Pivots and Prestige: Venture Capital Contracts with Experimentation*

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Abstract

We study venture capital finance with experimentation. An entrepreneur contracts with an investor and has private information about a project, which requires costly experimentation by both parties to succeed. In equilibrium, investors learn about the project from the arrival of exogenous information and from the entrepreneur's contract offers. The optimal contract features vesting and dilution, consistent with empirical evidence. Early payouts, pivots, and prestige projects emerge as signaling devices. Surprisingly, technological progress, which lowers the cost of experimentation or which increases the rate of learning, delays separation of types and worsens adverse selection. Liquidation rights for investors also delay separation.

***JEL Classification**:* G24, G32, D82, D83.

***Keywords**: Venture Capital, Experimentation, Signaling, Pivots, Prestige Projects.*

1 Introduction

Venture capital (VC) financing faces substantial uncertainty.[1] This uncertainty generates an information advantage for the entrepreneur, who knows her firm better than any outside investor. Consistent with this view, a long line of literature recognizes private information

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[1]See e.g. Kerr et al. (2014), who document that even conditional on investing, venture capitalists face significant residual uncertainty about the prospects of their investment.

as the central friction in VC financing.[2] When entrepreneurs have private information, they may signal this information by the contractual terms they offer the VC or by the projects they pursue. The degree of private information, however, is not static. As the startup progresses, it generates information, and both entrepreneurs and investors learn about its prospects. Thus, operating a startup is inherently an act of experimentation.[3] Over time, this experimentation changes the degree of adverse selection, and with it the optimal contract between entrepreneur and investor. Despite this natural interaction, how experimentation affects adverse selection, and whether this interaction can explain observed features in VC contracts, has remained elusive.[4]

In this paper, we analyze a dynamic contracting model in which experimentation changes both adverse selection and the optimal contract over time. Common features of VC contracts emerge as equilibrium outcomes. The entrepreneur's equity share features vesting and dilution. Early payouts, pivots, and prestige projects act as signaling devices.

In our model, the entrepreneur (she) offers a contract to an investor (he) in each period. This contract consists of an equity share, which grants the investor a stake in the project if it succeeds, and an immediate payout. The entrepreneur is privately, but imperfectly, informed about the quality of her project, which is either good or bad. She is either a high type, who knows that her project is likely to be good, or a low type. The project requires costly experimentation by both the entrepreneur and the investor to succeed.[5] When both experiment, the good project generates a breakthrough with positive probability, while the bad project generates no breakthrough.[6] Over time, entrepreneur and investor learn from the absence of breakthroughs and revise their beliefs about the project downwards. The high and low type hold different beliefs at any point in time, but their beliefs converge as they learn about the project. Thus, experimentation gradually reduces adverse selection.

The optimal equity share features vesting and dilution. Since the investor becomes more pessimistic about the project over time, the entrepreneur pledges successively larger equity shares to prevent him from abandoning the project. Thus, the entrepreneur's share is diluted

[2]That uncertainty begets private information has been long recognized in the literature on venture capital. See Gompers (1995) for an early contribution. See also Admati and Pfleiderer (1994), Gompers (1995), Ueda (2004), and Piacentino (2019), among many others.

[3]More generally, experimentation is an intrinsic feature of any innovative activity. As Hayek (1948) put it "the solution of the economic problem of society is [...] always a voyage of exploration into the unknown."

[4]While a prominent literature studies how experimentation affects VC contracts, it abstracts from adverse selection. See Bergemann and Hege (1998), Bergemann and Hege (2005), and Manso (2011).

[5]Thus, our model features double moral hazard. See e.g. Schmidt (2003), Casamatta (2003), Repullo and Suarez (2004), and Hellmann (2006). These papers do not feature experimentation.

[6]The bad project does not generate any breakthroughs. Thus, our model features an exponential bandit with good news, as in e.g. Bergemann and Hege (1998), Bergemann and Hege (2005), and Keller et al. (2005).

over time. Eventually, however, the low type entrepreneur starts liquidating, because she knows that her project is unlikely to succeed. Then, the investor updates his belief about the project upwards, because the likelihood that he faces the high type increases.[7] In response, the entrepreneur optimally lowers the investor's share and increases her own. This feature resembles a delayed vesting schedule: the entrepreneur's share initially decreases, but it starts to increase after sufficient time has passed. As Kaplan and Strömberg (2004) document, vesting and dilution feature prominently in VC contracts.

In equilibrium, the investor learns about the project from experimenting and from the entrepreneur's contract offers. This allows the entrepreneur to signal her type. We show that early payouts serve as signaling devices. Initially, the low and high type pool by offering the same contract, which consists of only an equity share. Although the high type can separate in any period, doing so is not optimal early on, because the cost of separating is too high. As time passes, however, experimentation reduces the amount of adverse selection, and therefore the cost of separating. Eventually, the high type separates by offering a payout. Then, the investor's belief about the project jumps upwards, because he learns that he is facing the high type. Again, the entrepreneur reduces the investor's share and increases her own. This resembles performance-contingent vesting (see again Kaplan and Strömberg (2004)) or a buyback.

We also show that pivots and prestige projects can act as signaling devices. Pivots are common among startups and many successful firms have emerged following a radical change in strategy.[8] The folklore explanation for pivots is simple: entrepreneurs realize that their idea is not working and radically change their approach. We instead show that by pivoting entrepreneurs can signal information.

Here is the intuition. Suppose that the entrepreneur can pay a fixed cost to draw another project with the same ex-ante likelihood of success. The entrepreneur's type does not change once the new project is drawn, e.g. because it represents entrepreneurial skill.[9] Since the

[7]Importantly, this is true even though the investor's belief that the project succeeds keeps decreasing. There are two conflicting effects here, and the "composition effect," i.e. the investor's changing belief about which type of entrepreneur she faces, dominates.

[8]Examples abound. Groupon initially started as a social network, Twitter emerged as a side project of an unsuccessful podcasting platform, and Instagram's founders initially developed an app which facilitated meet ups among whiskey enthusiasts. See `https://www.inc.com/jeff-haden/21-side-projects-that-became-million-dollar-startups-and-how-yours-can-too.html` (last accessed 10/13/19) for other examples.

[9]This is a reasonable assumption. For many VC firms, the quality of the founding team determines whether they invest in a startup. VCs expect founders to change ideas, but they believe that founder ability is key to eventual success. See Gompers et al. (2019), who find that founder ability is the most important factor in VC financing decisions. Similarly, Gladstone and Gladstone (2002) note that "Most venture capitalists consider management to be the key to every successful venture capital investment. [...] You can have a good idea and poor management and lose every time; conversely, you can have a poor idea

high type knows that her project is more likely to succeed, her value from pivoting is higher. Given the fixed cost, it may be optimal for the high type to pivot, but not for the low type. This renders separation feasible. We then characterize conditions such that optimal contract indeed features separation via a pivot.

Prestige projects are also common among early stage firms. Perhaps paradoxically, firms divert resources from their main project and use them to generate publicity and goodwill.[10] We show that prestige projects can serve as signaling devices, because they tempt low types to liquidate.

Suppose that the entrepreneur can divert resources towards a prestige project, which generates a higher outside option for her. For example, publicity may make it easier to obtain funding for another startup or to find outside employment.[11] This outside option is more appealing for the low type, who knows that her project is less likely to succeed. Once the high type implements the prestige project, the low type prefers to liquidate and takes the outside option.[12] Because of this, prestige projects can be used to signal, and we provide conditions such that the optimal contract implements prestige projects.

Recent technological progress has dramatically transformed venture capital financing. As Kerr et al. (2014) report, the cost of starting internet companies has decreased radically,[13] which has prompted VC firms to adopt a "spray and pray" approach and to fund a large number of startups with limited vetting and support. Simultaneously, cohort-based accelerators (such as Y-Combinator) have increased entry by relatively inexperienced founders. Arguably, experimentation about startups has sped up, and investors discover more quickly whether a startup is going to be successful. An important question is how these developments affect the adverse selection friction between the entrepreneur and the VC. Does technological progress alleviate adverse selection? Or does it cause adverse selection to persist longer?

Surprisingly, technological progress delays separation in our model. Adverse selection

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and a good management team and win every time."

[10]For example, WeWork, a co-working platform, founded an elementary school (see `https://www.reuters.com/article/us-wework-wegrow/wework-to-close-its-wegrow-elementary-school-in-new-york-next-year-idUSKBN1WQ28V`, last accessed 10/13/19), Uber offered helicopter rides (see `https://www.theverge.com/2019/10/3/20897427/uber-helicopter-trips-manhattan-jfk-airport-price`, last accessed 10/13/19), and Tesla's Elon Musk has sold a device which closely resembles, but is not, a flamethrower (see `https://www.boringcompany.com/not-a-flamethrower`, last accessed 10/13/19).

[11]This is consistent with Gompers et al. (2010), who document that and entrepreneur's past performance is an important consideration in funding decisions.

[12]Importantly, we show that the low type prefers to take the outside option even though the alternative is imitating the high type.

[13]They write "firms in these sectors that would have cost $5 million to set up a decade ago can be done for under $50,000 today. For example, open-source software lowers the costs associated with hiring programmers. In addition, fixed investments in high-quality infrastructure, servers, and other hardware are no longer necessary [...] because they can be rented in tiny increments from cloud computing providers"

persists longer and the venture capital market is less efficient. We cast these results in terms of comparative statics. First, as the cost of running the startup decreases for the entrepreneur, the high type separates later. Intuitively, the lower cost makes it more appealing for the low type to imitate, which increases the cost of separating and delays separation. Second, as VCs become less involved in a startup, their cost of experimentation decreases, which also delays separation by making imitation more appealing for the low type. Third, if breakthroughs for the good project arrive more quickly, i.e. learning speeds up, adverse selection may also persist longer. Finally, we consider the effect of accelerators. When less experienced entrepreneurs found startups, the likelihood of facing the high type decreases. Then, pooling becomes costlier for the high type, and she separates earlier.

Liquidation rights are commonly used to protect investors from the entrepreneur's information advantage (see again Kaplan and Strömberg (2004)). However, increasing investors' liquidation rights may backfire, because it delays separation. The effect is similar to the one of our example on prestige projects. When investors get favorable liquidation rights, the entrepreneur's value from abandoning the project is lower. Then, the low type is willing to continue longer, which makes it costlier to separate. This result is broadly consistent with Ewens et al. (2019), who estimate that investor liquidation preference is detrimental to firm value.

Technical Contribution Our model is a dynamic informed-principal problem with experimentation. In equilibrium, each entrepreneur type and the investor learn *differently* from the absence of breakthroughs and their beliefs follow different laws of motion. In addition, signaling occurs on the equilibrium path either by the low type liquidating or by offering a different contract from the high type. As time passes, learning diminishes the amount of adverse selection, which introduces subtle dynamic incentives. The optimal contract trades off separating today against pooling and separating at a lower cost tomorrow. Despite these apparent complexities, we characterize the ex-ante optimal contract. In fact, we characterize the entire set of optimal pooling and separating Perfect Bayesian Equilibria (PBE). We do this with only minimal restrictions: the entrepreneur lacks commitment and offers a contract in each period[14] and once beliefs are degenerate, they stay that way. Limited commitment is reasonable in our setting. Startups face rapid changes and significant uncertainty, which often render contractual commitments moot.[15] The latter assumption is common in dynamic adverse selection models. It serves to avoid situations in which the investor is offered

[14] We share this feature with the literature on relational contracts with adverse selection, i.e. Halac (2012), Malcomson (2016), Fahn and Klein (2017), and Kartal (2018).

[15] See Kaplan and Strömberg (2001), Kaplan and Strömberg (2003), Kaplan and Strömberg (2004) and Kerr et al. (2014).

a contract which she believes will be offered with probability zero.[16]

2 Literature

Our paper contributes to the literature on experimentation in venture capital financing. In seminal work, Bergemann and Hege (1998) and Bergemann and Hege (2005) study optimal contracts when the entrepreneur can divert investment in an experimentation setting. We extend this literature by considering private information on the entrepreneur's side and by characterizing how adverse selection changes as information arrives over time. In Bergemann and Hege's papers, the questions about signaling and separation do not arise. Hence, our results on early payouts, pivots, and prestige projects cannot be obtained in their frameworks.

Methodologically, the closest paper is Kaniel and Orlov (2018), which studies the relationship between a mutual fund family and a manager. As in our paper, there is experimentation about the manager's skill and information is revealed by both news arrival and retention/continuation decisions. In their paper, however, retention is the only signaling device, whereas in our paper, the terms of the contract also can be used to signal. Our results on vesting and dilution, early payouts, pivots, and prestige project do not appear in Kaniel and Orlov (2018).

Also closely related is Azarmsa and Cong (2018). They study a model of venture capital financing in which the entrepreneur reveals information at an interim date, at which an additional investment is required. They characterize how information is revealed and how information disclosure leads to a hold-up problem. Our paper shares a similar spirit, since in each period the investor must make payments to continue with the project. However, information in our setting is revealed both exogenously, via the arrival of breakthroughs, and indirectly, via the choice of contract and the liquidation decisions. While Azarmsa and Cong (2018) characterize the optimal security design and its interaction with optimal disclosure, we focus on how signaling reveals information over time.

Bouvard (2012) studies a real options model. In his paper, information arrives via perfect bad news, the entrepreneur has private information about the project, and she commits to a contract which specifies the duration of experimentation together with performance-contingent payments. In Bouvard's paper, signaling occurs through excessively delaying investment and through cash-flow rights. Our paper features fundamentally different economic forces, which lead to different predictions. In our setting, the project cannot be delayed, because we do not have a real option. Instead, the project has a chance to succeed

[16] See Osborne and Rubinstein (1990)'s "Never Dissuaded Once Convinced" condition. As is well-known, Bayes' rule does not apply to such situations.

in each period as long as the entrepreneur and the investor do not liquidate. As we show, signaling though cash-flow rights is possible, but never optimal. Instead, signaling occurs via payouts, pivots, or prestige projects, which play no role in Bouvard's paper.

Similar to Bouvard (2012), a number of papers study signaling via the length of experimentation, i.e. Grenadier et al. (2014), Dong (2016), and Thomas (2019). In these papers, the experimenter has private information and can choose to continue experimenting for an excessive amount of time to signal his type. In our setting, the entrepreneur chooses both how long to experiment and which contract to offer to investors. In equilibrium, her private information is revealed through both choices, and we characterize how signaling shapes the optimal contract.

To render our analysis tractable, we borrow from the literature on relational contracts with adverse selection (i.e. Halac (2012), Fahn and Klein (2017), and Kartal (2018)). Just as these papers, we assume that the entrepreneur, who acts as the principal in our setting, does not have commitment and optimizes period-by-period. The key difference is that in our model, all parties learn about the project by observing whether a success arrived, so that the degree of adverse selection changes over time. By contrast, in Halac (2012), Fahn and Klein (2017), and Kartal (2018), there is no exogenous information about the principal's type and agents can learn only by observing the principal's choices. Indeed, the arrival of information is crucial for our results. Without it, the high type would separate either immediately or never and there would be no dynamics.

3 Model

Environment An entrepreneur (she) needs to contract with an investor (he) to start a project. The project is either good or bad. It requires costly experimentation by both the entrepreneur and the investor. When both experiment, the good project generates a single payoff V, which realizes with probability $\lambda \in (0,1)$ in each period $t \in \{1,2,...\}$. The bad project never generates a payoff.[17] Once either stops experimenting, the project is irreversibly liquidated.

The entrepreneur is privately, but imperfectly, informed about the quality of the project. We denote the entrepreneur's type with $\theta \in \{l,h\}$. A high type entrepreneur knows that the project is good with ex-ante probability p_1^h, while a low type knows that the project is good with probability p_1^l, where $0 < p_1^l < p_1^h < 1$. The ex-ante likelihood of a high type is

[17] Thus, we have an exponential bandit with good news, as in Bergemann and Hege (1998), Bergemann and Hege (2005), and Keller et al. (2005).

$q_0 \in (0,1)$.[18] Both parties are risk-neutral and have a common discount factor $\delta \in (0,1)$.

