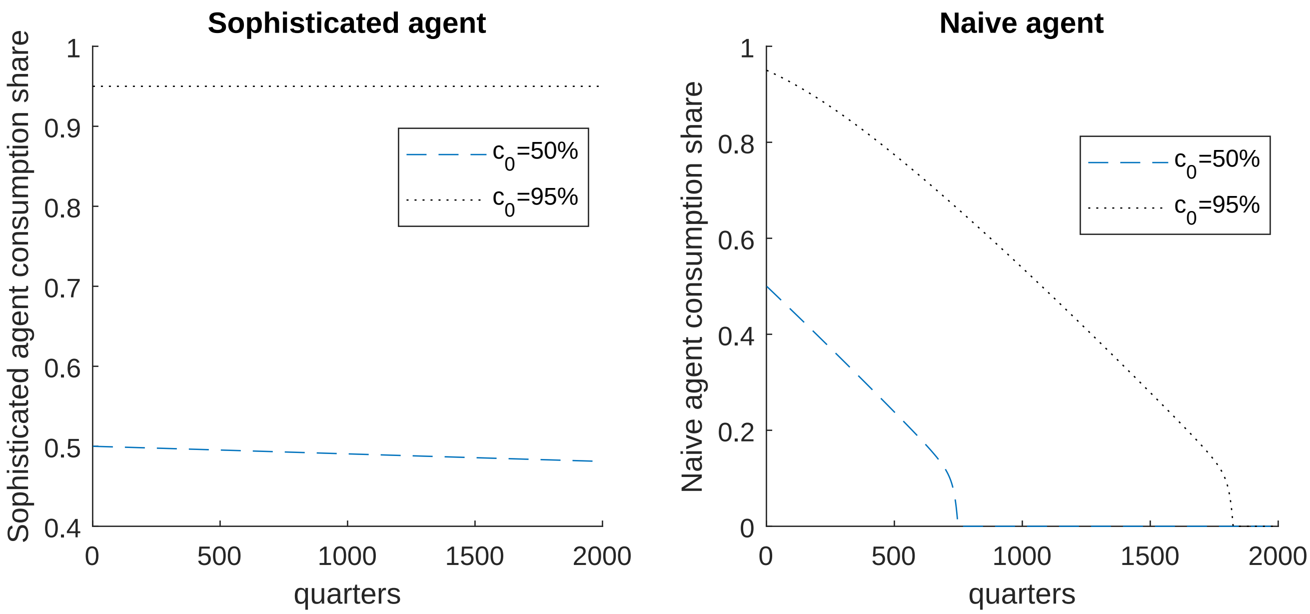


Figure 18: Consumption share evolution in an OLG economy ($\delta = 0.9$)

The figure shows the consumption share evolution of the sophisticated and naive time-inconsistent agents in an OLG economy with i.i.d. consumption growth over 2000 quarters (500 years) ahead. The starting endowment is 5%, 50% or 95% of the total aggregate consumption. The time inconsistency parameter is $\delta_A = 0.9$. The rest of the preference parameters are equal for the two investors: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

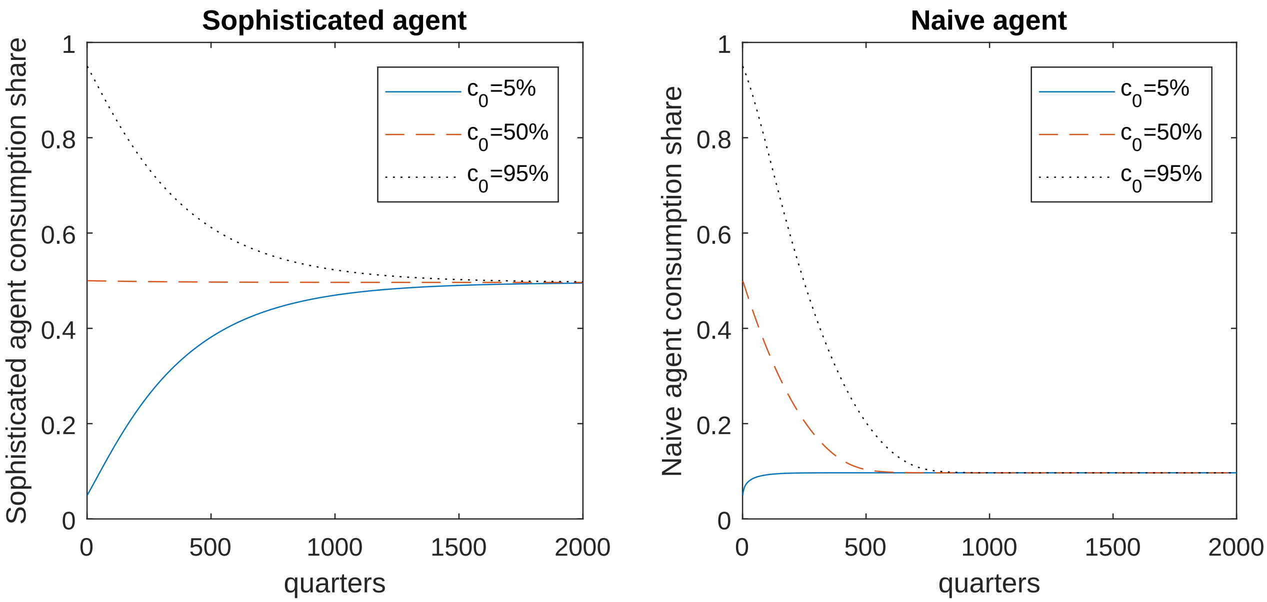


Figure 19: Consumption share evolution in an OLG economy ($\delta = 0.98$)

The figure shows the consumption share evolution of the sophisticated and naive time-inconsistent agents in an OLG economy with i.i.d. consumption growth over 2000 quarters (500 years) ahead. The starting endowment is 5%, 50% or 95% of the total aggregate consumption. The time inconsistency parameter is $\delta_A = 0.98$. The rest of the preference parameters are equal for the two investors: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