Contracts At the beginning of each period t, the entrepreneur chooses a liquidation probability $l_t^\theta \in [0,1]$. If she continues, the entrepreneur pays a cost $k > 0$ and offers the investor a contract $C_t^\theta = \left(d_t^\theta, \alpha_t^\theta\right)$. This contract consists of a payout[19] $d_t^\theta \geq 0$ and an equity share $\alpha_t^\theta \in [0,1]$. The equity share is contingent on the project succeeding, but d_t^θ is not contingent and paid immediately instead.[20] Given the contract, the investor chooses whether to continue $(e_t = 1)$ or whether to abandon the project $(e_t = 0)$.[21] Continuing has cost $c > 0$ for the investor. This cost can represent a cash investment which is made each period, the opportunity cost of already committed funds, or advising effort. If the project is liquidated, the entrepreneur and investor each receive an outside option of zero.[22]

Beliefs The high type entrepreneur, the low type entrepreneur, and the investor each have different beliefs about the likelihood that the project is good. Figure 1 shows how beliefs are updated.

Each entrepreneur type learns from the absence of successes. Type θ enters period t with belief p_t^θ. Without a success, she updates her belief to

$$p_{t+1}^\theta = \frac{(1-\lambda)\, p_t^\theta}{1-\lambda p_t^\theta}, \tag{1}$$

which is strictly decreasing over time. Equation (1) implies that $p_t^h > p_t^l$ for all t. That is, the high type believes her project is more likely to succeed at all times.

The investor learns from two sources. First, he may learn about entrepreneur's type from the contract offered. At the beginning of period t, he believes he is facing the high type with probability q_{t-1}. Upon observing the contract, he updates this belief to q_t. From the investor's perspective, the likelihood that the project is good is

$$p_t(q_t) = q_t p_t^h + (1-q_t)\, p_t^l. \tag{2}$$

[18] In the model, q_t is updated at the beginning of each period, while p_t is updated at the end. This is why our notation for the ex-ante probabilities, i.e., p_1^θ vs. q_0, differs.

[19] Or, equivalently, any costly action which does not affect the project value or likelihood of success.

[20] Once the equity is pledged, it is enforceable. Thus, the entrepreneur cannot renege once the project succeeds, unlike in e.g. Halac (2012).

[21] Thus, offering a contract for which the investor is not willing to experiment is the same as liquidating the project. We retain l_t^θ for notational clarity. In the relational contracting literature, from which we borrow, it is common to model separate acceptance and effort decisions, see e.g., Levin (2003). Because liquidating is irreversible, this is not necessary in our setting.

[22] The value of the outside option does not affect the qualitative properties of the contract. We set it to zero to simplify notation. We study the case when the entrepreneur can choose different projects, which have different outside options, in Section 7.2.

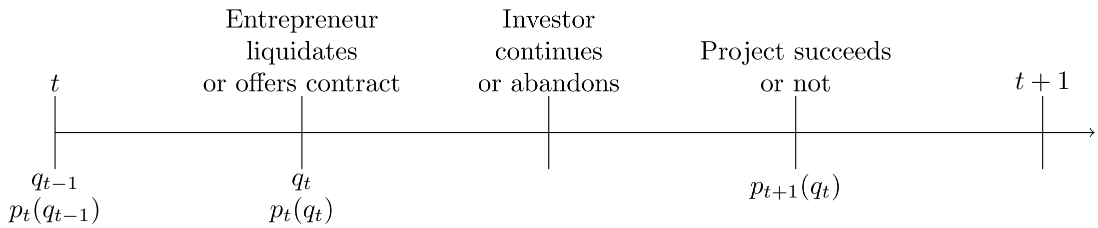


Figure 1: Timeline and Beliefs

Second, if the project does not succeed, he updates this belief to $p_{t+1}(q_t)$, using Bayes' rule in Equation (1).

Payoffs Denote with τ the period in which the project ends, either because it succeeds or because it is liquidated.[23] Denote with $\mathbf{1}_s$ the indicator function which is one if and only if the project succeeds in period s.

In period t, the payoffs for the type-θ entrepreneur and the investor are

$$\Pi_t^\theta = E_t^\theta \left[\sum_{s=t}^{\tau} \delta^{s-t} \left(\mathbf{1}_s (1-\alpha_s) V - k - d_s \right) \right] \tag{3}$$

and

$$U_t = E_t \left[\sum_{s=t}^{\tau} \delta^{s-t} \left(\mathbf{1}_s \alpha_s V - c + d_s \right) \right]. \tag{4}$$

Here, E_t^θ is the expectation given type θ's belief p_t^θ and E_t is the expectation given the investor's beliefs $p_t(q_t)$ and q_t.

For $t < \tau$, the entrepreneur's and investor's values can be written recursively as

$$\Pi_t^\theta = \left(1 - l_t^\theta\right) \left(\lambda p_t^\theta (1-\alpha_t) V - k - d_t^\theta + \delta \left(1 - \lambda p_t^\theta\right) \Pi_{t+1}^\theta \right) \tag{5}$$

and

$$U_t = (1 - l_t(q_{t-1})) \left(\lambda p_t(q_t) \alpha_t V - c + q_t d_t^h + (1-q_t) d_t^l + \delta (1 - \lambda p_t(q_t)) U_{t+1} \right), \tag{6}$$

where

$$1 - l_t(q_{t-1}) = q_{t-1} \left(1 - l_t^h\right) + (1 - q_{t-1}) \left(1 - l_t^l\right)$$

is the investor's expectation about the entrepreneur's liquidation probability.

[23] Since liquidation is irreversible, $e_t = 1$ for all $t < \tau$.

Equilibrium Concept We focus on Perfect Bayesian Equilibria. A Perfect Bayesian Equilibrium is a set of strategies and posterior beliefs, such that the strategies are sequentially rational at each history given the beliefs, and the beliefs are updated according to Bayes' rule whenever possible. Following Halac (2012), we require Bayesian updating both on and off the equilibrium path. Bayes' rule does not apply at histories at which the investor's belief about the entrepreneur is degenerate.[24] We again follow Halac (2012) and make the following assumption.[25]

Assumption 1 *If, at any history, the investor believes he is facing type θ with certainty, he will continue to believe so no matter which contract is offered.*

Throughout the paper, we refer to a Perfect Bayesian Equilibrium as *equilibrium*. We consider pooling and separating equilibria. In a pooling equilibrium, both types offer the same contract each period, but the low type may liquidate the project earlier and thereby reveal her type. In a separating equilibrium, types *separate in period t* if they pool until period $t-1$ and offer different contracts in period t. An equilibrium contract is *optimal* if it maximizes a weighted average of the low and high type's ex-ante values, where $\gamma \in [0,1]$ is the weight on the high type. We provide a formal equilibrium definition in Appendix A.

Parametric Assumptions To avoid uninteresting cases, we maintain the following assumptions throughout the paper.

Assumption 2 *In the first best, the good project is never liquidated, i.e.,*

$$\lambda V > k + c, \tag{7}$$

and in the pooling equilibrium, the low type does not immediately liquidate, i.e.,

$$\lambda p_1^l \left(1 - \frac{c}{\lambda p_1\left(q_0\right) V}\right) V > k. \tag{8}$$

Without Equation (7), the entrepreneur would immediately liquidate the project in any equilibrium. Without Equation (8), there may exist a pooling equilibrium in which either the low type or both types immediately liquidate the project. Then, the investor's belief evolution is trivial. He either learns nothing (if both liquidate) or immediately learns he is facing the high type (if only the low type liquidates).

[24] That is, $q_t \in \{0,1\}$. These histories arise after the high type successfully separates from the low type. See e.g. Section 5.2.

[25] This, or similar assumptions, are common in dynamic adverse selection models. See e.g. Osborne and Rubinstein (1990)'s "Never Dissuaded Once Convinced" condition.

4 Discussion

We now discuss how our modeling assumptions map to observed patterns in venture capital and how they relate to existing literature.

Venture Capital In our model, both the entrepreneur and investor exert effort. This modeling choice is consistent with a long line of literature on "double moral hazard" in venture capital (see Schmidt (2003), Casamatta (2003), Repullo and Suarez (2004), and Hellmann (2006)). It is also consistent with a substantial empirical literature, which documents that venture capital investors provide valuable services to entrepreneurs (see Sahlman (1990), Gorman and Sahlman (1989), Lerner (1995), and Hellmann and Puri (2000)). These services, which include providing advice, helping determine strategy, or helping recruit talent, are important for a firm's success (see e.g. Kortum and Lerner (2000) and Bernstein et al. (2016)). In addition to effort, we can interpret the investor's cost c as an opportunity cost of already committed funds or as investments under a given staging structure. We can interpret the investor's exit as the firm shutting down or being bought out, or as the founder being replaced (see Wasserman (2003)). We can interpret the arrival of success as an IPO.

Startup firms are subject to rapid changes and significant uncertainty, which often render contractual commitments moot (see Kaplan and Strömberg (2001), Kaplan and Strömberg (2003), Kaplan and Strömberg (2004) and Kerr et al. (2014)). Our modeling of contracts is consistent with this view. Instead of committing to a long-term contract at the beginning, the entrepreneur offers a sequence of contracts to the investor.[26] We can also understand the contract as being renegotiated each period (see Bengtsson and Sensoy (2015)).

Our model features binary outcomes (either success or no success). Thus, every payment contingent on success is equivalent to an equity share. While this is stylized, equity indeed makes up a majority of venture capitalists' compensation (see e.g. Kaplan and Strömberg (2004)).[27]

[26] Similar commitment issues underlie the literature on incomplete contracts and control rights (see Aghion and Tirole (1994), Aghion et al. (2004), and Dessein (2005)) and the literature on holdup problems (see Rajan (1992), Admati and Pfleiderer (1994), Burkart et al. (1997), Gompers and Lerner (2004), Azarmsa and Cong (2018), and Inderst and Vladimirov (2019)). The particular allocation of control rights is irrelevant in our model. If, as in the literature on hold-up, we assume that the investor makes all liquidation decisions, all results remain unchanged.

[27] Specifically, in their Table 1, the vast majority of financing is in the form of convertible preferred stock. This type of equity is equivalent to straight equity in our model. We keep the security design simple and instead focus on the role of learning and adverse selection. See Schmidt (2003), Casamatta (2003), and Hellmann (2006) for models in which convertible and straight equity differ.

Experimentation and Adverse Selection We assume that entrepreneur and investor learn about the firm over time. This is consistent with Kerr et al. (2014), who document that even conditional on investing, VCs face significant residual uncertainty, and with Ewens et al. (2018), who document that investors adjust their contracts as information becomes available. Our modeling of learning follows Bergemann and Hege (1998) and Bergemann and Hege (2005), who also assume that information arrives in the form of successes or their absence. Given the substantial skewness of returns in the venture capital industry, which features few startups with high profits and many startups with profits close to zero, this assumption is reasonable (see Scherer and Harhoff (2000) and Hall and Woodward (2010)).

An overwhelming part of the venture capital literature highlights the entrepreneur's information advantage as a source of frictions, going back at least to Gompers (1995). As investors learn about the firm, however, the information advantage disappears and the optimal contract changes (see Kaplan and Strömberg (2003) and Ewens et al. (2018) for evidence). This is exactly what happens in our model. As time passes, the projects of the good and bad type become indistinguishable. This evolution of the adverse selection friction is a key driver for our results.

Alternative Formulations In our model, the entrepreneur has private information, chooses the contract, and has all bargaining power. This is a common modeling choice (see e.g. Gale and Hellwig (1985), Innes (1990), and Bouvard (2012)), which can easily be relaxed. If, as in Axelson (2007), we assume that the investor has private information and chooses the contract, our entire analysis goes through, except that the roles of the entrepreneur's and investor's share are reversed.

Alternatively, we could interpret the model as contracting between a founder and an early employee, who is compensated by a significant equity stake. Such arrangements are common in startups (see Hand (2008)) and other industries (see Eisfeldt et al. (2018)). In this interpretation, the employee learns about the firm's prospects the longer he is employed and prefers to leave if the prospects become sufficiently unfavorable. None of our results change.

Recently, Robb and Robinson (2014) and Mann (2018) have documented that bank finance is a significant source of startup capital. Although we prefer the interpretation with equity and venture capitalists, our model is consistent with this alternative. If the investor is a bank, the cost c can represent monitoring effort. Since our outcome is binary, debt and equity contracts are equivalent. Thus, we can define a face value of debt F_t, which is to be repaid once the project succeeds, so that the entrepreneur's payout is $V - F_t$. This face value is renegotiated continuously, as in Hart and Moore (1998).

5 Analysis

We start with some notation. We continue denoting a generic payoff for type θ with Π_t^θ. We denote the payoff given belief q_t and contract C_t as $\Pi_t^\theta(q_t, C_t)$, irrespectively of whether this is on or off the equilibrium path. Finally, we denote with $\Pi_t^\theta(q_t)$ the *equilibrium* payoff given belief q_t. All proofs are in the Appendix.

5.1 Symmetric Information Benchmark

Suppose that the entrepreneur's type is public and that she offers a contract $\bar{C}_t^\theta = \left(\bar{d}_t^\theta, \bar{\alpha}_t^\theta\right)$. The investor's belief about the project is the same as the entrepreneur's, i.e., $p_t(q_t) = p_t^\theta$. The investor is willing to continue the project whenever

$$\lambda p_t^\theta \alpha_t^\theta V - c + \delta\left(1 - \lambda p_t^\theta\right) U_{t+1} \geq 0. \tag{9}$$

The left-hand side (LHS) is the investor's payoff from continuing,[28] which must exceed his outside option. The optimal contract leaves the investor indifferent between continuing or abandoning the project. That is, $U_t = 0$ for all t, the optimal share is

$$\bar{\alpha}_t^\theta = \frac{c}{\lambda p_t^\theta V}, \tag{10}$$

and $\bar{d}_t^\theta = 0$. Any other contract can be improved upon by the entrepreneur. If $U_t > 0$ for some t, lowering either d_t^θ or α_t^θ increases the entrepreneur's payoff without violating Equation (9).

The optimal equity share is increasing in time. As time passes without a success, the investor becomes more pessimistic about the project, i.e. p_t^θ decreases. Then, the entrepreneur must pledge a larger share to ensure that the investor continues. Moreover, the low type pledges a larger share than the high type, i.e. $\bar{\alpha}_t^l > \bar{\alpha}^h$, because the likelihood that the low type's project succeeds is lower.

The entrepreneur's payoff in period t is

$$\Pi_t^\theta = \left(1 - l_t^\theta\right)\left(\lambda p_t^\theta V - c - k + \delta\left(1 - \lambda p_t^\theta\right)\Pi_{t+1}^\theta\right). \tag{11}$$

When t becomes large, the entrepreneur's value becomes negative, because the project is unlikely to succeed. Then, she liquidates. Since the low type's project is less likely to succeed than the high type's, the low type liquidates earlier.

[28] Recall that at the time when the investor decides whether to continue, d_t^θ has already been paid and is therefore sunk.

Lemma 1 *The type-θ entrepreneur offers share*

$$\bar{\alpha}_t^\theta = \frac{c}{\lambda p_t^\theta V}$$

and liquidates the project whenever $\lambda p_t^\theta V - c - k \leq 0$. Let τ^θ be the period in which liquidation occurs under symmetric information. We have $\tau^l \leq \tau^h$ and $\bar{\alpha}_t^l > \bar{\alpha}_t^h$.

In the following, we denote the high and low type's symmetric information payoffs as $\Pi_t^h(1)$ and $\Pi_t^l(0)$.[29]

5.2 Cashless Entrepreneur

With private information, offering the symmetric information contracts is not incentive compatible. Since $\bar{\alpha}_t^h < \bar{\alpha}_t^l$, the low type prefers to imitate the high type, because then she can offer a lower equity share. This is the source of adverse selection in our model.

We first consider a cashless entrepreneur who cannot provide payouts to the investor unless the project succeeds. That is, we set $d_t^\theta = 0$ for all t and θ. Then, the optimal equity contract is pooling. Both types offer the same equity share, but the low type liquidates earlier than the high type and thereby reveals her type. The entrepreneur's share first decreases and then increases. This resembles dilution, i.e. as the project continues the entrepreneur's share becomes increasingly diluted, and vesting, i.e. after enough time has passed, the entrepreneur's shares vest and her stake in the firm increases. Both of these features are prominent in VC contracts (see Kaplan and Strömberg (2001)).

Proposition 2 *The following pooling contract is optimal. There exist two periods $\underline{\tau}^l \leq \bar{\tau}^l$, such that both types continue for $t < \underline{\tau}^l$. The low type liquidates with positive probability for $t \geq \underline{\tau}^l$ and liquidates with certainty in period $\bar{\tau}^l$. For all $t < \bar{\tau}^l$, both types offer equity share*

$$\alpha_t^P = \frac{c}{\lambda p_t(q_t) V}. \tag{12}$$

This share is increasing in time for $t < \underline{\tau}^l$ and decreasing for $t \geq \underline{\tau}^l$. After the low type liquidates, the high type offers $\bar{\alpha}_t^h$ and continues until period τ^h.

The low type knows that her project is less likely to succeed. Thus, when offering the same contract as the high type, her value from continuing is lower. After enough time without a success, the low type starts liquidating with positive probability, while the high

[29] That is, if $\bar{C}_t^\theta$ is the optimal symmetric information contract, then $\Pi_t^h(1) = \Pi_t^h(1, \bar{C}_t^h)$ and $\Pi_t^l(0) = \Pi_t^l(0, \bar{C}_t^l)$.

type continues. Thus, even though both types offer the same contract, the investor learns the entrepreneur's type over time. He updates his belief according to

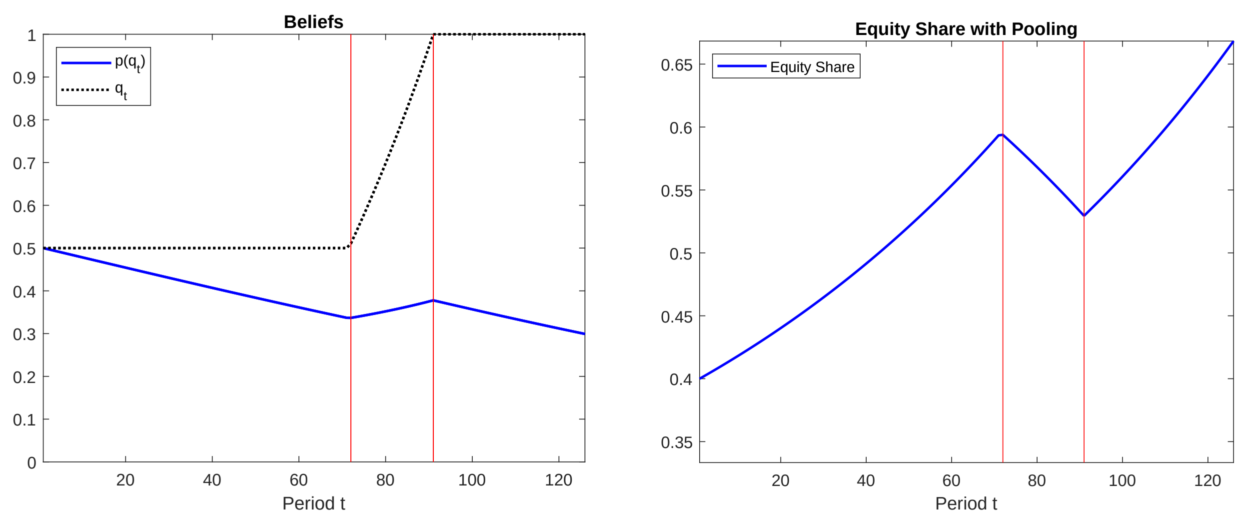
$$q_t = \frac{q_{t-1}}{q_{t-1} + (1 - q_{t-1}) \left(1 - l_t^l\right)}. \tag{13}$$

The belief is constant when the low type does not liquidate ($l_t^l = 0$) and increasing when she does ($l_t^l > 0$). Since the low type never liquidates before period $\underline{\tau}_l$, we have $q_t = q_0$ for any $t < \underline{\tau}^l$.

The investor continues the project whenever

$$\lambda p_t (q_t) \alpha_t^P V - c + \delta (1 - \lambda p_t (q_t)) U_{t+1} \geq 0. \tag{IC_I}$$

The optimal equity share (in Equation (12)) leaves the investor indifferent between continuing and abandoning the project. Any higher share is suboptimal, because both types can lower the share until the investor's incentive compatibility (IC) condition (IC_I) binds.



(a) Pooling Equilibrium: Investor's Beliefs

(b) Pooling Equilibrium: Investor's Equity Share

Figure 2: On each panel, the left vertical line indicates $\underline{\tau}^l$ and the right vertical line indicates $\bar{\tau}^l$. Before $\underline{\tau}^l$, the investor does not update his beliefs about the entrepreneur's type, because neither type liquidates. Between $\underline{\tau}^l$ and $\bar{\tau}^l$, the low type liquidates with positive probability, so that q_t increases. This causes $p_t(q_t)$, the investor's belief about the project, to increase (left panel). Between $\underline{\tau}^l$ and $\bar{\tau}^l$, the optimal pooling share decreases (right panel), because the investor becomes more optimistic about the project, which resembles a vesting schedule for the entrepreneur.

The optimal equity share is increasing in time when the low type does not liquidate, before period $\underline{\tau}^l$, because the investor's belief $p_t(q_0)$ is decreasing.[30] To keep him indifferent,

[30] Recall that both p_t^h and p_t^l decrease without a success. Keeping q_t at q_0, $p_t(q_t)$ decreases as well.

his share must increase. Starting from period $\underline{\tau}^l$, the low type liquidates with positive probability and the equity share is decreasing. Intuitively, the low type must be indifferent between liquidating and continuing, i.e. $\Pi_t^l = \Pi_{t+1}^l = 0$, and Equation (5) reduces to

$$\lambda p_t^l \left(1 - \alpha_t^P\right) V = k.$$

The belief p_t^l decreases over time, so the equity share must decrease to keep the low type indifferent. The equilibrium liquidation probability l_t^l, together with Bayes rule in Equation (13), ensure that the investor continues the project in any such period.[31] When the low type liquidates, q_t increases, and thus the investor is willing to continue despite receiving a lower share.

In period $\bar{\tau}^l$, the low type liquidates with certainty and the investor learns whether he is facing the high type. The high type then offers the symmetric information contract $\bar{\alpha}_t^h$. Figure 2 illustrates the dynamics of the investor's equity share and beliefs.[32]

In addition to pooling equilibria, there exist equilibria in which the high type separates by offering an inefficiently large equity share. However, as we show next, all separating equilibria are suboptimal.

Proposition 3 *For any $t < \bar{\tau}^l$, there exists an equilibrium in which the high type separates in period t by offering a share $\alpha_t^h > \alpha_t^P$. Any such equilibrium is suboptimal.*

In a separating equilibrium, the following IC conditions must hold. The low type prefers to reveal her type instead of offering the high type's equity share, i.e.

$$\Pi_t^l(0) \geq \lambda p_t^l \left(1 - \alpha_t^h\right) V - k + \delta \left(1 - \lambda p_t^l\right) \Pi_{t+1}^l (1). \tag{IC_l}$$

Similarly, the high type prefers to offer α_t^h, so that

$$\Pi_t^h(0) \leq \lambda p_t^h \left(1 - \alpha_t^h\right) V - k + \delta \left(1 - \lambda p_t^h\right) \Pi_{t+1}^h (1). \tag{IC_h}$$

The continuation values following separation are the symmetric information values $\Pi_{t+1}^h (1)$ and $\Pi_t^l (0)$. If the low type imitates the high type, she optimally offers $\bar{\alpha}_{t+1}^h$ the next period,

[31] That is, we have for $\underline{\tau}^l \leq t < \bar{\tau}^l$,

$$\alpha_t^P = \frac{\lambda p_t^l V - k}{\lambda p_t^l V} = \frac{c}{\lambda p_t (q_t)} V.$$

[32] The investor's share is α_t, which is depicted in Figure 2, and the entrepreneur's share is $1 - \alpha_t$.

and we denote her value with $\Pi_{t+1}^{l}(1)$, while if the high type imitates the low type, she optimally offers $\bar{\alpha}_t^l$ and receives $\Pi_t^h(0)$.[33]

Intuitively, pledging a large share dissuades the low type from imitating, because she has to give up a larger portion of the project's value if it succeeds. Then, the low type prefers to be discovered and to continue offering $\bar{\alpha}_t^l$. The high and low type's values satisfy a variant of single crossing in any period $t < \bar{\tau}^l$,

$$\frac{\Pi_t^h(1) - \Pi_t^h(0)}{p_t^h} \geq \frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l}. \tag{14}$$

That is, the value of being perceived as the high type is larger for the high type than for the low type.[34] Because single crossing holds, the high type can separate in any period.

However, separating via a higher share, is relatively costly. The high type's project is more likely to succeed, and thus she is more likely to pay the investor. If she increases the share by ε, she reduces the low type's value from imitating by $\lambda p_t^l V\varepsilon$ and her own value by $\lambda p_t^h V\varepsilon$. Thus, to reduce the low type's value by one, the high type has to give up value $p_t^h/p_t^l > 1$. Because of this, the high type's cost of deterring the low type exceeds her benefit from separating. Then, the high type prefers to pool in all periods.

5.3 Equilibrium

We now consider the case with payouts and show that early payouts can be used to signal. For tractability, we impose the following parametric assumption.

Assumption 3 *We have* $\left(1 - \lambda p_1^h\right)\left(1 - \lambda p_1^h\right) > 1 - \lambda$.

Assumption 3 implies that the degree of adverse selection, as measured by the difference in the high and low type's beliefs, $p_t^h - p_t^l$, is decreasing over time.[35] It holds whenever the initial probabilities p_1^l and p_1^h are sufficiently small.

[33] This is because of Assumption 1. After types separate, the investor never changes his belief, and he will accept any contract which promises a share of at least $\bar{\alpha}_t^h$ (if $q = 1$) or $\bar{\alpha}_t^l$ (if $q = 0$). The optimal contract for type θ is then the symmetric information contract of Section 5.1.

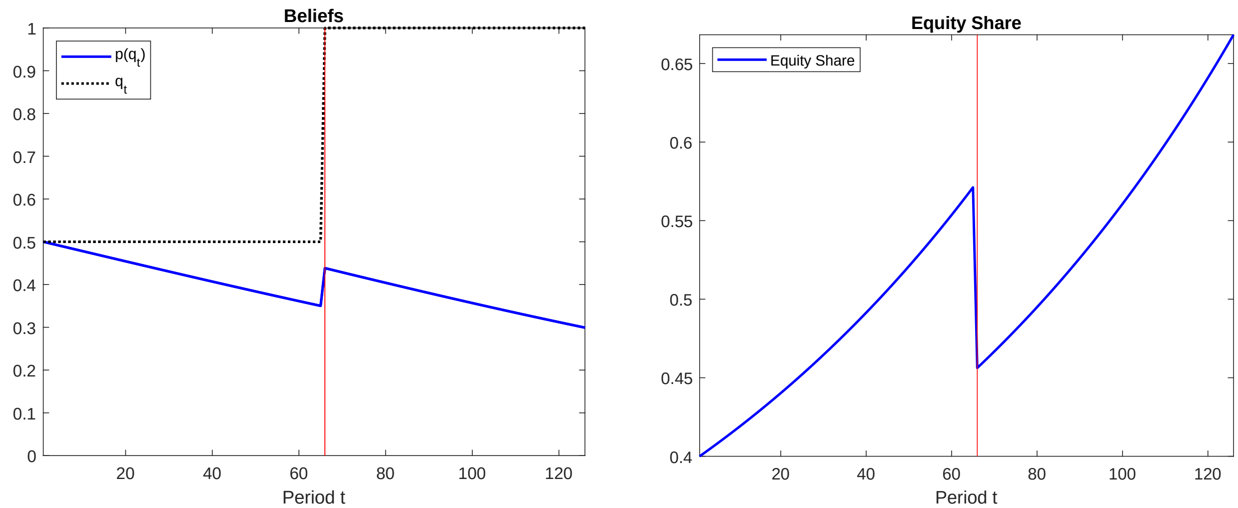
[34] Intuitively, if type θ is being perceived as the high type in a future period $s > t$, she can offer a share $\bar{\alpha}_s^h$, while when she is perceived as the low type, she offers share $\bar{\alpha}_s^l$. Type θ's value of being perceived as the high type is thus $\lambda p_s^\theta(\bar{\alpha}_s^l - \bar{\alpha}_s^h)$. This value is larger for the high type, whose project is more likely to succeed. After discounting, this leads to Equation (14).

[35] This is because

$$p_{t+1}^h - p_{t+1}^l = \frac{1-\lambda}{\left(1 - \lambda p_t^h\right)\left(1 - \lambda p_t^h\right)} \left(p_t^h - p_t^l\right).$$

Since $p_t^\theta < p_1^\theta$, it is sufficient to show this condition holds at $t = 1$. We use Assumption 3 to establish a single-crossing condition in Lemma 26. Note that $p_t^h - p_t^l$ is decreasing at time t whenever p_t^h and p_t^l are sufficiently small. Thus, even without the assumption, $p_t^h - p_t^l$ is decreasing when t is sufficiently large. Assumption 3 merely ensures that this is true for all t.

In the optimal contract, the high type separates by offering a payout in period τ_S, but separation is inefficiently delayed. Intuitively, the high type prefers to wait until the degree of adverse selection has decreased, since separating earlier is too costly.



(a) Separating Equilibrium: Investor's Beliefs

(b) Separating Equilibrium: Investor's Equity Share

Figure 3: The vertical line indicates the time at which the high type separates. Before, the investor's belief about the project is decreasing. Once the investor learns that he is facing the high type, the beliefs q_t and $p_t(q_t)$ both jump upwards (left panel). Before separation, the high type successively pledges higher shares to the investor. Once the high type separates, the investor's share drops and the entrepreneur's share increases (right panel).

Proposition 4 *If p_1^l/p_1^h and q_0 are sufficiently small, and if γ is sufficiently large, the optimal contract is separating in period τ_S. It consists of a payment $d_{\tau_S}^h$ and the high type's symmetric information share $\bar{\alpha}_{\tau_S}^h$. Before period τ_S, both types offer the pooling contract of Proposition 2. If either p_1^l/p_1^h or q_0 are large, or if γ is small, pooling is optimal.*

With payouts, the pooling equilibrium of Proposition 2 still exists. Moreover, it is optimal among all pooling equilibria.[36] Suppose that the high type separates in period t by offering a contract $C_t^h = \left(d_t^h, \alpha_t^h\right)$. The relevant IC conditions are

$$\Pi_t^l(0) \geq \lambda p_t^l \left(1 - \alpha_t^h\right) V - d_t^h - k + \delta \left(1 - \lambda p_t^l\right) \Pi_{t+1}^l (1) \qquad (\tilde{IC}_l)$$

for the low type and

$$\Pi_t^h(0) \leq \lambda p_t^h \left(1 - \alpha_t^h\right) V - d_t^h - k + \delta \left(1 - \lambda p_t^h\right) \Pi_{t+1}^h (1) \qquad (\tilde{IC}_h)$$

[36] Intuitively, any pooling equilibrium in which there are positive payouts $d_t^P > 0$ leaves rents to the investor and can be improved upon by setting the payouts to zero.

for the high type. The first condition states that the low type's value from offering C_t^h and imitating the high type must be lower than the value from revealing her type. Specifically, once her type is discovered, the low type offers her symmetric information contract $\bar{C}_t^l = (0, \bar{\alpha}_t^l)$.[37] The second condition states that offering C_t^h must indeed be optimal for the high type. As before, once separated, the continuation values are the symmetric information values $\Pi_{t+1}^h(1)$ and $\Pi_t^l(0)$.

Separating via the equity share is costly for the high type. As we described in Section 5.2, to reduce the low type's payoff by one, the high type gives up a payoff of $p_t^h/p_t^l > 1$. By contrast, if she separates via the payout d_t^h, her cost of reducing the low type's payoff is one. Thus, separating via a payout is cheaper and the equity share is not distorted, i.e. $\alpha_t^h = \bar{\alpha}_t^h$. The low type's IC constraint binds in an optimal contract and the payout reduces to[38]

$$d_t^h = \Pi_t^l(1) - \Pi_t^l(0). \tag{15}$$

That is, the payment equals the value of imitating for the low type, which is given by the difference in continuation values at beliefs $q_t = 1$ and $q_t = 0$.

Because the cost of separating is smaller, the high type may prefer to separate rather than pooling forever. This is true when pooling is relatively costly, i.e. when p_1^l is small relative to p_1^h, and when q_0 is small. Intuitively, when the investor believes he is unlikely to be facing the high type, i.e. q_0 is small, the pooling equity share α_t^P is large. Then, the high type must give up a large portion of the project when she continues pooling. Similarly, whenever the low type's project is unlikely to succeed, i.e. p_1^l is small relative to p_1^h, the pooling share is relatively large. In both cases, pooling is relatively costly for the high type.