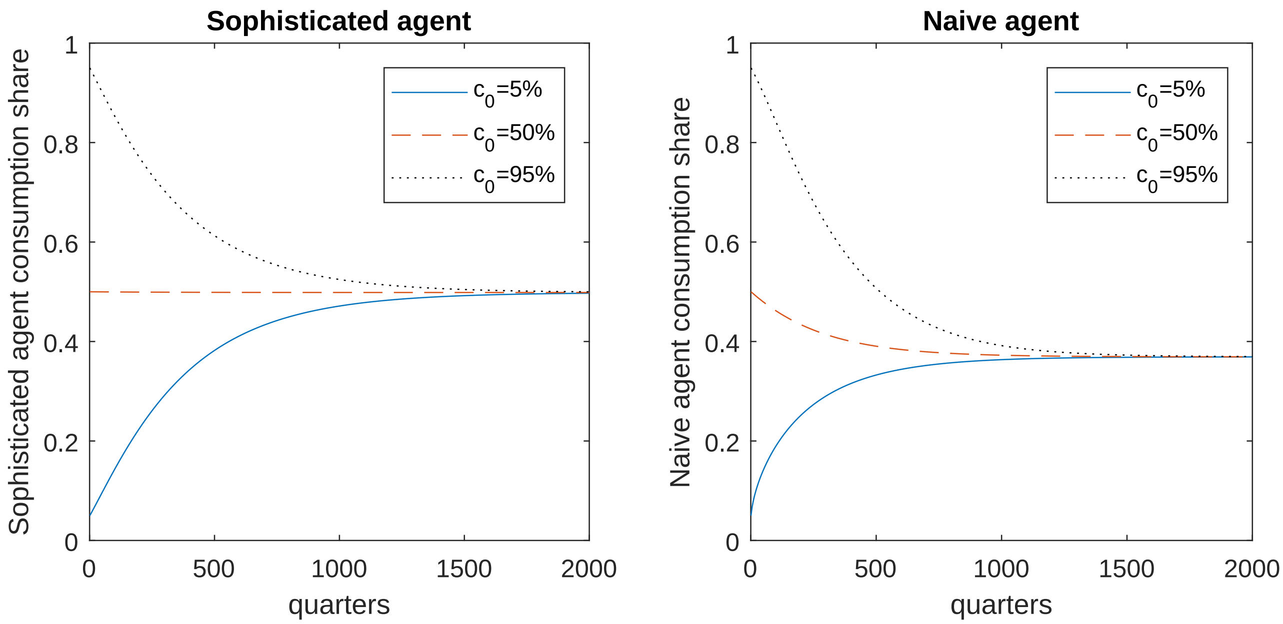


Figure 20: Consumption share evolution in a survival economy ($\delta = 0.8$)

The figure shows the consumption share evolution of the naive time-inconsistent agents in a survival economy with i.i.d. consumption growth over 200 quarters (50 years) ahead. The starting endowment is 50% or 95% of the total aggregate consumption. The time inconsistency parameter is $\delta_A = 0.8$. The rest of the preference parameters are equal for the two investors: $\beta = 0.998, \gamma = 10, \psi = 1.5$.

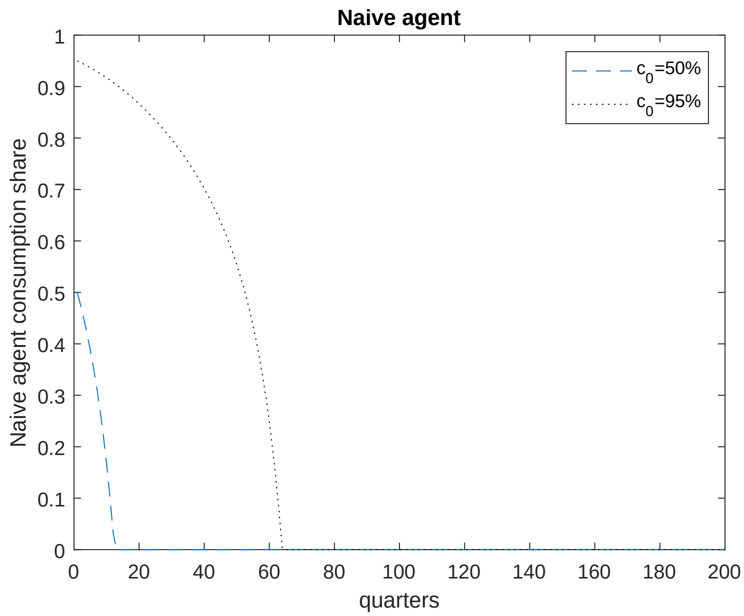
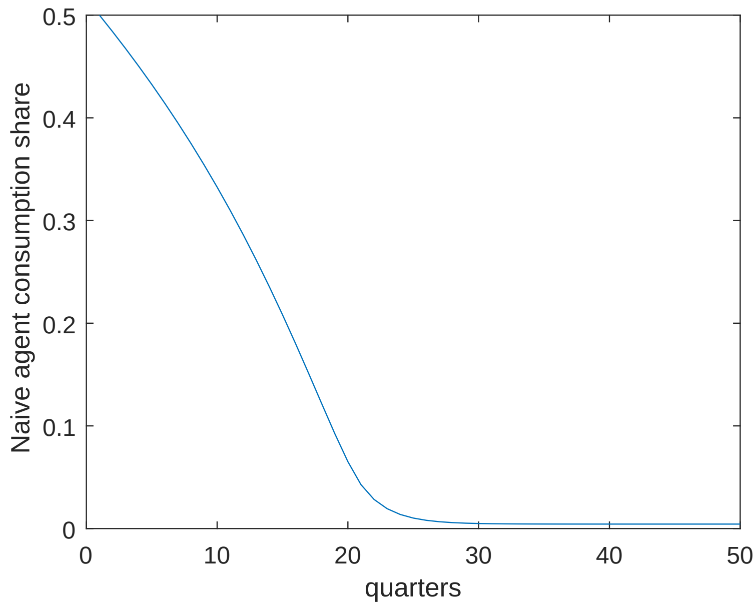


Figure 21: Consumption share evolution in an OLG economy ($\delta = 0.8$)

The figure shows the consumption share evolution of the naive time-inconsistent agents in an OLG economy with i.i.d. consumption growth over 50 quarters (12.5 years) ahead. The starting endowment is 50% of the total aggregate consumption. The time inconsistency parameter is $\delta_A = 0.8$. The rest of the preference parameters are equal for the two investors: $\beta = 0.998, \gamma = 10, \psi = 1.5$.