In equilibrium, separation is inefficiently delayed. The high type prefers to separate whenever the loss from pooling exceeds the cost of separating d_t^h. As time passes, the pooling equilibrium becomes progressively worse, and, compared to separating, the high type must pledge successively larger shares.[39] Simultaneously, the cost of separating d_t^h decreases, because the low type's project becomes less likely to succeed. After sufficient time has passed, the high type prefers to separate.

The low type, by contrast, always prefers to pool until the project is liquidated. Whenever γ, the weight on the high type's payoff, is small, the optimal contract is pooling, while when the weight is large, it is separating. Figure 3 illustrates the dynamics of the equity share

[37]This is straightforward. If the low type were to offer any other contract with $\alpha_t^l > \bar{\alpha}_t^l$ or $d_t^l > 0$ which reveals her type, that contract would be suboptimal.

[38]This follows from plugging $\alpha_t^h = \bar{\alpha}_t^h$ into Equation ($\tilde{IC}_l$).

[39]The ratio $\alpha_t^P/\bar{\alpha}_t^h$ is monotonically increasing, which implies that the high type's "adverse selection discount" becomes progressively worse.

and the investor's beliefs when separation is optimal.

Interpretation When the high type separates, she pays the investor, and simultaneously reduces the investor's equity share and increases her own. This resembles performance-contingent vesting: in period τ_S, the entrepreneur takes a costly action and is rewarded with immediate vesting of shares. As Kaplan and Strömberg (2004) document, such contingent vesting schemes are common in VC contracts. Alternatively, we can interpret this early payout as a buyback. That is, the entrepreneur pays the investor to repurchase a fraction of his shares, so that her own share increases and the investor's share declines. In reality, such buybacks occur most often among later stage startups, as is the case in our model.[40]

6 Applied Results

Our results explain important features of venture capital contracts. First, the entrepreneur's share decreases over time, which is consistent with e.g. Kaplan and Strömberg (2003) and Kaplan and Strömberg (2004). In our model, the investor becomes more pessimistic as time passes and the entrepreneur must pledge successively larger shares to prevent the investor from leaving. Second, the optimal contract features vesting. After sufficient time has passed, the investor expects the low type to liquidate, and he becomes more optimistic over time even without observing a breakthrough. Then, the entrepreneur's share increases, which can be implemented by a delayed vesting schedule. Finally, the separating equilibrium of Proposition 4 can be implemented via a milestone with a contingent vesting clause. In period τ_S, the entrepreneur takes a costly action, and her share in the company increases. As Kaplan and Strömberg (2004) document, contingent vesting provisions are prominent in VC contracts.

Technological advances in recent decades have dramatically changed how VCs finance startups (see Kerr et al. (2014) and Ewens et al. (2018)). Three aspects are particularly relevant to our model. First, the cost of experimentation has declined for both entrepreneurs and investors. For example, could computing services have lowered the cost of operating IT startups by orders of magnitude. As a result, venture capital firms have adopted a "spray-and-pray" approach, now funding a large number of startups with limited vetting and oversight. Second, cohort-based accelerators (such as Y-Combinator) have increased entry by relatively inexperienced founders. Third, existing advances have made follow-up innovations easier, so that learning about startups has sped up. We now investigate how

[40] See e.g. https://www.wsj.com/articles/ditch-the-venture-model-say-founders-who-buy-out-early-investors-to-make-a-clear-break-1531827001 for a prominent case.

these aspects affect adverse selection in our model.

Cost of Experimenting When the entrepreneur's cost of experimenting k decreases, both the low and high type are willing to continue longer. This increases the information rent that the high type must give up to dissuade the low type from imitating. As a result, separation becomes costlier and the high type separates later. Thus, adverse selection persists longer. We illustrate this result in Figure 4a, which is obtained by solving the model numerically.

To evaluate how VCs "spray-and-pray" approach affects adverse selection, consider a reduction in the investor's cost to experiment c. For example, the investor may exert less effort in advising the entrepreneur, making experimentation less costly for him. Again, τ_S increases and adverse selection persists longer. Intuitively, as c decreases, the optimal equity share decreases as well. This is true both when the two types pool (see Equation (12)) and when they separate (see Equation (10)). This decrease, in turn, makes both types of entrepreneur more willing to continue. Just as in the previous case, separation becomes costlier for the high type. Figure 4b illustrates the result.

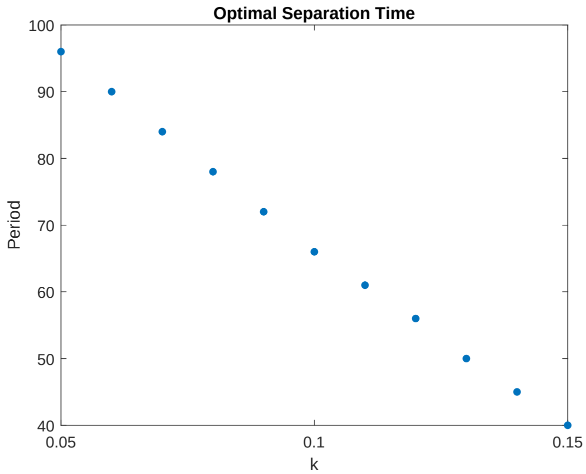
Entrepreneur Quality Accelerators have allowed relatively inexperienced founders to start firms and to receive funding. In our model, this corresponds to a decrease in q_0, the ex-ante quality of the entrepreneur. One might expect that funding lower quality startups leads to more adverse selection. However, the effect is more subtle: as q_0 decreases, the high type separates earlier. Intuitively, as q_0 decreases, the high type's "adverse selection discount," which she suffers in the pooling equilibrium, becomes worse. The cost of separating from the low type, however, is independent of q_0. Thus, the high type separates earlier. We illustrate this result in Figure 4c.

Faster Learning As technology increases λ, the likelihood of breakthroughs, learning about the entrepreneur's type speeds up. However, the high type does not necessarily separate earlier. The effect is subtle, because the cost of separating d_t (in Equation (15)) is non-monotone in λ. The low type's expected value from imitating the high type, $\lambda p_t^l \left(\bar{\alpha}_t^l - \bar{\alpha}_t^h\right) V$, increases in both λ and t. As λ increases, this expression increases for any fixed t,[41] which increases the cost of separation. However, as λ increases, breakthroughs arrive earlier on average, which lowers the low type's expected value of imitating and therefore the cost of

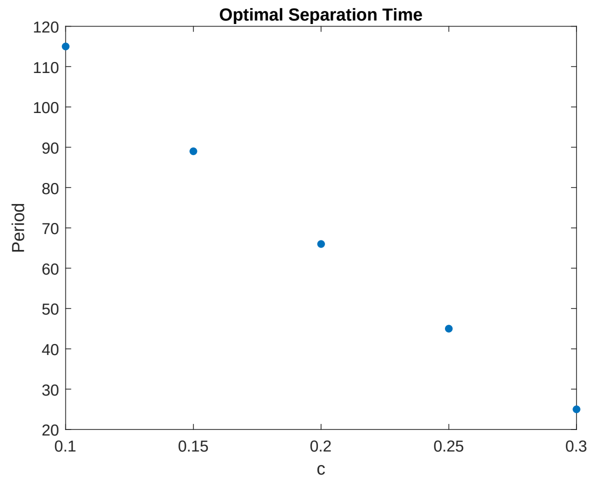
[41] Specifically, we have

$$\lambda p_t^l \left(\bar{\alpha}_t^l - \bar{\alpha}_t^h\right) V = c\left(1 - \frac{p_t^l}{p_t^h}\right).$$

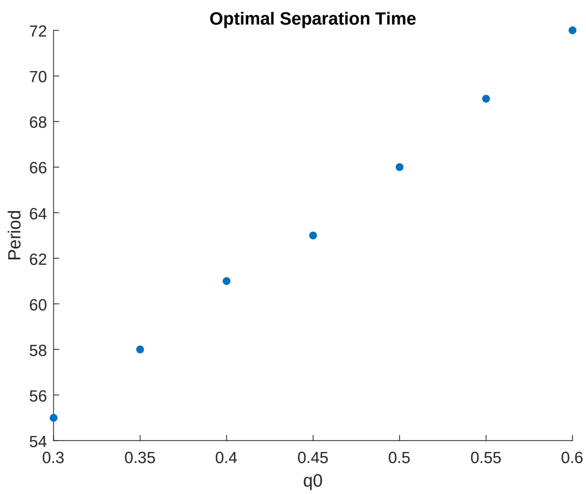
As λ increases, p_t^l / p_t^h decreases for any fixed t, because learning speeds up.



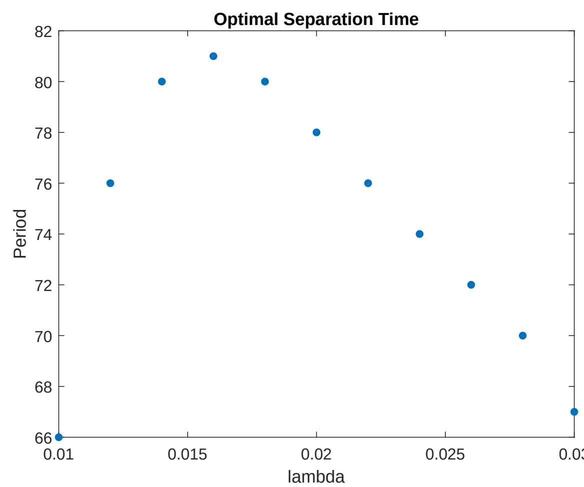
(a) As the cost of experimenting k decreases, the high type separates later from the low type.



(b) As experimentation becomes cheaper for the investor, the high type separates later.



(c) As the expected quality of the entrepreneur decreases, the high type separates earlier.



(d) As the likelihood of breakthrough increases, the high type does not necessarily sparate earlier.

Figure 4: Comparative Statics

separating. Because of these conflicting effects, the cost of separating may increase or decrease in as λ increases. When the cost increases, the high type prefers to separate later, to reduce the cost of doing so. Figure 4d shows that the optimal separation time is indeed non-monotone in λ.

Liquidation Rights To protect investors from the entrepreneur's information advantage, many VC contracts grant them favorable liquidation rights, which allow them to capture a larger part of the firm's value in bankruptcy (see Kaplan and Strömberg (2003)). However, liquidation rights come with a downside because they delay separation.

Suppose that the firm has value $V_L < V$ if it is liquidated. Let $\alpha_L \in [0,1]$ index the investor's liquidation rights, so that he captures value $\alpha_L V_L$, while the entrepreneur captures the remainder $(1-\alpha_L)\, V_L$. Stronger liquidation rights make the investor less willing to continue, because the outside option of forcing liquidation becomes more appealing. To ensure that the investor continues, the entrepreneur must pledge a larger share, which is now given by

$$\alpha_t^P = \frac{c + \alpha_L V_L}{\lambda p_t\,(q_t)}$$

in the pooling contract and

$$\bar{\alpha}_t^\theta = \frac{c + \alpha_L V_L}{\lambda p_t^\theta}$$

once her type is revealed. This makes the low type less willing to continue, which should make separation easier to achieve. At the same time, however, increasing the investor's liquidation rights decreases the entrepreneur's value of liquidating the project, which is given by $(1-\alpha_L)\, V_L$. Intuitively, as α_L increases, low type's value from imitating the high type increases, because she expects to continue the project longer. As Figure 5 shows, the second effect dominates. Then, separation becomes more costly as α_L increases and the high type separates later. This result is broadly consistent with Ewens et al. (2019), who estimate a matching model between VCs and entrepreneurs and find that the investor's liquidation preference reduces firm value.

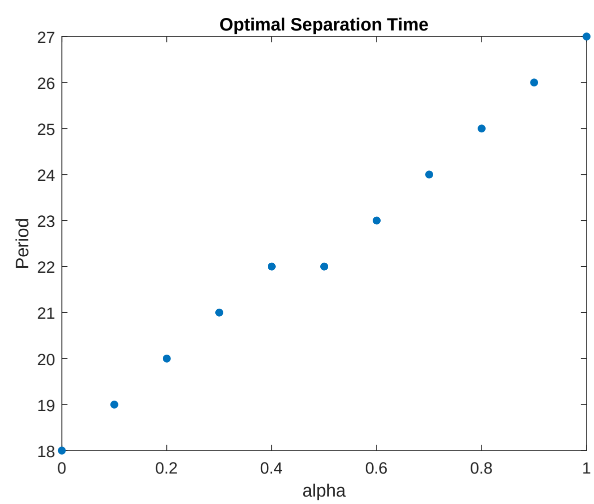


Figure 5: As the investor's liquidation preference increases, the high type separates later.

Overall, our comparative statics suggest that common wisdom about technology and contractual provisions may be misleading. Recent technological progress, which has speed up learning and reduced the cost of experimentation, may actually have worsened adverse selection between founders and VCs, by delaying the time at which low types are revealed. Liquidation preferences for investors may have a similar downside. Suitably modified, all the results in this section carry over to the pooling equilibrium of Section 5.2, when the entrepreneur has no cash to pay the investor before a breakthrough.[42] They do not rely on the high type being able to separate.

7 Signaling with Strategy

7.1 Pivots

Pivots are common among startups.[43] When entrepreneurs realize that their project is not succeeding, they may abandon it and pivot to a new one. In this section, we show that pivots can also signal information. For the sake of clarity, we focus on pivots as the only separating device. Thus, we assume that the entrepreneur cannot pay investors, as in Section 5.2.

The entrepreneur now has a real option to pivot. That is, she can abandon the current project and start a new one at fixed cost $F > 0$, which is visible to investors. The entrepreneur's type is fixed across projects, e.g. because it represents entrepreneurial ability.[44] Thus, the new project of type θ has likelihood p_1^θ of being good. Since the high type's project is more likely to be successful, i.e. $p_1^h > p_1^l$, her value from pivoting is higher than the low type's. Thus, the fixed cost may dissuade the low type from pivoting, but not the high type, which allows the high type to separate.

In a separating equilibrium, the following IC conditions must hold. First, the low type prefers not to imitate the high type by pivoting as well, i.e.

$$\Pi_1^l(1) - F \leq \Pi_t^l(0). \tag{IC_l^{Piv}}$$

The LHS is the low type's value from pivoting. She starts a new project, so that her belief is p_1^l, and the investor believes he is facing the high type, i.e. $q = 1$. When the low type does not pivot, she continues her initial project, which has a likelihood of p_t^l of being good,

[42] We omit describing them in detail to save space. A description and accompanying figures are available from the authors upon request.

[43] See the examples in the introduction.

[44] This assumption is consistent with reality. Many VC firms prioritize the quality of the founding team in their financing decisions over the particular business idea. They anticipate that the founders may change direction, but believe that the founders' quality is most important for the eventual outcomes. See Gompers et al. (2019), who find that "the management team is the most important factor VCs consider in choosing portfolio company investments" and that "ability is the most mentioned factor." See also the broader discussion in Gompers and Lerner (2001) and Kaplan et al. (2009).

but the investor knows he is facing the low type.

The high type's IC constraint is similarly given by

$$\Pi_1^h(1) - F \geq \Pi_t^h(0), \qquad (IC_h^{Piv})$$

i.e. the high type prefers to pivot rather than continuing her initial project and being perceived as the low type.

When the fixed cost is too low, the low type imitates, while when it is too high, pivoting is too costly for the high type. Overall, separating via a pivot is feasible whenever

$$F \in \left[\Pi_1^l(1) - \Pi_t^l(0), \Pi_1^h(1) - \Pi_t^h(0)\right]. \tag{16}$$

The interval on the RHS is non-empty for all t, because the high type's value from pivoting is higher.[45] To rule out uninteresting cases and to ensure tractability, make the following assumptions, in addition to Assumptions 1-3.