B Stochastic discount factor

2.1 Sophisticated time-inconsistent agent

2.1.1 CRRA preferences

We first consider the optimization problem of the sophisticated time-inconsistent agent. She is aware of her self-control problems and therefore realizes that she values future strings of consumption differently today from the way she will value them tomorrow. The first subindex in notation notes the period from which the agent values each variable, while the second subindex shows the actual time period of the variable.

$$V_{0,0} = \max_{C_{0,0},W_{0,1}} U(C_{0,0}) + \delta\beta E_0[V_{0,1}(W_{0,1})] \tag{80}$$

$$\text{s.t. } \Pi_0 C_{0,0} + E_0[\Pi_1 W_{0,1}] \leq \Pi_0 W_{0,0} \tag{81}$$

$$V_{0,1}(W_{0,1}) = \frac{1}{\alpha} C_{0,1}^{\alpha} + \beta V_{0,2}(W_{0,2}) \tag{82}$$

The continuation function $V_{0,1}(W_{0,1})$ shows the value that self 0 gives to future consumption. Throughout the paper the first subindex denotes the time at which an agent makes a plan about a given variable and the second subindex shows the time at which the value takes place. For instance, $V_{0,1}$ denotes the value function at time 1 according to the plan that self 0 makes at time 0.

Equation (82) shows that the time discount factor that self 0 plans to use is $\beta > \delta\beta$. This means that self 0 gives larger relative weight on immediate consumption at time 0 than she plans to give on immediate consumption at time 1. The agent, however, realizes that this function differs from the actual $V_{1,1}(W_{0,1})$ that self 1 will optimize. Hence, she will incorporate this in her plan for time 1 and the planned consumption and wealth will be equal to the optimal consumption and wealth that self 1 will solve for: $C_{0,1} = C_{1,1}$ and $W_{0,1} = W_{1,1}$. The Lagrangian is given as follows:

$$\mathcal{L} = U(C_{0,0}) + \delta\beta E_0[V_{0,1}(W_{0,1})] + \lambda_{0,0}\left(W_{0,0} - C_{0,0} - E_0\left[\frac{\Pi_1}{\Pi_0} W_{0,1,s}\right]\right) \tag{83}$$

The first-order conditions at each state s are:

$$\frac{\partial \mathcal{L}}{\partial C_{0,0}} = C_{0,0}^{\alpha-1} - \lambda_{0,0} = 0 \tag{84}$$

$$\Rightarrow \lambda_{0,0} = C_{0,0}^{\alpha-1} \tag{85}$$

$$\frac{\partial \mathcal{L}}{\partial W_{0,1,s}} = \delta\beta \frac{\partial V_{0,1}(W_{0,1,s})}{\partial W_{0,1,s}} - p_s \left[\frac{\Pi_{1,s}}{\Pi_0}\right] \lambda_{0,0} = 0 \tag{86}$$

$$\Rightarrow \lambda_{0,0} = \delta\beta \frac{\frac{\partial V_{0,1}(W_{0,1})}{\partial W_{0,1,s}}}{p_s \frac{\Pi_{1,s}}{\Pi_0}}, \tag{87}$$

where p_s is the probability of state s realizing at time t. From equations (85) and (87) it follows that the stochastic discount factor is:

$$\frac{\Pi_{1,s}}{\Pi_0} = \delta\beta \frac{\frac{\partial V_{0,1}(W_{0,1,s})}{\partial W_{0,1,s}}}{p_s C_{0,0}^{\alpha-1}} \tag{88}$$

Analogously, the maximization problem for a time-inconsistent sophisticated agent at time 1 is:

$$V_{1,1} = \max_{C_{1,1},W_{1,2}} U(C_{1,1}) + \delta\beta E_1[V_{1,2}(W_{1,2})] \tag{89}$$

$$\text{s.t. } \Pi_1 C_{1,1} + E_1[\Pi_2 W_{1,2}] \leq \Pi_1 W_{1,1} \tag{90}$$

$$V_{1,2}(W_{1,2}) = \frac{1}{\alpha} C_{1,2}^{\alpha} + \beta V_{1,3}(W_{1,3}) \tag{91}$$

The Lagrangian is:

$$\mathcal{L} = U(C_{1,1}) + \delta\beta E_1[V_{1,2}(W_{1,2})] + \lambda_{1,1}\left(W_{1,1} - C_{1,1} - E_1\left[\frac{\Pi_2}{\Pi_1} W_{1,2,s}\right]\right) \tag{92}$$

where according to the Envelope theorem applied to the value function $V_{1,1}^*$ at time 1 at its optimum we get:

$$\frac{\partial V_{1,1}^*(W_{1,1,s})}{\partial W_{1,1,s}} = \frac{\partial \mathcal{L}}{\partial W_{1,1,s}} \quad \Rightarrow \quad \frac{\partial V_{1,1}^*(W_{1,1,s})}{\partial W_{1,1,s}} = \frac{\partial V_{1,1}^*(W_{1,1,s})}{\partial C_{1,1,s}} = C_{1,1,s}^{\alpha-1} \tag{93}$$

Hence we can represent the SDF of a sophisticated time-inconsistent agent as the SDF of a time-consistent agent times an adjustment term:

$$\frac{\Pi_{1,s}}{\Pi_0} = \delta\beta \frac{\frac{\partial V_{0,1}(W_{0,1,s})}{\partial W_{0,1,s}}}{p_s C_{0,0}^{\alpha-1}} = \delta\beta \left(\frac{C_{1,1,s}}{C_{0,0}}\right)^{\alpha-1} \frac{\frac{\partial V_{0,1}(W_{0,1,s})}{\partial W_{0,1,s}}}{\frac{\partial V_{1,1}(W_{1,1,s})}{\partial W_{1,1,s}}} \tag{94}$$