Assumption 4 *Both types pivot rather than liquidate, i.e.*

$$\Pi_1^l(0) - F \geq 0, \tag{17}$$

and separation is feasible, i.e.

$$F \geq \Pi_1^l(1) - \Pi_1^l(0). \tag{18}$$

Equation (17) is not crucial, but allows us to reduce the number of cases we have to consider.[46] Without Equation (18), the low type always imitates the high type by pivoting at the same time, so that separating is not feasible.[47]

Proposition 5 *Suppose that δ and F are sufficiently small and that γ is sufficiently large. Then, the optimal contract features pooling in all periods $t < \tau_S$, and in period τ_S, the high type separates via a pivot.*

[45] This is another variant of single crossing. The value of pivoting vs. continuing and being perceived as the low type is higher for the high type, i.e.,

$$\Pi_1^l(1) - \Pi_t^l(0) \leq \Pi_1^h(1) - \Pi_t^h(0).$$

[46] Without the assumption, the high and/or the low type may prefer to liquidate rather than to pivot in the first best, which yields different value functions for the equilibrium.

[47] For separating to be feasible, there also has to exist a period t in which $F \leq \Pi_1^h(1) - \Pi_t^h(0)$. Since $\Pi_t^h(0)$ is decreasing in t and eventually reaches zero, a sufficient condition is $F \leq \Pi_1^h(1)$. This condition holds because of Equation (17), since we have $\Pi_1^h(1) \geq \Pi_1^l(1) \geq \Pi_1^l(0)$.

Early on, pivoting is not incentive compatible for the high type, because the value $\Pi_t^h(0)$ is relatively large. Intuitively, pivoting is not worth it when the high type is still optimistic about the project. As time passes, $\Pi_t^h(0)$ decreases and the high type eventually prefers to pivot. As long as t is not too large, the low type's IC condition holds as well, because her value from pivoting is lower.[48] As more time passes, however, the low type prefers to pay the fixed cost and imitate the high type, because her value from separating, $\Pi_t^l(0)$, becomes too small. As long as F is sufficiently small, the high type prefers to pivot before this happens.[49] Then, separating is optimal whenever γ, the weight on the high type's value, is sufficiently large.[50]

7.2 Prestige Projects

Many early stage firms divert resources towards prestige projects in order to generate publicity or goodwill. These prestige projects range from helping deliver surplus food to nonprofits, to attempting to colonize Mars.[51] As we show next, such prestige projects can act as signaling devices.

The entrepreneur can now implement a publicly observable prestige project in each period. Doing so reduces the payoff of the original project and generates a higher outside option for the entrepreneur. Specifically, when implementing the prestige project, a breakthrough yields value $V - V_0$ and the outside option is $\pi > 0$. For example, generating prestige makes it more likely that the entrepreneur can fund another startup[52] or obtain outside employment, but to do so the entrepreneur has to divert resources from her main project. The entrepreneur decides whether to implement the prestige project at the same time she offers the contract to the investor.[53] For the sake of clarity, we assume that the prestige project is the only signaling device and that the entrepreneur cannot pay the investor, as in Section 5.2.

The high type can use the prestige project to separate. After enough time has passed,

[48] We can see this from the IC conditions (IC_l^{Piv}) and (IC_h^{Piv}), which satisfy a single crossing type condition.

[49] Importantly, it is possible to pick F sufficiently small without violating Equation (18).

[50] The assumption that δ is small allows us to simplify the derivations.

[51] DoorDash, a food delivery platform, has started donating drivers' time to deliver surplus food from restaurants to nonprofits (see `https://thespoon.tech/with-project-dash-doordash-uses-logistics-to-rescue-over-1-million-pounds-of-surplus-food/`, last accessed 10/13/19). SpaceX, an aerospace manufacturer, has a stated, if lofty, goal to colonize Mars with one million inhabitants (see `https://www.businessinsider.com/elon-musk-mars-iac-2017-transcript-slides-2017-10`, last accessed 10/13/19).

[52] This is broadly consistent with Gompers et al. (2010), who find that an entrepreneur's past success is an important factor for VC financing decisions.

[53] The particular timing is irrelevant, as long as the decision to implement the prestige project occurs before the investor's continuation decision.

the low type is pessimistic about the likelihood of success and her value from continuing to experiment is small. Then, she liquidates immediately upon implementing the prestige project, since taking the higher outside option is more valuable than continuing. Because of this, she cannot mimic the high type. Since the prestige project has a lower value, separating is costly and can be suboptimal. Intuitively, if both V_0 and π are very large, both types prefer to liquidate instead of continuing with the prestige project. As we show in the proposition below, for certain parameter values, separating via a prestige project is optimal.[54]

Proposition 6 *Suppose that γ is sufficiently large. Then, there exists a pair (π, V_0) such that the optimal contract features pooling in all periods $t < \tau_S$, and in period τ_S, the high type separates by implementing a prestige project and the low type liquidates.*

Formally, the following IC constraints hold when the high type separates at time t. First, the low type prefers not to implement the prestige project, i.e.,

$$\Pi_t^l(1) - \lambda p_t^l V_0 \leq \Pi_t^l(0). \tag{IC_l^{Pres}}$$

The LHS is the low type's value from imitating the high type, which consists of her continuation value $\Pi_t^l(1)$ and the loss in value if the project succeeds, while the RHS is the low type's value from separating. The high type's IC constraint is similarly given by

$$\Pi_t^h(1) - \lambda p_t^h V_0 \geq \Pi_t^h(0). \tag{IC_h^{Pres}}$$

Separating is feasible whenever

$$\frac{\Pi_t^h(1) - \Pi_t^h(0)}{p_t^h} \geq \lambda V_0 \geq \frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l}. \tag{19}$$

A variant of the single-crossing condition in Equation (14) holds in this extension. Thus, the interval in Equation (19) is nonempty, and for each period, there exists a V_0 that separates types. When the low type liquidates upon implementing the prestige project, her IC constraint becomes

$$\Pi_t^l(1) - \lambda p_t^l V_0 \leq \pi.$$

Whenever the outside option π is sufficiently close to $\Pi_t^l(1)$, separating can be achieved relatively cheaply. That is, the loss in value V_0 which induces the low type to separate is relatively small. This shows, intuitively, that there exists a pair (π, V_0) which makes

[54] Broadly, V_0 cannot be too large, because then the high type never prefers to separate, but it also cannot be too small, because otherwise the low type is never dissuaded from mimicking. The range of feasible V_0 is affected by the outside option π. It is larger whenever π is smaller.

separation optimal for the high type. As in the main model, the low type prefers to never separate. Thus, separating is ex-ante optimal whenever γ, the weight on the high type's value, is sufficiently large.

8 Conclusion

[TBD]

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A Equilibrium Definition

Since the game ends as soon as $e_t = 0$, the only relevant history is the one in which the investor has chosen $e_s = 1$ for all $s < t$. We thus do not need to keep track of e_t. Let $\mathcal{C} = [0, \infty) \times [0, 1]$ be the space of possible contracts in any given period, where each contract is given by a pair $C_t = (d_t, \alpha_t)$.

A history for entrepreneur type θ in period t is given by $h^{\theta t} = \{\theta, C_1, C_2, ..., C_{t-1}\} \in H^{\theta t}$. A history for the investor is given by $h^t = \{C_1, C_2, ..., C_{t-1}, C_t\} \in H^t$. A strategy for a type θ entrepreneur is pair $\sigma_t^\theta = \left(g_t^\theta \left(h^t, C\right), l_t^\theta \left(h^t\right)\right)$, where $g_t^\theta : H^{\theta t} \to \Delta\left(\mathcal{C}\right)$ is the probability she offers contract C given history h^t and $l_t^\theta : H^{\theta t} \to [0, 1]$ is the probability she liquidates the project. A strategy for the investor is $\sigma_t : H^t \to [0, 1]$, which maps $\left(h^t, C_t\right)$ into a distribution over $e_t \in \{0, 1\}$.

A Perfect Bayesian Equilibrium consists of strategies σ^θ and σ, and beliefs p_t^θ, $p_t\left(q_t\right)$, and q_t, such that for all $t \leq \tau$, σ_t^h, σ_t^l, and σ_t are sequentially rational at all histories and beliefs satisfy Bayes' rule whenever possible.

This definition is consistent with the following extensive form stage game in each period t:

- Stage 1: The entrepreneur chooses l_t^θ.
- Stage 2: If the entrepreneur has not liquidated, she chooses C_t conditional on $h^{\theta t}$.
- Stage 3: The investor observes C_t (and the fact that the game has not ended yet) and chooses e_t conditional on h^t (which includes C_t).
- Stage 4: The project succeeds or not. If not, the game proceeds to period $t+1$.

B Proofs

B.1 Proof of Lemma 1

To establish that liquidation is optimal whenever $\lambda p_t^\theta V - c - k \leq 0$, suppose by way of contradiction that the period payoff is strictly positive when the entrepreneur liquidates. Then $\bar{\Pi}_t^\theta > 0$, because the entrepreneur always has the option of liquidating in the next period, so that $\bar{\Pi}_{t+1}^\theta \geq 0$. Thus, liquidating in period t cannot be optimal. Conversely, if the period payoff is non-positive, it will be strictly negative in all future periods, because the belief p_t^θ is strictly decreasing. Thus, it must be the case that $\bar{\Pi}_{t+1}^\theta < 0$. If the entrepreneur continues the project, she earns a negative value. Thus, liquidating is optimal.

Period τ^θ is defined as the first period in which $\lambda p_t^\theta V - c - k$ becomes negative, or, equivalently, the first period for which

$$p_t^\theta \leq \frac{c+k}{\lambda V}.$$

Since $p_t^l < p_t^h$ for all t, we have $\tau^l < \tau^h$.

That the optimal equity share is given $\bar{\alpha}_t^\theta$ has been established in the text.

B.2 Proof of Proposition 2

We start with some preliminaries. First, no pooling equilibrium with inefficient liquidation can be optimal.

Lemma 7 *There exists no optimal pooling equilibrium in which* $l_t^l = l_t^h = 1$*, but* $\Pi_t^h(1) > 0$.

Proof. To characterize alternative equilibria with higher payoffs, we must consider a number of cases. Let $\Pi_t^l(q_{t-1})$ be the low type's value in period t under the strategies $l_t^{l\prime} = 0$, $l_t^{h\prime} = 0$, $\alpha_t' = \alpha_t^P(q_{t-1})$, $l_{t+1}^{l\prime} = l_t^l$, etc. Note that we have $\Pi_t^h(1) \geq \Pi_t^l(1) \geq \Pi_t^l(q_{t-1})$. Thus, if $\Pi_t^l(q_{t-1}) \geq 0$, both types simply continue and offer contract $\alpha_t' = \alpha_{t-1}^P$. Then, the belief in the alternative equilibrium is $q' = q_{t-1}$, which ensures that the investor's IC condition holds. The payoffs are then larger: $\Pi_t^{h\prime} > \Pi_t^{l\prime} = \Pi_t^l(q_{t-1}) \geq 0$.[55]

Suppose instead that $\Pi_t^l(q_{t-1}) < 0 \leq \Pi_t^l(1) \leq \Pi_t^h(1)$. Then, by continuity, there exists a belief q' such that $\Pi_t^l(q') = 0$. Consider the following alternative equilibrium: $l_t^{l\prime}, l_t^{h\prime} \in (0,1)$ such that

$$1 - l_t^{l\prime} = \left(1 - l_t^{h\prime}\right) q_{t-1} \left(\frac{1-q'}{q'}\right)$$

and

$$\alpha_t' = \frac{c}{\lambda p_t(q') V}.$$

The liquidation probabilities induce the belief q'. This yields the same payoff as in equilibrium for the low type, but a strictly larger payoff for the high type. Finally, suppose that $\Pi_t^h(1) > 0 = \Pi_t^l(1)$. Then, picking $l_t^{h\prime} = 0$ and $l_t^{l\prime} = 1$ is a Pareto improvement. ■

In equilibrium, the high type receives a larger payoff.

Lemma 8 *In any pooling equilibrium, we have* $\Pi_t^h \geq \Pi_t^l$*. If* $l_t^l < 1$*, then* $\Pi_t^h > \Pi_t^l$.

Proof. Since choosing l_t^l is (weakly) suboptimal for the high type, his value satisfies

$$\Pi_t^h \geq \sum_{s=t}^{\infty} \delta^{s-t} \left[\prod_{t \leq u \leq s-1} \left(1 - \lambda p_u^h\right)\left(1 - l_u^l\right)\right] \left(1 - l_s^l\right) \left(\lambda p_s^h \left(1 - \alpha_s^P\right) V - k\right). \tag{20}$$

Using the updating rule in Equation (1) repeatedly yields

$$\prod_{t \leq u \leq s-1} \left(1 - \lambda p_u^h\right) \lambda p_s^h = \lambda p_t^h (1-\lambda)^{s-t},$$

so that the RHS of Equation (20) equals

$$p_t^h \sum_{s=t}^{\infty} (\delta(1-\lambda))^{s-t} \left[\prod_{t \leq u \leq s} \left(1 - l_u^l\right)\right] \lambda \left(1 - \alpha_s^P\right) V \\ - \sum_{s=t}^{\infty} \delta^{s-t} \left[\prod_{t \leq u \leq s-1} \left(1 - \lambda p_u^h\right)\left(1 - l_u^l\right)\right] \left(1 - l_s^l\right) k.$$

[55]Recall that in any pooling equilibrium $\Pi_t^h > \Pi_t^l$, by Lemma 8.

We similarly obtain for the low type

$$\begin{aligned}\Pi_t^l &= p_t^l \sum_{s=t}^{\infty} (\delta(1-\lambda))^{s-t} \left[\prod_{t \le u \le s} \left(1 - l_u^l\right) \right] \lambda \left(1 - \alpha_s^P\right) V \\ &\quad - \sum_{s=t}^{\infty} \delta^{s-t} \left[\prod_{t \le u \le s-1} \left(1 - \lambda p_u^l\right) \left(1 - l_u^l\right) \right] \left(1 - l_s^l\right) k.\end{aligned}$$

Since $p_t^h > p_t^l$ for all t, combining the two expressions yields $\Pi_t^h \ge \Pi_t^l$ and the inequality is strict if $l_t^l < 1$. The lemma implies that if the low type does not liquidate, the high type will not liquidate either. We will exploit this fact throughout. ■

Since the high type receives a higher payoff, she liquidates later.

Corollary 9 *Whenever $l_t^l = 0$, we have $l_t^h = 0$. Whenever $l_t^h > 0$, we have $l_t^l = 1$. There exists no equilibrium in which $l_t^l, l_t^h \in (0,1)$.*

Proof. Liquidating with probability $l_t^l \in (0,1)$ is optimal for the low type if and only if

$$\Pi_t^l = 0,$$

where Π_t^l is the equilibrium value of the low type. Similarly, liquidating with probability $l_t^h \in (0,1)$ is optimal for the high type if and only if

$$\Pi_t^h = 0.$$

Lemma 8 then implies the results. ■

Moreover, the constraint $\Pi_t^\theta \ge \Pi_t^\theta(0)$ does not bind in equilibrium whenever both types continue.

Lemma 10 *For all $t < \bar{\tau}^l$, we have $\Pi_t^\theta > \Pi_t^\theta(0)$.*

Proof. We have, using a similar calculation as in Lemma 8,

$$\Pi_t^h - \Pi_t^h(0) = \lambda p_t^h V \sum_{s=t}^{\bar{\tau}^l - 1} (\delta(1-\lambda))^{s-t} \left(\bar{\alpha}_t^l - \alpha_s^P\right) + \delta^{\bar{\tau}^l - t} \Pi_{t \le u < \bar{\tau}^l - 1} \left(1 - \lambda p_u^h\right) \left(\Pi_{\bar{\tau}^l}^h(1) - \Pi_{\bar{\tau}^l}^h(0)\right)$$

and[56]

$$\Pi_t^l - \Pi_t^l(0) \ge \lambda p_t^l V \sum_{s=t}^{\bar{\tau}^l} (\delta(1-\lambda))^{s-t} \Pi_{t \le u < s-1} \left(1 - l_u^l\right) \left(\bar{\alpha}_s^l - \alpha_s^P\right).$$

For all $t < \bar{\tau}^l$, we have $q_0 \le q_t$ and therefore $\alpha_t^P < \bar{\alpha}_t^l$. We also have $l_t^l < 1$ and $\Pi_t^h(1) \ge \Pi_t^h(0)$ for all t. Thus, both expressions are strictly positive. ■

We are now done with preliminaries and ready to prove the proposition. We first show that the contract in Proposition 2 is optimal among all pooling contracts.

[56] Note that by construction, $\Pi_{\bar{\tau}^l}^l = \Pi_{\bar{\tau}^l}^l(0) = 0$.