Assuming that consumption is homogenous of degree one in wealth ($C = \varphi W$) we can rewrite the stochastic discount factor as follows:

$$\frac{\Pi_{1,s}}{\Pi_0} = \delta\beta \left(\frac{C_{1,1,s}}{C_{0,0}}\right)^{\alpha-1} \frac{\left[\frac{1}{\alpha} + \beta\frac{1}{\alpha} E_0\left[\frac{V_{2,2}}{C_{1,1}}\right]^{\alpha}\right] \times \alpha\varphi_{0,1,s}^{\alpha} W_{1,1,s}^{\alpha-1}}{\left[\frac{1}{\alpha} + \delta\beta\frac{1}{\alpha} E_0\left[\frac{V_{2,2}}{C_{1,1}}\right]^{\alpha}\right] \times \alpha\varphi_{0,1,s}^{\alpha} W_{1,1,s}^{\alpha-1}} = \delta\beta \left(\frac{C_{1,1,s}}{C_{0,0}}\right)^{\alpha-1} \frac{V_{0,1}}{V_{1,1}} \tag{95}$$

Since $V_{0,1} > V_{1,1}$ there exists a specification of parameters under which the sophisticated time-inconsistent agent can outcrowd the time-consistent agent with time discount parameter β.

2.1.2 Epstein-Zin preferences

A sophisticated time-inconsistent agent solves the following problem

$$V_{0,0} = \max_{C_{0,0},W_{0,1,s}} \left[C_{0,0}^{\rho} + \delta\beta E_0[(V_{0,1}(W_{0,1}))^{\alpha}]^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{96}$$

$$\text{s.t. } \Pi_0 C_{0,0} + E_0[\Pi_1 W_{0,1}] \leq \Pi_0 W_{0,0} \tag{97}$$

$$V_{0,1} = \left[C_{1,1}^{\rho} + \beta E_1[(V_{0,2}(W_{0,2}))^{\alpha}]^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}, \tag{98}$$

where $V_{0,1}$ is the value that the self 0 gives to future consumption . Hence, the Lagrangian and the first-order conditions w.r.t. $C_{0,0}$ and $W_{0,1,s}$ in each state s are:

$$\mathcal{L} = \left[C_{0,0}^{\rho} + \delta\beta E_0[(V_{0,1}(W_{0,1}))^{\alpha}]^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} + \lambda_{0,0}\left(W_{0,0} - C_{0,0} - E_0\left[\frac{\Pi_1}{\Pi_0}W_{0,1}\right]\right) \tag{99}$$

$$\frac{\partial \mathcal{L}}{\partial C_{0,0}} = \frac{\partial V_{0,0}}{\partial C_{0,0}} - \lambda_{0,0} = \frac{1}{\rho}V_{0,0}^{1-\rho}\rho C_{0,0}^{\rho-1} - \lambda_{0,0} = 0 \tag{100}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial W_{0,1,s}} &= \frac{\partial V_{0,0}}{\partial W_{0,1,s}} - p_s\left[\frac{\Pi_{1,s}}{\Pi_0}\right]\lambda_{0,0} = \\ &= \frac{1}{\rho}V_{0,0}^{1-\rho}\frac{\rho}{\alpha}\delta\beta\left(E_0[V_{0,1}(W_{0,1})^{\alpha}]\right)^{\frac{\rho-\alpha}{\alpha}}\alpha V_{0,1}(W_{0,1,s})^{\alpha-1}\frac{1}{\rho}V_{0,1}(W_{0,1,s})^{1-\rho}\frac{\partial V_{0,1}(W_{0,1,s})^{\rho}}{\partial W_{0,1,s}} \\ &- p_s\left[\frac{\Pi_{1,s}}{\Pi_0}\right]\lambda_{0,0} = 0\end{aligned} \tag{101}$$

where $\lambda_{0,0}$ is the Lagrangian multiplier at time 0 and p_s is the probability of state s occurring at time t. From equations (100) and (101) it follows that the stochastic discount factor at state s will be:

$$\frac{\Pi_{1,s}}{\Pi_0} = \frac{\frac{\partial V_{0,0}}{\partial W_{0,1,s}}}{\frac{\partial V_{0,0}}{\partial C_{0,0}}} = \frac{\frac{1}{\rho}\frac{\partial V_{0,1}(W_{0,1,s})^{\rho}}{\partial W_{0,1,s}}}{p_s C_{0,0}^{\rho-1}}\left(\frac{V_{0,1}(W_{0,1,s})}{E_0[V_{0,1}(W_{0,1})^{\alpha}]^{\frac{1}{\alpha}}}\right)^{\alpha-\rho} \tag{102}$$

At time $t = 1$ the sophisticated time-inconsistent agent maximizes:

$$V_{1,1} = \max_{C_{1,1},W_{2,2,s}} \left[C_{1,1}^{\rho} + \delta\beta E_1[(V_{1,2}(W_{1,2}))^{\alpha}]^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{103}$$

$$\text{s.t. } \Pi_1 C_{1,1} + E_1[\Pi_2 W_{2,2}] \leq \Pi_1 W_{1,1} \tag{104}$$

$$\Rightarrow C_{1,1} = W_{1,1} - E_1\left[\frac{\Pi_2}{\Pi_1}W_{0,0}\right] \tag{105}$$

Thus, the Lagrangian is:

$$\mathcal{L} = \left[C_{1,1}^{\rho} + \delta\beta E_1[(V_{1,2}(W_{1,2}))^{\alpha}]^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} + \lambda_{1,1}\left(W_{1,1} - C_{1,1} - E_1\left[\frac{\Pi_2}{\Pi_1}W_{2,2}\right]\right) \tag{106}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial C_{1,1}} = \frac{\partial V_{1,1}}{\partial C_{1,1}} - \lambda_{1,1} = \frac{1}{\rho}V_{1,1}^{1-\rho}\rho C_{1,1}^{\rho-1} - \lambda_{1,1} = 0$$

$$\Rightarrow \lambda_{1,1} = V_{1,1}^{1-\rho}C_{1,1}^{\rho-1} \tag{107}$$

By the envelope theorem applied to the value function $V_{1,1}^*$ at time 1 at its optimum it follows that:

$$\frac{\partial V_{1,1}^*(W_{1,1,s})}{\partial W_{1,1,s}} = \frac{\partial \mathcal{L}}{\partial W_{1,1,s}} \quad \Rightarrow \quad \frac{\partial V_{1,1}^*(W_{1,1,s})}{\partial W_{1,1,s}} = \frac{\partial V_{1,1}^*(W_{1,1,s})}{\partial C_{1,1,s}} = \lambda_{1,1} = V_{1,1}^{1-\rho} C_{1,1,s}^{\rho-1} \tag{108}$$