Proposition 11 *Any other pooling equilibrium yields weakly lower payoffs for both types than the equilibrium in Proposition 2.*

Specifically, the next series of Lemmas establishes that $U_t = 0$, that

$$\alpha_t^P = \frac{c}{\lambda p_t(q_t) V}$$

for all $t < \bar{\tau}^l$, and that no equilibrium with different payoffs can be optimal.

Lemma 12 *Suppose that $1 = l_t^l \geq l_t^h$. Then, $U_t = 0$ and if $l_t^h < 1$, the high type offers the optimal contract $\bar{\alpha}_s^h$ for all $s \geq t$.*

Proof. That $U_t = 0$ follows directly from Equation (21). Since $l_t^l = 1$, we have $q_t = 1$. Then, any equilibrium must have the high type offer $\bar{\alpha}_s^h$ for $s \geq t$. Overpaying the investor, i.e. $\alpha_s' > \bar{\alpha}_s^h$, is not optimal for the high type and liquidating for $t < s < \tau^h$ cannot be optimal either. ■

Thus, if $l_t^l = 1$, the contract is uniquely pinned down for $s \geq t$. We can therefore restrict attention to periods in which $l_t^l < 1$ and, by Corollary 9, $l_t^h = 0$. We keep this restriction throughout the remainder of the section.

Lemma 13 *Any optimal pooling contract features $U_t = 0$ and*

$$\alpha_t^P = \frac{c - \delta(1 - \lambda p_t(q_t)) U_{t+1}}{\lambda p_t(q_t) V}$$

whenever $\Pi_t^l > 0$.

Proof. The investor experiments whenever

$$U_t = (1 - l_t(q_{t-1})) \left(\lambda p_t(q_t) \alpha_t^P V - c + \delta(1 - \lambda p_t(q_t)) U_{t+1} \right) \geq 0. \tag{21}$$

Suppose that $U_t > 0$, $\Pi_t^l > 0$, and $l_t^l = l_t^h = 0$ for some t.[57] We can generate an improvement for the entrepreneur by picking equity share $\alpha_t' = \alpha_t^P - \varepsilon$, where, ε is chosen sufficiently small to ensure that the investor's value remains positive. This clearly increases both types' payoffs. ■

We next show that for any pooling equilibrium in which $\Pi_t^l = 0$ and $U_t > 0$, there exists another equilibrium in which $U_t = 0$ and which yields at least weakly increases the payoffs to both types in period t. We distinguish two cases, when $\Pi_{t+1}^l > 0$ and when $\Pi_{t+1}^l = 0$.

Lemma 14 *Suppose that $\Pi_t^l = \Pi_{t+1}^l = 0$. Then, any pooling equilibrium in which*

$$\alpha_t^P > \frac{c}{\lambda p_t(q_{t-1}) V}$$

is not optimal.

[57] Recall that $l_t^l = 0$ implies $l_t^h = 0$ by Corollary 9.

Proof. Consider the following alternative equilibrium

$$\begin{aligned}
\alpha_t' &= \frac{c}{\lambda p_t\left(q_{t-1}\right)V} \\
l_t^{l\prime} &= l_t^{h\prime} = 0 \\
q_t' &= q_{t-1} \\
\alpha_{t+1}^{'} &= \alpha_t^P \\
l_{t+1}^{\theta\prime} &= l_t^\theta \text{ for } \theta = l, h \\
q_{t+1}' &= q_t \\
&\ldots
\end{aligned}$$

We now verify that the alternative contract is indeed an equilibrium and improves the entrepreneur's payoffs. First, in any equilibrium, we have $U_{t+1} \geq 0$. Thus, the investor experiments whenever his share exceeds $c/\left(\lambda p_t\left(q_t'\right)V\right)$. In particular, he experiments at α_t' given belief $q_t' = q_{t-1}$. Second, since $\Pi_{t+1}^{l\prime} = \Pi_t^l = 0$ and $\alpha_t' < \alpha_t^P$, it must be the case that $\Pi_t^{l\prime} > \Pi_t^l = 0$ and therefore $l_t^{l\prime} = 0$ is optimal. A similar argument holds for type h, which implies that $\Pi_t^{h\prime} > \Pi_t^h \geq 0$ and that $l_t^{h\prime} = 0$ is optimal.[58] Third, we have $l_t^{l\prime} = l_t^{h\prime} = 0$ and Bayesian updating implies that $q_t' = q_{t-1}$. Finally, since the continuation game in period $t+1$ in the alternative equilibrium is the same as the continuation game in period t under the original equilibrium, all conditions are satisfied from period $t+1$ onward. Thus, we have constructed an equilibrium which improves the entrepreneur's payoffs. ■

Lemma 15 *If* $\Pi_t^l = \Pi_{t+1}^l = 0$ *and*

$$\alpha_t^P \leq \frac{c}{\lambda p_t\left(q_{t-1}\right)V},$$

then there exists another pooling equilibrium which yields the same payoffs to both types in period t and which satisfies

$$\alpha_t^P = \frac{c}{\lambda p_t\left(q_t\right)V}.$$

Proof. Consider the following alternative equilibrium. We pick

$$\begin{aligned}
\alpha_t' &= \alpha_t^P \\
q_t' &: \alpha_t^P = \frac{c}{\lambda p_t\left(q_t'\right)V} \\
l_t^{h\prime} &= l_t^h \\
1 - l_t^{l\prime} &= \left(1 - l_t^h\right) q_{t-1}\left(\frac{1 - q_t'}{q_t'}\right) \\
\alpha_{t+1}' &= \alpha_{t+1}^P \\
q_{t+1}' &= q_{t+1} \\
l_{t+1}^{h\prime} &= l_{t+1}^h \\
1 - l_{t+1}^{l\prime} &= \left(1 - l_{t+1}^h\right) q_t'\left(\frac{1 - q_{t+1}'}{q_{t+1}'}\right).
\end{aligned}$$

That is, we keep the equity share the same, i.e. $\alpha_t^{'} = \alpha_t^P$. However, we change the likelihood of

[58] Of course, it is possible that $l_t^{h\prime} = l_t^h = 0$, which happens whenever $\Pi_t^h > 0$.

termination $l_t^{l\prime}$ so that the belief q_t' satisfies

$$\alpha_t^P = \frac{c}{\lambda p_t\left(q_t'\right)V}$$

under Bayes' rule.[59] Note that this implies $q_t' \geq q_{t-1}$. In period $t+1$, we keep the equity share and beliefs the same as in the original equilibrium, but we again adjust type l's likelihood of liquidation so that $q_{t+1}' = q_{t+1}$.[60] From period $t+2$ onward, the strategies and beliefs in the alternative equilibrium are the same as in the original one.

Let us confirm that the alternative equilibrium exists. First, since $q_t' \geq q_{t-1}$, the investor's IC condition in Equation (21) holds in period t given equity share α_t' and belief q_t'. Similarly, his IC condition in period $t+1$ holds because $q_{t+1}' = q_{t+1}$ and $\alpha_{t+1}' = \alpha_{t+1}^P$. Second, we have $\Pi_{t+1}^{l\prime} = \Pi_{t+1}^l = 0$, which holds because $\alpha_{t+1}' = \alpha_{t+1}^P$ and because the continuation strategies after time $t+1$ are the same as in the original equilibrium. We also have $\Pi_t^{l\prime} = \Pi_t^l = 0$, because $\alpha_t' = \alpha_t^P$ and $\Pi_{t+1}^{l\prime} = \Pi_{t+1}^l$.[61] Since the low type is indifferent in period t, we can freely pick $l_t^{l\prime}$ to ensure that the investor's belief is indeed q_t'. Similarly, we can pick $l_{t+1}^{l\prime}$ such that $q_{t+1}' = q_{t+1}$. ■

Now, we consider the case when $\Pi_t^l = 0$ and $\Pi_{t+1}^l > 0$. We will show that either (i) this case is equivalent to the previous one, where $\Pi_t^l = \Pi_{t+1}^l = 0$, or (ii) we can pick an alternative equilibrium in which $U_t = 0$.

Lemma 16 *Suppose that $\Pi_t^l = 0$ and $\Pi_{t+1}^l > 0$. Then, there exists another pooling equilibrium which yields the same payoffs to both types and in which either $\Pi_{t+1}^l = 0$ or $U_t = 0$.*

Proof. Consider the following alternative equilibrium

$$\begin{aligned}
\alpha_t' &= \alpha_t^P - \varepsilon \\
\alpha_{t+1}' &= \alpha_{t+1}^P + \varepsilon / \left(\delta\left(1-\lambda\right)\right) \\
q_t' &= q_t \\
q_{t+1}' &= q_{t+1} \\
l_s^{\theta\prime} &= l_s^\theta \text{ for } \theta = l, h \text{ and } s = t, t+1.
\end{aligned}$$

Let us confirm that this is indeed an equilibrium. Type l's payoff in the alternative equilibrium is

$$\begin{aligned}
\Pi_t^l\left(q_t, \alpha_t'\right) &= \left(1 - l_t^{l\prime}\right)\left(\lambda p_t^l\left(1-\alpha_t^p+\varepsilon\right)V - k + \delta\left(1-\lambda p_t^l\right)\Pi_{t+1}^l\left(q_{t+1}, \alpha_{t+1}'\right)\right) \\
&= \left(1 - l_t^{l\prime}\right)\left(\lambda p_t^l\left(1-\alpha_t^p+\varepsilon\right)V - k + \delta\left(1-\lambda p_t^l\right)\Pi_{t+1}^l\left(q_{t+1}, \alpha_{t+1}^P\right)\right. \\
&\quad \left. -\delta\left(1-\lambda p_t^l\right)\lambda p_{t+1}^l \frac{\varepsilon V}{\delta\left(1-\lambda\right)}\right).
\end{aligned}$$

[59]This is always possible, since we are considering the case when $l_t^h < 1$.

[60]That is, the investor's beliefs in the alternative and original equilibrium coincide.

[61]A similar argument for the high type yields $\Pi_t^{h\prime} = \Pi_t^h$ and $\Pi_{t+1}^{h\prime} = \Pi_{t+1}^h$. Thus, both types' payoffs are unchanged.

Using the updating rule in Equation (1) yields

$$\begin{aligned}\Pi_t^l\left(q_t,\alpha_t'\right) &= \left(1-l_t^l\right)\left(\lambda p_t^l\left(1-\alpha_t^p+\varepsilon\right)V-k+\delta\left(1-\lambda p_t^l\right)\Pi_{t+1}^l\left(q_{t+1},\alpha_{t+1}^P\right)\right.\\ &\quad \left.-\delta\left(1-\lambda\right)\lambda p_t^l\frac{\varepsilon V}{\delta\left(1-\lambda\right)}\right)\\ &= \Pi_t^l.\end{aligned}$$

Thus, type l receives the same payoff as in equilibrium. A similar calculation for type h yields

$$\Pi_t^h\left(q_t,\alpha_t'\right)=\Pi_t^h.$$

Thus, $l_t^{\theta\prime}=l_t^\theta$ for $\theta=l,h$ is optimal. Since the liquidation probabilities are the same, the beliefs are the same as well, i.e. $q_t'=q_t$. Notice that $\Pi_{t+1}^{\theta\prime}$ is decreasing in ε. Thus, if we pick ε sufficiently large, we have $\Pi_{t+1}^{l\prime}=0$ and we can then pick $l_{t+1}^{l\prime}$ to ensure that $q_{t+1}'=q_{t+1}$.

Finally, it remains to check whether the investor's incentive compatibility constraint holds in period t in the alternative equilibrium. If this is not true, i.e. for the ε for which $\Pi_{t+1}^l=0$, we have $U_t<0$, then, since U_t is continuous in ε, there exists another ε' such that $U_t=0$. We have thus established the result in the statement of the lemma. ■

So far, we have shown that for each t, any optimal pooling contract must feature either $U_t=0$ or $\alpha_t^p=\frac{c}{\lambda p_t(q_t)V}$. The following Lemma shows concludes this part of our argument by showing that $U_t=0$ for all t.

Lemma 17 *Suppose that either $U_t=0$ or $\alpha_t^p=c/\left(\lambda p_t\left(q_t\right)V\right)$ for all t. Then, for all t, we have $U_t=0$ and*

$$\alpha_t^p=\frac{c}{\lambda p_t\left(q_t\right)V}.$$

Proof. Suppose $U_t>0$. Then, we have

$$U_t=\delta\left(1-\lambda p_t\left(q_t\right)\right)U_{t+1}$$

and thus $U_{t+1}>0$. Proceeding inductively, we must have $U_s>0$ for all $s\geq t$. But this is impossible. Under the contract α_t^P, the low type will eventually liquidate with probability one, which leaves the investor with a continuation value of zero, since either the game ends or the high type offers his optimal symmetric information contract. Therefore, we must have $U_t=0$ for all t. But this immediately implies that

$$\alpha_t^p=\frac{c}{\lambda p_t\left(q_t\right)V}$$

for all t. ■

We have now shown that there is no pooling equilibrium which yields a strictly higher payoff to any type than the equilibrium of Proposition 2. We next show that the equilibrium of Proposition 2 exists. A necessary condition is that given

$$\alpha_t^P=\frac{c}{\lambda p_t\left(q_t\right)V},$$

the following conditions are satisfied. For all t and $\theta \in \{l, h\}$,

$$\begin{aligned} \Pi_t^\theta &\geq \Pi_t^\theta(0) \\ \Pi_t^l &> 0 \Rightarrow l_t^h = l_t^l = 0 \\ l_t^\theta &> 0 \Rightarrow \Pi_t^\theta = 0 \end{aligned} \tag{22}$$

and q_t satisfies Equation (13).[62] The first equation says that deviating to any other contract (in which case we can set the off-path belief to zero) makes each type worse off than staying in equilibrium. The two following equations ensure that the liquidation decisions are optimal for both types. Note that we do not have to consider the investor's incentives, since in this equilibrium, we have $U_t = 0$ for all t.

Lemma 18 *Let $\underline{\tau}^l$ be the first period for which*

$$\lambda p_t^l \left(1 - \frac{c}{\lambda p_t(q_0) V}\right) V - k \leq 0$$

and let $\bar{\tau}^l$ be the first period for which

$$\lambda p_t^l \left(1 - \bar{\alpha}_t^h\right) V - k \leq 0.$$

We have $1 < \underline{\tau}^l \leq \bar{\tau}^l$. Consider the following strategies and beliefs. For any $t < \underline{\tau}^l$, we have $q_t = q_0$ and $l_t^l = l_t^h = 0$. If $\underline{\tau}^l < \bar{\tau}^l$, then for any $\underline{\tau}^l \leq t < \bar{\tau}^l$, we have $l_t^h = 0$, l_t^l satisfies

$$q_t = \frac{q_{t-1}}{q_{t-1} + (1 - q_{t-1})\left(1 - l_t^l\right)},$$

and q_t satisfies

$$\lambda p_t^l \left(1 - \frac{c}{\lambda p_t(q_t) V}\right) V = k. \tag{23}$$

Finally, we have $l_{\bar{\tau}^l}^l = 1$. Then, the belief q_t is strictly increasing for all $\underline{\tau}^l \leq t < \bar{\tau}^l$ and the conditions in Equation (22) are satisfied.

Intuitively, we adjust liquidation probabilities so that the low type remains indifferent between continuing and liquidating, given that q_t is updated using Bayes rule. As we show, this is possible and satisfies all relevant incentive constraints.

Proof. Condition (8) implies that

$$\lambda p_1^l \left(1 - \alpha_1^P\right) V - k > 0.$$

Thus, $l_1^l = l_1^h = 0$, $q_1 = q_0$, and $\Pi_1^l > 0$. For any t such that $l_s^l = 0$ for all $s \leq t$, we have $q_t = q_0$

[62] Recall that we are restricting attention to times at which $l_t^l < 1$, since by Lemma 12, the contract is pinned down if $l_t^l = 1$. Thus, Bayes' rule applies.

and the low type's period payoff is

$$\lambda p_t^l \left(1 - \frac{c}{\lambda p_t (q_0) V}\right) V - k.$$

This expression crosses zero exactly once from above, since $p_t(q_0)$ is strictly decreasing in t and vanishes as t becomes large. Thus, $\underline{\tau}^l$ exists and we have $\underline{\tau}^l > 1$. We have $\Pi_t^l > 0$ for $t < \underline{\tau}^l$ and thus $l_t^l = l_t^h = 0$ is optimal for any such t.

A similar argument implies that $\bar{\tau}^l$ exists. That $\underline{\tau}^l \leq \bar{\tau}^l$ is straightforward, because $\alpha_t^P \geq \alpha_t^h$ for all t.

In period $\underline{\tau}^l$, the low type liquidates with strictly positive probability. Here is the argument. If she continues with certainty, then the equilibrium must feature a belief $q_{t+1} = q_0$ and a contract $\alpha_{t+1}^P = c / (\lambda p_{t+1}(q_0) V)$, etc. But then, the low type's period payoff for any period $s > t$ in which she continues is strictly negative, so that $\Pi_{t+1}^l \leq 0$. This, in turn implies that

$$\Pi_t^l = \left(1 - l_t^l\right) \left(\lambda p_t^l \left(1 - \alpha_t^P\right) V - k + \delta \left(1 - \lambda p_t^l\right) \Pi_{t+1}^l\right) < 0,$$

so that the low type's decision to continue in period t must be suboptimal. Thus, no such equilibrium can exist, and we have $\Pi_{\underline{\tau}^l}^l = 0$.

Now, consider the case when $\bar{\tau}^l > \underline{\tau}^l$[63] and recall that by construction of $\bar{\tau}^l$, we have

$$\lambda p_{\bar{\tau}^l}^l \left(1 - \alpha_{\bar{\tau}^l}^h\right) V \leq k. \tag{24}$$

Then, for any $\underline{\tau}^l \leq t < \bar{\tau}^l$, there exists a unique belief $q_t \in (q_0, 1)$ such that

$$\lambda p_t^l \left(1 - \frac{c}{\lambda p_t (q_t) V}\right) V = k. \tag{25}$$

Using the updating rule in Equation (13), we can find a unique l_t^l which induces belief q_t given q_{t-1}. We can now inductively construct a sequence $\left\{l_t^l, q_t\right\}$ such that Equation (25) holds in each period $\underline{\tau}^l \leq t < \bar{\tau}^l$. In period $\bar{\tau}^l$, we pick $l_{\bar{\tau}^l}^l = 1$. If $\underline{\tau}^l = \bar{\tau}^l$, we also pick $l_{\bar{\tau}^l}^l = 1$.

In periods $\underline{\tau}^l \leq t < \bar{\tau}^l$, we have $\Pi_t^h > 0$ and therefore it is optimal for the high type to continue. For the low type, the indifference condition $\Pi_t^l = 0$ must hold. This is true. In period $\bar{\tau}^l$, the low type receives zero value, i.e. $\Pi_{\bar{\tau}^l}^l = 0$, and in any period $\underline{\tau}^l \leq t < \bar{\tau}^l$, Equation (25) implies that her period payoff is zero. Backwards induction then implies that $\Pi_t^l = 0$.

In period $\bar{\tau}^l$, it is optimal for the low type to liquidate with certainty. We must distinguish two cases. Suppose that $l_{\bar{\tau}^l}^h < 1$. Then, Equation (13) implies that $q_{\bar{\tau}^l} = 1$. If the low type continues instead, she receives a payoff of

$$\lambda p_{\bar{\tau}^l}^l \left(1 - \alpha_{\bar{\tau}^l}^h\right) V - k + \delta \left(1 - \lambda p_{\bar{\tau}^l}^l\right) \Pi_{\bar{\tau}^l + 1}^l.$$

Equation (24) implies that the deviation payoff is negative. Thus, the low type indeed prefers to liquidate. Now, suppose that $l_{\bar{\tau}^l}^h = 1$, so that the game ends with certainty in period $\bar{\tau}^l$. In that case, Equation (24) guarantees that the low type's deviation payoff is negative for any off-path

[63] This implies that $\lambda p_t^l \left(1 - \bar{\alpha}_t^h\right) V > k$ at $t = \underline{\tau}^l$.

belief. ■

We have now established that the proposed strategies satisfy the necessary conditions in Equation (22).[64] It remains to show that the pooling equilibrium yields at least a weakly higher payoff to both types than any separating equilibrium. To prove the result, we must first characterize separating equilibria. Therefore, we defer this proof. It can be found in Corollary 24 below.

We conclude by showing that q_t is strictly increasing for $\underline{\tau}^l \leq t < \bar{\tau}^l$. This follows, because $p_t(q)$ is strictly decreasing in t for any fixed q and strictly increasing in q for any fixed t. Thus, to satisfy the indifference condition in Equation (25) in consecutive periods, q_t must be strictly increasing.

Finally, the equilibrium in Lemma 18 is unique, provided we fix the equity share offered. To establish this, we need the following auxiliary result.

Lemma 19 *For $\underline{\tau}^l \leq t < \bar{\tau}^l$, we have*

$$\alpha_t^P = \frac{\lambda p_t^l V - k}{\lambda p_t^l V}$$

and $\Pi_t^l = 0$. For $t < \underline{\tau}^l$, we have

$$\alpha_t^P < \frac{\lambda p_t^l V - k}{\lambda p_t^l V}.$$

Proof. For $t < \underline{\tau}^l$, the low type continues with certainty and $q_t = q_0$, so that

$$\alpha_t^P = \frac{c}{\lambda p_t(q_0) V}.$$

The inequality

$$\frac{c}{\lambda p_t(q_0) V} \geq \frac{\lambda p_t^l V - k}{\lambda p_t^l V},$$

is equivalent to

$$c \geq \lambda p_t(q_0) V - k\left(q_0 \frac{p_t^h}{p_t^l} + (1 - q_0)\right). \tag{26}$$

Since $p_t(q_0)$ is decreasing in t and p_t^h/p_t^l is increasing, the RHS is decreasing. We will exploit this fact throughout the proof.

First, we show that there exists no $\underline{\tau}^l \leq t < \bar{\tau}^l$ for which $\Pi_t^l > 0$. Assume towards a contradiction that there exists such a period and let $\hat{t}$ be the largest one. This implies that $\Pi_{\hat{t}+1}^l = 0$, $l_{\hat{t}}^l = 0$, $q_{\hat{t}} = q_{\hat{t}-1}$, and

$$\Pi_{\hat{t}}^l = \lambda p_{\hat{t}}^l \left(1 - \frac{c}{\lambda p_{\hat{t}}(q_{\hat{t}-1}) V}\right) V - k > 0.$$

By construction of $\underline{\tau}^l$, we have

$$\frac{c}{\lambda p_{\underline{\tau}^l}^l(q_0) V} \geq \frac{\lambda p_{\underline{\tau}^l}^l V - k}{\lambda p_{\underline{\tau}^l}^l V}.$$

[64] Recall that $\alpha_t^P = \frac{c}{\lambda p_t(q_t)V}$ guarantees that the investor is always willing to experiment. Thus, we do not need to consider the investor's incentive compatibility constraints. Similarly, we do not need to consider deviations in the contract offered, i.e. $\alpha_t' \neq \alpha_t^P$, since the off-path belief $q' = 0$ renders such deviations unprofitable for either type.

If $\Pi_t^l > 0$ for all $\underline{\tau}^l \leq t < \hat{t}$, then we have $q_{\hat{t}} = q_0$. But since the RHS in Equation (26) is decreasing in time, this implies that

$$\frac{c}{\lambda p_{\hat{t}}(q_0)V} > \frac{\lambda p_{\hat{t}}V - k}{\lambda p_{\hat{t}}V} \tag{27}$$

and therefore $\Pi_{\hat{t}}^l < 0$, a contradiction.

Otherwise, there exists a $\underline{\tau}^l \leq \tilde{t} < \hat{t}$ such that $\Pi_{\tilde{t}}^l = 0$ and $\Pi_t^l > 0$ for all $\tilde{t} < t \leq \hat{t}$. Then, we have $l_t^l = 0$ for any such t and $q_{\hat{t}} = q_{\tilde{t}}$. Moreover,

$$\Pi_{\tilde{t}}^l = \lambda p_{\tilde{t}}^l \left(1 - \alpha_{\tilde{t}}^P\right) V - k + \delta \left(1 - \lambda p_{\tilde{t}}^l\right) \Pi_{\tilde{t}+1}^l = 0$$

implies that

$$\frac{c}{\lambda p_{\tilde{t}}^l(q_{\tilde{t}})V} \geq \frac{\lambda p_{\tilde{t}}^l V - k}{\lambda p_{\tilde{t}}^l V},$$

since $\Pi_{\tilde{t}+1}^l \geq 0$. Because the belief does not change between $\tilde{t}$ and $\hat{t}$, a variant of Equation (26) implies that Inequality (27) holds again, and we get a contradiction.

That

$$\alpha_t^P = \frac{\lambda p_t^l V - k}{\lambda p_t^l V}$$

for $\underline{\tau}^l \leq t < \bar{\tau}^l$ follows because $\Pi_t^l = \Pi_{t+1}^l = 0$ for any such t, so that

$$\lambda p_t^l \left(1 - \alpha_t^P\right) V - k + \delta \left(1 - \lambda p_t^l\right) \cdot 0 = 0.$$

To show the second part of the lemma, note simply that by construction, $\underline{\tau}^l$ is the first period in which Inequality (26) holds. ■

Corollary 20 *Suppose that*

$$\alpha_t^P = \frac{c}{\lambda p_t(q_t)V}.$$

Then, the equilibrium of Lemma 18 is unique, i.e. there does not exist another pooling equilibrium with the same equity share but different liquidation probabilities.

Proof. Any equilibrium which features liquidation before period $\underline{\tau}^l$ violates the entrepreneur's incentive constraints. By Lemma 19, we have for any $\underline{\tau}^l \leq t < \bar{\tau}^l$,

$$\frac{c}{\lambda p_t(q_t)} = \frac{\lambda p_t^l V - k}{\lambda p_t^l V}.$$

Thus, the sequence of beliefs $\{q_t\}$ is unique and so is the sequence of liquidation probabilities $\{l_t^l\}$. Any other choice of liquidation probabilities will either violate the low type's incentive constraint for some t or violate Bayesian updating in Equation (13). Finally, there is no equilibrium in which the low type continues past $\bar{\tau}^l$, because even if $q_t = 1$, her value from continuing is negative. ■

B.3 Proof of Proposition 3

We now construct separating equilibria and show that any separating equilibrium is suboptimal. To show existence, we must ensure that the low type does not mimic the high type and vice versa. For this, we need to consider the continuation payoff of the high type when $q = 0$, i.e. the investor believes he is facing the low type, and the low type's continuation payoff when $q = 1$. If $q = 0$, the high type's continuation contract is $\bar{\alpha}^l_{t+1}$. Any lower share leads to the investor abandoning the project while any higher share is suboptimal.[65] We denote the high type's continuation value in that case as $\Pi^h_{t+1}(0) = \Pi^h_{t+1}\left(0, \bar{\alpha}^l_{t+1}\right)$. Similarly, if the low type succeeds in mimicking the high type, he optimally offers $\bar{\alpha}^h_{t+1}$ and receives a value of $\Pi^l_{t+1}(1) = \Pi^l_{t+1}\left(1, \bar{\alpha}^h_{t+1}\right)$. When the investor's beliefs are degenerate, the project is liquidated at a deterministic time. We denote with $\tau^{\theta\prime}$ the liquidation times after a deviation. That is $\tau^{l\prime}$ is the low type's liquidation time if $q' = 1$ and $\tau^{h\prime}$ is the high type's liquidation time when $q' = 0$. We have $\tau^{l\prime} \geq \tau^l$ and $\tau^{h\prime} \leq \tau^h$.

If $t < \tau^l$, combining the two incentive constraints yields the necessary condition

$$\alpha^h_t \in \left[\bar{\alpha}^l_t + \delta(1-\lambda)\frac{\Pi^l_{t+1}(1) - \Pi^l_{t+1}(0)}{\lambda p^l_{t+1} V}, \bar{\alpha}^l_t + \delta(1-\lambda)\frac{\Pi^h_{t+1}(1) - \Pi^h_{t+1}(0)}{\lambda p^h_{t+1} V}\right], \tag{28}$$

while if $\tau^l \leq t < \tau^{l\prime}$, we have

$$\alpha^h_t \in \left[\frac{\lambda p^l_t V - k}{\lambda p^l_t V} + \delta(1-\lambda)\frac{\Pi^l_{t+1}(1) - \Pi^l_{t+1}(0)}{\lambda p^l_{t+1} V}, \bar{\alpha}^l_t + \delta(1-\lambda)\frac{\Pi^h_{t+1}(1) - \Pi^h_{t+1}(0)}{\lambda p^h_{t+1} V}\right], \tag{29}$$

because the low type liquidates if his type is revealed. If $t = \tau^{l\prime}$, the low type liquidates even if she successfully imitates the high type and therefore the high type simply offers the symmetric information contract, i.e. $\alpha^h_t = \bar{\alpha}^h_t$.

Finally, there is no equilibrium in which the high type separates in period $t > \tau^{l\prime}$. Any such equilibrium requires pooling in period $\tau^{l\prime}$. But even if the belief under pooling were $q_{\tau^{l\prime}} = 1$, the low type would liquidate with certainty. Thus, period $t > \tau^{l\prime}$ cannot be reached.

The intervals in Equation (28) and (29) are nonempty, because the high and low type's values satisfy a variant of single crossing. We prove this in Lemma 21 below.[66]

Lemma 21 *We have*

$$\frac{\Pi^h_t(1) - \Pi^h_t(0)}{p^h_t} \geq \frac{\Pi^l_t(1) - \Pi^l_t(0)}{p^l_t}$$

for all $t < \tau^h$.

Proof. The low type's gain from imitating the high type vs. revealing her type in a given period

[65] If $q = 0$, then the investor experiments whenever $\lambda p^l_t(\alpha_t V_t - c) + \delta\left(1 - \lambda p^l_t\right) U_{t+1} \geq 0$. The optimal contract for type h induces $U_t = 0$ for all t, just as in the symmetric information benchmark.

[66] Specifically, in Equation (29), we have $\lambda p^l_t\left(1 - \bar{\alpha}^l_t\right) V \leq k$, which implies that

$$\frac{\lambda p^l_t V - k}{\lambda p^l_t V} \leq \bar{\alpha}^l_t.$$

Together with Lemma 21, this ensures that the interval in Equation (29) is nonempty.

is

$$\Delta_t^l = \begin{cases} \lambda p_t^l(\bar{\alpha}_t^l - \bar{\alpha}_t^h)V & \text{if } t < \tau^l, \\ \lambda p_t^l \left(1 - \bar{\alpha}_t^h\right) V - k & \text{if } \tau^l \leq t < \tau^{l\prime}, \\ 0 & \text{if } \tau^{l'} \leq t. \end{cases} \tag{30}$$

If the project succeeds before τ_l, the low type pays the investor $\bar{\alpha}_t^h V$ instead of $\bar{\alpha}_t^l V$. If the project succeeds after τ_l, she receives an additional continuation value since she would have liquidated the project otherwise.

Similarly, the gain for the high type from being indeed perceived as the high type is

$$\Delta_t^h = \begin{cases} \lambda p_t^h(\bar{\alpha}_t^l - \bar{\alpha}_t^h)V & \text{if } t < \tau^{h\prime}, \\ \lambda p_t^h \left(1 - \bar{\alpha}_t^h\right) V - k & \text{if } \tau^{h\prime} \leq t < \tau^h, \\ 0 & \text{if } \tau^h \leq t. \end{cases} \tag{31}$$

Thus, we have

$$\Pi_t^l(1) - \Pi_t^l(0) = E_t^l \left[\sum_{s=t}^{\tau^{l\prime}-1} \delta^{s-t} \Delta_s^l \right]$$

and

$$\Pi_t^h(1) - \Pi_t^h(0) = E_t^h \left[\sum_{s=t}^{\tau^h-1} \delta^{s-t} \Delta_s^h \right].$$

To prove the result, we distinguish two cases. Suppose first that $\tau^l \leq \tau^{l\prime} \leq \tau^{h\prime} \leq \tau^h$. Then, we have

$$\begin{aligned} \Pi_t^h(1) - \Pi_t^h(0) &\geq E_t^h \left[\sum_{s=t}^{\tau^{h\prime}-1} \delta^{s-t} \Delta_s^h \right] \\ &= \lambda p_t^h \sum_{s=t}^{\tau^{h\prime}-1} \left(\delta\left(1-\lambda\right)\right)^{s-t} (\bar{\alpha}_s^l - \bar{\alpha}_s^h)V, \end{aligned}$$

and

$$\begin{aligned} \Pi_t^l(1) - \Pi_t^l(0) &\leq E_t^l \left[\sum_{s=t}^{\tau^{l\prime}-1} \delta^{s-t} \lambda p_s^l \left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right) V \right] \\ &= \lambda p_t^l \sum_{s=t}^{\tau^{l\prime}-1} \left(\delta\left(1-\lambda\right)\right)^{s-t} \left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right) V. \end{aligned}$$

Since $\tau_h' \geq \tau_l'$ and $\bar{\alpha}_t^l > \bar{\alpha}_t^h$ for all t, the above expressions imply

$$\frac{\Pi_t^h(1) - \Pi_t^l(0)}{p_t^h} \geq \frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l},$$

which is what we set out to prove.