To see this alternatively we can substitute the optimal $C_{1,1}^*$ that follows from constraint (105) in equation (103), where (105) holds when $V_{1,1}^*$ is at its optimum. Considering the two partial derivatives gives:

$$\frac{\partial V_{1,1}(W_{1,1,s})}{\partial W_{1,1,s}} = \frac{\partial V_{1,1}(W_{1,1,s})}{\partial W_{1,1,s}} = V_{1,1}^{1-\rho} C_{1,1,s}^{\rho-1} \tag{109}$$

Thus,

$$\frac{\partial V_{1,1}(W_{1,1,s})}{\partial W_{1,1,s}} = \frac{1}{\rho} V_{1,1}^{1-\rho} \frac{\partial V_{1,1}(W_{1,1,s})^\rho}{\partial W_{1,1,s}} = V_{1,1}^{1-\rho} C_{1,1,s}^{\rho-1} \quad \Rightarrow \quad \frac{1}{\rho} \frac{\partial V_{1,1}(W_{1,1,s})^\rho}{\partial W_{1,1,s}} = C_{1,1,s}^{\rho-1} \tag{110}$$

In order to rewrite the stochastic discount factor of the sophisticated time-inconsistent agent in terms of the optimal consumption share at time 1, $C_{1,1}$, we multiply the stochastic discount factor by an adjustment term:

$$\frac{\Pi_{1,s}}{\Pi_0} = \frac{\frac{1}{\rho}\frac{\partial V_{0,1}(W_{0,1,s})^\rho}{\partial W_{0,1,s}}}{p_s C_{0,0}^{\rho-1}} \left(\frac{V_{0,1}(W_{0,1,s})}{E_0[V_{0,1}(W_{0,1})^\alpha]^{\frac{1}{\alpha}}} \right)^{\alpha-\rho} \times \frac{\frac{1}{\rho}\frac{\partial V_{1,1}(W_{1,1,s})^\rho}{\partial W_{1,1,s}}}{\frac{1}{\rho}\frac{\partial V_{1,1}(W_{1,1,s})^\rho}{\partial W_{1,1,s}}} \tag{111}$$

$$= \delta\beta \left(\frac{C_{1,1,s}}{C_{0,0}} \right)^{\rho-1} \left(\frac{V_{0,1}(W_{0,1,s})}{E_1[(V_{0,1}(W_{1,1}))^\alpha]^{\frac{1}{\alpha}}} \right)^{\alpha-\rho} \frac{\frac{\partial V_{0,1}(W_{0,1,s})^\rho}{\partial W_{0,1,s}}}{\frac{\partial V_{1,1}(W_{1,1,s})^\rho}{\partial W_{1,1,s}}} \tag{112}$$

Note that, the envelope theorem can be applied to $V_{1,1}^*$ at its optimum since this is the function that the agent maximizes and $C_{1,1}^*$ is the optimal solution. However, the envelope theorem cannot be applied to $V_{0,1}$ at the optimum of $V_{1,1}^*$ since $C_{1,1}^*$ does not maximize $V_{0,1}$. Assuming consumption is homogenous of degree one in wealth ($C = \varphi W$) and noting that for the sophisticated time-inconsistent agent her consumption plan equal the optimal consumption ($C_{0,1} = C_{1,1}$ and $W_{0,1} = W_{1,1}$) we can substitute the partial derivatives in equation (112) as follows:

$$\frac{\Pi_{1,s}}{\Pi_0} = \delta\beta \left(\frac{C_{1,1,s}}{C_{0,0}} \right)^{\rho-1} \left(\frac{V_{0,1}(W_{1,1,s})}{E_0[(V_{1,1}(W_{0,1}))^\alpha]^{\frac{1}{\alpha}}} \right)^{\alpha-\rho} \frac{\left[1 + \beta E_0\left[\left(\frac{V_{2,2}}{C_{1,1}}\right)^\alpha\right]^{\frac{\rho}{\alpha}}\right] \times \rho\varphi_{0,1,s}^\rho W_{1,1,s}^{\rho-1}}{\left[1 + \beta\delta E_0\left[\left(\frac{V_{2,2}}{C_{1,1}}\right)^\alpha\right]^{\frac{\rho}{\alpha}}\right] \times \rho\varphi_{0,1,s}^\rho W_{1,1,s}^{\rho-1}}$$

$$= \delta\beta \left(\frac{C_{1,1,s}}{C_{0,0}} \right)^{\rho-1} \left(\frac{V_{0,1}(W_{1,1,s})}{E_0[(V_{0,1}(W_{1,1}))^\alpha]^{\frac{1}{\alpha}}} \right)^{\alpha-\rho} \left(\frac{V_{0,1}(W_{1,1,s})}{V_{1,1}(W_{1,1,s})} \right)^\rho \tag{113}$$

Since $(V_{0,1}(W_{1,1,s}))^\rho > (V_{1,1}(W_{1,1,s}))^\rho$ there are specifications of the parameters under which the time-inconsistent agent can dominate over a time-consistent agent with time discount factor β.

2.2 Naive time-inconsistent agent

We next consider the optimization problem of the naive time-inconsistent agent. She does not realize her self-control problems and therefore values future strings of consumption today the same way she will value them tomorrow. Thus she believes she will not be time-inconsistent tomorrow, but instead will discount future cash flows rationally.

2.2.1 CRRA preferences

$$V_{0,0} = \max_{C_{0,0},W_{0,1}} U(C_{0,0}) + \delta\beta E_0[V_{0,1}(W_{0,1})] \tag{114}$$

$$\text{s.t. } \Pi_0 C_{0,0} + E_0[\Pi_1 W_{0,1}] \leq \Pi_0 W_{0,0} \tag{115}$$

$$V_{0,1}(W_{0,1}) = \frac{1}{\alpha} C_{0,1} + \beta V_{0,2}(W_{0,2}) \tag{116}$$

The continuation function $V_{0,1}(W_{0,1})$ shows the value that self 0 gives to future consumption. The only time discount factor that she plans to use is $\beta > \delta\beta$ which shows that self 0 gives larger relative weight on immediate consumption at time 0 than she plans to give on immediate consumption at time 1. The agent, however, does not realize that this function differs from the actual $V_{1,1}(W_{1,1})$ that self 1 will optimize. The Lagrangian and first-order conditions are given as follows:

$$\mathcal{L} = U(C_{0,0}) + \delta\beta E_0[V_{0,1}(W_{1,1})] + \lambda_{0,0}\left(W_{0,0} - C_{0,0} - E_0\left[\frac{\Pi_1}{\Pi_0} W_{0,1,s}\right]\right) \tag{117}$$

$$\frac{\partial \mathcal{L}}{\partial C_{0,0}} = C_{0,0}^{\alpha-1} - \lambda_{0,0} = 0 \tag{118}$$

$$\Rightarrow \lambda_{0,0} = C_{0,0}^{\alpha-1} \tag{119}$$

$$\frac{\partial \mathcal{L}}{\partial W_{0,1,s}} = \delta\beta \frac{\partial V_{0,1}(W_{0,1,s})}{\partial W_{0,1,s}} - p_s \left[\frac{\Pi_{1,s}}{\Pi_0}\right] \lambda_{0,0} = 0 \tag{120}$$

$$\Rightarrow \lambda_{0,0} = \delta\beta \frac{\frac{\partial V_{0,1}(W_{0,1})}{\partial W_{0,1,s}}}{p_s \frac{\Pi_{1,s}}{\Pi_0}} \tag{121}$$

From equations (119) and (121) it follows that the stochastic discount factor is:

$$\frac{\Pi_{1,s}}{\Pi_0} = \delta\beta \frac{\frac{\partial V_{0,1}(W_{1,1,s})}{\partial W_{1,1,s}}}{p_s C_{0,0}^{\alpha-1}} = \delta\beta \left(\frac{C_{0,1}}{C_{0,0}}\right)^{\alpha-1} \tag{122}$$

2.2.2 Epstein-Zin preferences

A naive time-inconsistent agent solves the following problem

$$V_{0,0} = \max_{C_{0,0}, W_{0,1,s}} \left[C_{0,0}^{\rho} + \delta\beta E_0[(V_{0,1}(W_{0,1}))^{\alpha}]^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}} \tag{123}$$

$$\text{s.t. } \Pi_0 C_{0,0} + E_0[\Pi_1 W_{0,1}] \leq \Pi_0 W_{0,0} \tag{124}$$

$$V_{0,1} = \left[C_{0,1}^{\rho} + \beta E_1[(V_{0,2}(W_{0,2}))^{\alpha}]^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}}, \tag{125}$$

where $V_{0,1}$ is the value that the self 0 gives to future consumption . Since this agent is not aware of her self-control problem she does not consider the optimal consumption $C_{1,1}$ in her optimization problem, but instead considers $C_{0,1}$ (the consumption that self 0 views as optimal). Hence, the Lagrangian and the first-order conditions w.r.t. $C_{0,0}$ and every $W_{0,1,s}$ in state s are:

$$\mathcal{L} = \left[C_{0,0}^{\rho} + \delta\beta E_0[(V_{0,1}(W_{0,1}))^{\alpha}]^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}} + \lambda_{0,0} \left(W_{0,0} - C_{0,0} - E_1 \left[\frac{\Pi_1}{\Pi_0} W_{0,1} \right] \right) \tag{126}$$

$$\frac{\partial \mathcal{L}}{\partial C_{0,0}} = \frac{\partial V_{0,0}}{\partial C_{0,0}} - \lambda_{0,0} = \frac{1}{\rho} V_{0,0}^{1-\rho} \rho C_{0,0}^{\rho-1} - \lambda_{0,0} = 0 \tag{127}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial W_{0,1,s}} &= \frac{\partial V_{0,0}}{\partial W_{0,1,s}} - p_s \left[\frac{\Pi_{1,s}}{\Pi_0} \right] \lambda_{0,0} = \\ &= \frac{1}{\rho} V_{0,0}^{1-\rho} \frac{\rho}{\alpha} \delta\beta \left(E_0[V_{0,1}(W_{0,1})^{\alpha}] \right)^{\frac{\rho-\alpha}{\alpha}} \alpha V_{0,1}(W_{0,1,s})^{\alpha-1} \frac{1}{\rho} V_{0,1}(W_{0,1,s})^{1-\rho} \frac{\partial V_{0,1}(W_{0,1,s})^{\rho}}{\partial W_{0,1,s}} \\ &- p_s \left[\frac{\Pi_{1,s}}{\Pi_0} \right] \lambda_{0,0} = 0 \end{aligned} \tag{128}$$

where $\lambda_{0,0}$ is the Lagrangian multiplier at time 0. From equations (127) and (128) it follows that the stochastic discount factor at state s will be:

$$\frac{\Pi_{1,s}}{\Pi_0} = \frac{\frac{\partial V_{0,0}(W_{0,0})}{\partial W_{0,1,s}}}{\frac{\partial V_{0,0}(W_{0,0})}{\partial C_{0,0}}} = \frac{\frac{1}{\rho} \frac{\partial V_{0,1}(W_{0,1,s})^{\rho}}{\partial W_{0,1,s}}}{p_s C_{0,0}^{\rho-1}} \left(\frac{V_{0,1}(W_{0,1,s})}{E_1[V_{0,1}(W_{0,1})^{\alpha}]^{\frac{1}{\alpha}}} \right)^{\alpha-\rho} \tag{129}$$

By the envelope theorem applied at the optimum of $V_{0,1}^{*}$ we know that:

$$\frac{\partial V_{0,1}(W_{0,1,s})}{\partial C_{0,1,s}} = \frac{\partial V_{0,1}(W_{0,1,s})}{\partial W_{0,1,s}} = \lambda_{0,1} \tag{130}$$

$$\Rightarrow V_{0,1}^{1-\rho} C_{0,1}^{\rho-1} = \frac{1}{\rho} V_{0,1}^{1-\rho} \frac{\partial V_{0,1}(W_{0,1,s})^{\rho}}{\partial W_{0,1,s}} \tag{131}$$

Hence, the stochastic discount factor of the naive time-inconsistent agent is given as follows:

$$\frac{\Pi_{1,s}}{\Pi_0} = \delta\beta \left(\frac{C_{0,1,s}}{C_{0,0}} \right)^{\rho-1} \left(\frac{V_{0,1}(W_{0,1,s})}{E_1[(V_{0,1}(W_{0,1,s}))^{\alpha}]^{\frac{1}{\alpha}}} \right)^{\alpha-\rho} \tag{132}$$

C Pareto problem and equilibrium derivation

The two-agent Pareto problem can be represented as the optimization of a social planner who maximizes utilities of the investors of both types. I present the case with two time-consistent agents. However, as shown by Lucas and Stokey (1984), Kan (1995), and Backus, Routledge, and Zin (2009) a recursive formulation of the problem exists. Applying Theorem 3 from Lucas and Stockey (1984) it follows that the Pareto optimal allocation is given by the following Bellman equation:

$$J(C_t, V_{B,t}) = \max_{\{C_{A,t}, V_{B,t+1}\}} \Big[(1-\beta_A)C_{A,t}^{\rho_A} + \beta_A E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{\rho_A}{\alpha_A}}\Big]^{\frac{1}{\rho_A}} \tag{133}$$

$$\text{s.t. } V_{B,t}(C_{B,t}, V_{B,t+1}) \geq V_{B,t} \tag{134}$$

$$C_{A,t} + C_{B,t} = C_t, \tag{135}$$

where $V_{B,t}$ is the so-called promised utility to agent B at time t. The resulting value function for agent A is then given by $V_{A,t} = J(C_t, V_{B,t})$. Since there is monotonicity in preferences, the utility-promise constraint is binding and hence, constraint (134) can be replaced by $V_{B,t}(C_{B,t}, V_{B,t+1}) = V_{B,t}$.

To derive the equilibrium condition we formulate and maximize the Lagrangian on the constraints in equation (133):

$$\begin{aligned}\mathcal{L} = \max_{\{C_{A,t}, V_{B,t+1}, \lambda_t\}} &\Big[(1-\beta_A)C_{A,t}^{\rho_A} + \beta_A E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{\rho_A}{\alpha_A}}\Big]^{\frac{1}{\rho_A}} \\ &+ \lambda_t\bigg(\Big[(1-\beta_B)C_{B,t}^{\rho_B} + \beta_B E_t[V_{B,t+1}^{\alpha_B}]^{\frac{\rho_B}{\alpha_B}}\Big]^{\frac{1}{\rho_B}} - V_{B,t}\bigg)\end{aligned} \tag{136}$$

where λ_t is the Lagrange multiplier. Now, given the state variables C_t and $V_{B,t}$, we find the decision variables $C_{A,t}$ and $V_{B,t+1}$ such that the first-order conditions are satisfied:

$$\frac{\partial \mathcal{L}}{\partial C_{A,t}} = \frac{1}{\rho_A} J(C_t, V_{B,t})^{1-\rho_A} \rho_A (1-\beta_A) C_{A,t}^{\rho_A - 1} - \lambda_t \frac{1}{\rho_B} V_{B,t}^{1-\rho_B} \rho_B (1-\beta_B) C_{B,t}^{\rho_B - 1} = 0 \tag{137}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial V_{B,t+1}} &= \frac{1}{\rho_A} J(C_t, V_{B,t})^{1-\rho_A} \beta_A \frac{\rho_A}{\alpha_A} \Big(E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{1}{\alpha_A}}\Big)^{\rho_A - \alpha_A} \alpha_A J(C_{t+1}, V_{B,t+1})^{\alpha_A - 1} \Big(\frac{\partial J_{t+1}}{\partial V_{B,t+1}}\Big) \\ &+ \lambda_t \frac{1}{\rho_B} V_{B,t}^{1-\rho_B} \beta_B \frac{\rho_B}{\alpha_B} \Big(E_t[V_{B,t+1}^{\alpha_B}]^{\frac{1}{\alpha_B}}\Big)^{\rho_B - \alpha_B} \alpha_B V_{B,t+1}^{\alpha_B - 1} = 0\end{aligned} \tag{138}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = \Big[(1-\beta_B)C_{B,t}^{\rho_B} + \beta_B E_t[V_{B,t+1}^{\alpha_B}]^{\frac{\rho_B}{\alpha_B}}\Big]^{\frac{1}{\rho_B}} - V_{B,t}, \tag{139}$$

where $\frac{\partial J_{t+1}}{\partial V_{B,t+1}} = -\lambda_{t+1}$, as the function J_{t+1} depends on the promised utility to agent B, $V_{B,t+1}$ only through constraint (134). Thus, changing $V_{B,t+1}$ by a certain amount leads to a decrease in J_{t+1} that equals that amount times λ_{t+1}.

From equation (137) we get:

$$\lambda_t = \frac{J(C_t, V_{B,t})^{1-\rho_A}(1-\beta_A)C_{A,t}^{\rho_A-1}}{V_{B,t}^{1-\rho_B}(1-\beta_B)C_{B,t}^{\rho_B-1}} \text{ at time t} \tag{140}$$

$$\lambda_{t+1} = \frac{J(C_{t+1}, V_{B,t+1})^{1-\rho_A}(1-\beta_A)C_{A,t+1}^{\rho_A-1}}{V_{B,t+1}^{1-\rho_B}(1-\beta_B)C_{B,t+1}^{\rho_B-1}} \text{ at time t+1.} \tag{141}$$

Then we simplify equation (138) and substitute equations (140) and (141) in it:

$$J(C_t, V_{B,t})^{1-\rho_A}\beta_A\Big(E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{1}{\alpha_A}}\Big)^{\rho_A-\alpha_A} J(C_{t+1}, V_{B,t+1})^{\alpha_A-1}(-\lambda_{t+1})+$$
$$+\lambda_t V_{B,t}^{1-\rho_B}\beta_B\Big(E_t[V_{B,t+1}^{\alpha_B}]^{\frac{1}{\alpha_B}}\Big)^{\rho_B-\alpha_B} V_{B,t+1}^{\alpha_B-1} = 0$$
$$J(C_t, V_{B,t})^{1-\rho_A}\beta_A\Big(E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{\frac{1}{\alpha_A}}\Big)^{\rho_A-\alpha_A} J(C_{t+1}, V_{B,t+1})^{\alpha_A-1}\frac{J(C_{t+1}, V_{B,t+1})^{1-\rho_A}(1-\beta_A)C_{A,t+1}^{\rho_A-1}}{V_{B,t+1}^{1-\rho_B}(1-\beta_B)C_{B,t+1}^{\rho_B-1}} =$$
$$= \frac{J(C_t, V_{B,t})^{1-\rho_A}(1-\beta_A)C_{A,t}^{\rho_A-1}}{V_{B,t}^{1-\rho_B}(1-\beta_B)C_{B,t}^{\rho_B-1}} V_{B,t}^{1-\rho_B}\beta_B\Big(E_t[V_{B,t+1}^{\alpha_B}]^{\frac{1}{\alpha_B}}\Big)^{\rho_B-\alpha_B} V_{B,t+1}^{\alpha_B-1}$$

$$\Leftrightarrow\ \beta_A\left(\frac{C_{A,t+1}}{C_{A,t}}\right)^{\rho_A-1}\left(\frac{J(C_{t+1}, V_{B,t+1})}{E_t[J(C_{t+1}, V_{B,t+1})^{\alpha_A}]^{1/\alpha_A}}\right)^{\alpha_A-\rho_A} = \beta_B\left(\frac{C_{B,t+1}}{C_{B,t}}\right)^{\rho_B-1}\left(\frac{V_{B,t+1}}{E_t[V_{B,t+1}^{\alpha_B}]^{1/\alpha_B}}\right)^{\alpha_B-\rho_B} \tag{142}$$

Equation (142) gives the equilibrium condition. I solve a normalized version of the model with all variables divided by aggregate consumption. I denote the value functions as $v_{i,t} = V_{i,t}/C_t$ and consumption shares as $c_{i,t} = C_{i,t}/C_t$. Hence, the equilibrium condition can be written as:

$$\beta_A \left(\frac{c_{A,t+1} C_{t+1}}{c_{A,t} C_t} \right)^{\rho_A - 1} \left(\frac{j(v_{B,t+1}) C_{t+1}}{E_t[j(v_{B,t+1})^{\alpha_A} C_{t+1}^{\alpha_A}]^{1/\alpha_A}} \right)^{\alpha_A - \rho_A} = \beta_B \left(\frac{c_{B,t+1} C_{t+1}}{c_{B,t} C_t} \right)^{\rho_B - 1} \left(\frac{v_{B,t+1} C_{t+1}}{E_t[v_{B,t+1}^{\alpha_B} C_{t+1}^{\alpha_B}]^{1/\alpha_B}} \right)^{\alpha_B - \rho_B} \tag{143}$$

$$\Leftrightarrow \beta_A \left(\frac{c_{A,t+1} C_{t+1}}{c_{A,t} C_t} \right)^{\rho_A - 1} \left(\frac{j(v_{B,t+1}) C_{t+1}}{E_t[j(v_{B,t+1})^{\alpha_A} C_{t+1}^{\alpha_A}]^{1/\alpha_A}} \right)^{\alpha_A - \rho_A} \frac{C_t^{\alpha_A - \rho_A}}{C_t^{\alpha_A - \rho_A}} =$$
$$= \beta_B \left(\frac{c_{B,t+1} C_{t+1}}{c_{B,t} C_t} \right)^{\rho_B - 1} \left(\frac{v_{B,t+1} C_{t+1}}{E_t[v_{B,t+1}^{\alpha_B} C_{t+1}^{\alpha_B}]^{1/\alpha_B}} \right)^{\alpha_B - \rho_B} \frac{C_t^{\alpha_B - \rho_B}}{C_t^{\alpha_B - \rho_B}} \tag{144}$$

$$\Leftrightarrow \beta_A \left(\frac{c_{A,t+1}}{c_{A,t}} \right)^{\rho_A - 1} \left(\frac{C_{t+1}}{C_t} \right)^{\rho_A - 1} \left(\frac{C_{t+1}}{C_t} \right)^{\alpha_A - \rho_A} \left(\frac{j(v_{B,t+1})}{E_t[j(v_{B,t+1})^{\alpha_A} (C_{t+1}/C_t)^{\alpha_A}]^{1/\alpha_A}} \right)^{\alpha_A - \rho_A} =$$
$$= \beta_B \left(\frac{c_{B,t+1}}{c_{B,t}} \right)^{\rho_B - 1} \left(\frac{C_{t+1}}{C_t} \right)^{\rho_B - 1} \left(\frac{C_{t+1}}{C_t} \right)^{\alpha_B - \rho_B} \left(\frac{v_{B,t+1}}{E_t[v_{B,t+1}^{\alpha_B} (C_{t+1}/C_t)^{\alpha_B}]^{1/\alpha_B}} \right)^{\alpha_B - \rho_B} \tag{145}$$

$$\Leftrightarrow \beta_A \left(\frac{c_{A,t+1}}{c_{A,t}} \right)^{\rho_A - 1} \left(\frac{C_{t+1}}{C_t} \right)^{\alpha_A - 1} \left(\frac{j(v_{B,t+1})}{E_t[j(v_{B,t+1})^{\alpha_A} (C_{t+1}/C_t)^{\alpha_A}]^{1/\alpha_A}} \right)^{\alpha_A - \rho_A} =$$
$$= \beta_B \left(\frac{c_{B,t+1}}{c_{B,t}} \right)^{\rho_B - 1} \left(\frac{C_{t+1}}{C_t} \right)^{\alpha_B - 1} \left(\frac{v_{B,t+1}}{E_t[v_{B,t+1}^{\alpha_B} (C_{t+1}/C_t)^{\alpha_B}]^{1/\alpha_B}} \right)^{\alpha_B - \rho_B} \tag{146}$$