Suppose now that $\tau^l \leq \tau^{h\prime} < \tau^{l\prime} \leq \tau^h$. We have

$$\begin{aligned}
\Pi_t^h(1) - \Pi_t^h(0) &\geq E_t^h\left[\sum_{s=t}^{\tau^{l\prime}-1} \delta^{s-t}\Delta_s^h\right] \\
&= E_t^h\left[\sum_{s=t}^{\tau^{h\prime}-1} \delta^{s-t}\lambda p_s^h\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V + \sum_{s=\tau^{h\prime}}^{\tau^{l\prime}-1} \delta^{s-t}\left(\lambda p_s^h\left(1-\bar{\alpha}_s^h\right)V - k\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\Pi_t^l(1) - \Pi_t^l(0) &= E_t^l\left[\sum_{s=t}^{\tau^{l\prime}-1} \delta^{s-t}\Delta_s^l\right] \\
&= E_t^l\left[\sum_{s=t}^{\tau^{l}-1} \delta^{s-t}\lambda p_s^h\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V + \sum_{s=\tau^{l}}^{\tau^{l\prime}-1} \delta^{s-t}\left(\lambda p_s^h\left(1-\bar{\alpha}_s^h\right)V - k\right)\right].
\end{aligned}$$

If $t \leq s < \tau^l$, both types continue. Then, we have

$$\frac{E_t^h\left[\Delta_s^h\right]}{p_t^h} = (1-\lambda)^{s-t}\lambda\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V = \frac{E_t^l\left[\Delta_s^l\right]}{p_t^l}.$$

If $\tau^l \leq s < \tau^{h'}$, then the high type always continues, while the low type liquidates if her type is known. Therefore, we have

$$\lambda p_s^l\left(1-\bar{\alpha}_s^l\right)V \leq k$$

and

$$\lambda p_s^l\left(1-\bar{\alpha}_s^h\right)V \geq k,$$

which together imply

$$\lambda p_s^l\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V \geq \lambda p_s^l\left(1-\bar{\alpha}_s^h\right)V - k.$$

This inequality, in turn, implies that

$$\begin{aligned}
\frac{E_t^h\left[\Delta_s^h\right]}{p_t^h} &= \frac{1}{p_t^h}\left[\prod_{t\leq u<s-1}\left(1-\lambda p_u^h\right)\right]\lambda p_s^h\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V \\
&= (1-\lambda)^{s-t}\lambda\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V \\
&= \frac{1}{p_t^l}\left[\prod_{t\leq u<s-1}\left(1-\lambda p_u^l\right)\right]\lambda p_s^l\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V \\
&\geq \frac{1}{p_t^l}E_t^l\left[\lambda p_s^l\left(1-\bar{\alpha}_s^h\right)V - k\right] \\
&= \frac{E_t^l\left[\Delta_s^l\right]}{p_t^l}.
\end{aligned}$$

If $\tau^{h'} \leq s < \tau^{l'}$, then both types liquidate if $q_s = 0$ and continue if $q_s = 1$. We have

$$\begin{aligned} \frac{E_t^h\left[\Delta_s^h\right]}{p_t^h} &= \frac{1}{p_t^h}\left[\prod_{t\leq u<s-1}\left(1-\lambda p_u^h\right)\right]\left(\lambda p_s^h\left(1-\bar{\alpha}_s^h\right)V-k\right) \\ &= \delta\left(1-\lambda\right)^{s-t}\lambda\left(1-\bar{\alpha}_s^h\right)V-\frac{1-\delta}{p_t^h}\left[\prod_{t\leq u<s-1}\left(1-\lambda p_u^h\right)\right]k. \end{aligned}$$

An analog expression holds for the low type. Since $p_t^h > p_t^l$ for all t, we have

$$\frac{1}{p_t^h}\prod_{t\leq u<s-1}\left(1-\lambda p_u^h\right) < \frac{1}{p_t^l}\prod_{t\leq u<s-1}\left(1-\lambda p_u^l\right)$$

and therefore

$$\frac{E_t^h\left[\Delta_s^h\right]}{p_t^h} \geq \frac{E_t^l\left[\Delta_s^l\right]}{p_t^l}.$$

Combining the three cases yields the result.

Finally, for $\tau^{l\prime} \leq t < \tau^h$, the result is obvious. The low type always liquidates and receives zero, while the high type continues if his type is known and receives a strictly positive payoff. ■

The above Lemma establishes that for each $t < \tau^h$, there is an equilibrium in which the high type separates in period t. In the optimal separating equilibrium, the low type's IC constraint binds, i.e.,

$$\alpha_t^h = \bar{\alpha}_t^l + \delta\frac{1-\lambda}{\lambda V}\frac{\Pi_{t+1}^l\left(1\right)-\Pi_{t+1}^l\left(0\right)}{p_{t+1}^l} \tag{32}$$

if $t < \tau^l$ and[67]

$$\alpha_t^h = \frac{\lambda p_t^l V - k}{\lambda p_t^l V} + \delta\frac{1-\lambda}{\lambda V}\frac{\Pi_{t+1}^l\left(1\right)}{p_{t+1}^l} \tag{33}$$

if $\tau^l \leq t < \tau^h$.

We now show that in any separating equilibrium, both types would at least weakly prefer to offer the pooling contract. We split the argument into two cases, depending on whether the low type continues once her type is revealed. The low type is at least weakly better off in the pooling equilibrium compared to the separating equilibrium. Thus, we only need to show that the high type is better off.

Lemma 22 *Any equilibrium in which the high type separates in period $t < \min\left\{\tau^l, \tau^h - 1\right\}$ is suboptimal. The entrepreneur can strictly improve by pooling in period t and separating in period $t+1$.*

Proof. The high type's payoff from separating in period t is

$$\Pi_t^h = \lambda p_t^h\left(1-\alpha_t^h\right)V - k + \delta\left(1-\lambda p_t^h\right)\Pi_{t+1}^h\left(1\right),$$

where $\Pi_{t+1}^h\left(1\right)$ is the symmetric information continuation payoff, which we defined in Section 5.1,

[67]Note that $\Pi_{t+1}^l\left(0\right) = 0$ in this case.

while her payoff from pooling in period t and separating in period $t+1$ is

$$\Pi_t^{h\prime} = \lambda p_t^h \left(1-\alpha_t^P\right) V - k + \delta \left(1-\lambda p_t^h\right) \Pi_{t+1}^{h\prime},$$

where $\Pi_{t+1}^{h\prime}$ is her payoff from offering the separating contract in period $t+1$.[68]

Suppose first that $t < \tau^l - 1 \leq \tau^{l\prime}$. Then, the low type continues after her type is revealed, both in the initial separating contract and in the alternative one. Using Equation (32), we can write

$$\Pi_t^h = \lambda p_t^h \left(V - \frac{c}{\lambda p_t^l}\right) - k - \delta\left(1-\lambda\right) p_t^h \frac{\Pi_{t+1}^l\left(1\right) - \Pi_{t+1}^l\left(0\right)}{p_{t+1}^l} + \delta\left(1-\lambda p_t^h\right)\Pi_{t+1}^h\left(1\right)$$

and

$$\begin{aligned}
\Pi_t^{h\prime} &= \lambda p_t^h \left(V - \frac{c}{\lambda p_t\left(q_t\right)}\right) - k \\
&\quad + \delta\left(1-\lambda p_t^h\right)\left(\lambda p_{t+1}^h\left(V - \frac{c}{\lambda p_{t+1}^l}\right) - k\right. \\
&\quad \left. - \delta\left(1-\lambda\right) p_{t+1}^h \frac{\Pi_{t+2}^l\left(1\right) - \Pi_{t+2}^l\left(0\right)}{p_{t+2}^l} + \delta\left(1-\lambda p_{t+1}^h\right)\Pi_{t+2}^h\left(1\right)\right).
\end{aligned}$$

We can now plug in the expression

$$\frac{\Pi_{t+1}^l\left(1\right) - \Pi_{t+1}^l\left(0\right)}{p_{t+1}^l} = \frac{1}{p_{t+1}^l}\left(\lambda p_{t+1}^l\left(\frac{c}{\lambda p_{t+1}^l} - \frac{c}{\lambda p_{t+1}^h}\right) + \delta\left(1-\lambda p_{t+1}^l\right)\left(\Pi_{t+2}^l(1) - \Pi_{t+2}^l(0)\right)\right),$$

plug in the symmetric information value

$$\Pi_{t+1}^h\left(1\right) = \lambda p_{t+1}^h V - c - k + \delta\left(1 - \lambda p_{t+2}^h \Pi_{t+2}^h\left(1\right)\right),$$

and use the Bayesian updating rule in Equation (1). This yields, after some algebra,

$$\Pi_t^{h\prime} - \Pi_t^h = \lambda p_t^h \left(\frac{c}{\lambda p_t\left(q_t\right)} - \frac{c}{\lambda p_t^l}\right) > 0.$$

Thus, pooling in period t and separating in period $t+1$ yields a strictly larger payoff for the high type.

If $t = \tau^l - 1 < \tau^{l\prime}$, the low type liquidates in period $t+1$ if her type is revealed. This changes the high type's payoff from separating later. The separating contract in period $t+1$ is given by Equation (33) and we have

$$\frac{\Pi_{t+1}^l\left(1\right) - \Pi_{t+1}^l\left(0\right)}{p_{t+1}^l} = \frac{1}{p_{t+1}^l}\left(\lambda p_{t+1}^l\left(V - \frac{c}{\lambda p_{t+1}^h}\right) - k + \delta\left(1-\lambda p_{t+1}^l\right)\Pi_{t+2}^l(1)\right).$$

[68] For $t < \tau^{l\prime}$, the high type does not liquidate when offering the pooling contract. See Proposition 2.

A similar argument as in the previous case yields

$$\Pi_t^{h\prime} - \Pi_t^h = \lambda p_t^h \left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t(q_t)} \right) > 0.$$

■

Note that the alternative equilibrium we construct is only meaningful if the high type does not liquidate in period $t+1$. This complication occurs when $t = \tau^h - 1$. Then, the pooling and separating contracts coincide, i.e.

$$\alpha_t^P = \alpha^h = \frac{\lambda p_t^l V - k}{\lambda p_t^l V}.$$

This is because we have $\tau^h \geq \tau^{l\prime}$, so the low type will liquidate with certainty in period $t+1$ under both the pooling and separating contracts. In this case, pooling and separating contracts yields the same payoffs to both types, and they both induce liquidation. The distinction in that case is thus purely notational.

Now, we consider the case when the low type liquidates if her type is known and the separating contract is given by Equation (33).

Lemma 23 *Any equilibrium in which the high type separates in period $\tau^l \leq t < \tau^{l'}$ is suboptimal. If instead the entrepreneur pools in period t and separates in period $t+1$, her payoff is at least weakly larger.*

Proof. Suppose first that $t < \tau^{l\prime} - 1$. Using Equation (33), the high type's payoff from separating is

$$\Pi_t^h = \lambda p_t^h \left(1 - \frac{\lambda p_t^l V - k}{\lambda p_t^l V} \right) V - k - \delta (1-\lambda) p_t^h \frac{\Pi_{t+1}^l (1)}{p_{t+1}^l} + \delta \left(1 - \lambda p_t^h \right) \Pi_{t+1}^h (1),$$

and her payoff from pooling in period t and separating in period $t+1$ is

$$\begin{aligned} \Pi_t^{h\prime} &= \lambda p_t^h \left(1 - \alpha_t^P \right) V - k + \delta \left(1 - \lambda p_t^h \right) \Pi_{t+1}^{h\prime} \\ &= \lambda p_t^h \left(1 - \alpha_t^P \right) V - k \\ &\quad + \delta \left(1 - \lambda p_t^h \right) \left(k \left(\frac{p_{t+1}^h}{p_{t+1}^l} - 1 \right) \right. \\ &\quad \left. - \delta (1-\lambda) p_{t+1}^h \frac{\Pi_{t+2}^l (1)}{p_{t+2}^l} + \delta \left(1 - \lambda p_{t+1}^h \right) \Pi_{t+2}^h (1) \right). \end{aligned}$$

Using

$$\Pi_{t+1}^l (1) = \lambda p_{t+1}^l \left(V - \frac{c}{\lambda p_{t+1}^h} \right) - k + \delta \left(1 - \lambda p_{t+1}^l \right) \Pi_{t+2}^l (1)$$

and substituting the high type's symmetric information value $\Pi^h_{t+1}(1)$, we can write

$$\begin{aligned}\Pi^h_t &= \lambda p^h_t \left(1 - \frac{\lambda p^l_t V - k}{\lambda p^l_t V}\right) V - k \\ &\quad -\delta(1-\lambda) p^h_t \left(\lambda V - \frac{c}{p^h_{t+1}} - \frac{k}{p^l_{t+1}} + \delta(1-\lambda)\frac{\Pi^l_{t+2}(1)}{p^l_{t+2}}\right) \\ &\quad +\delta\left(1-\lambda p^h_t\right)\left(\lambda p^h_{t+1} V - c - k + \delta\left(1 - \lambda p^h_{t+1}\right)\Pi^h_{t+2}(1)\right),\end{aligned}$$

which yields

$$\Pi^{h\prime}_t - \Pi^h_t = \lambda p^h_t \left(\frac{\lambda p^l_t V - k}{\lambda p^l_t V} - \alpha^P_t\right) V$$

after some algebra. Lemma 19 then implies that[69] $\Pi^{h\prime}_t \geq \Pi^h_t$.

Finally, if $t = \tau^{l\prime} - 1$, the low type liquidates in period $t+1$ even after mimicking the high type. In that case, we have $\Pi^l_{t+1}(1) = 0$. A similar calculation as above yields

$$\Pi^{h\prime}_t - \Pi^h_t = \lambda p^h_t \left(\frac{\lambda p^l_t V - k}{\lambda p^l_t V} - \alpha^P_t\right) V - \delta(1-\lambda)\frac{p^h_t}{p^l_{t+1}}\left(\lambda p^l_{t+1}\left(V - \frac{c}{\lambda p^h_{t+1}}\right) - k\right).$$

The first term is positive because of Lemma 19. The second term is positive because the low type prefers to liquidate even if $q_{t+1} = 1$, which implies that

$$\lambda p^l_{t+1}\left(V - \frac{c}{\lambda p^h_{t+1}}\right) \leq k.$$

Thus, we again have $\Pi^{h\prime}_t - \Pi^h_t \geq 0$. ■

Corollary 24 *The pooling equilibrium yields an at least weakly higher payoff for the high type and a strictly higher payoff for the low type than any separating equilibrium.*

Proof. We can apply the two previous Lemmas inductively. Separating in period t is dominated by separating in period $t+1$, which is dominated by separating in period $t+2$, etc. Thus, separating in period t is dominated by pooling in period $\tau^{l\prime} - 1$. In period $\tau^{l\prime}$, the low type liquidates with certainty in the pooling equilibrium, so pooling and separating contracts are identical. ■

This concludes our proof of Proposition 3.

B.4 Proof of Proposition 4

Combining the incentive constraints above yields the necessary condition

$$d^h_t \in \left[\Pi^l_t(1) - \Pi^l_t(0), \Pi^h_t(1) - \Pi^h_t(0)\right]. \tag{34}$$

[69] The inequality binds for $t > \underline{\tau}^l$.

In Lemma 21, we have shown that the entrepreneur's values satisfy the single-crossing condition

$$\frac{\Pi_t^h(1)-\Pi_t^h(0)}{p_t^h} \geq \frac{\Pi_t^l(1)-\Pi_t^l(0)}{p_t^l}.$$

Since $p_t^h > p_t^l$, Equation (14) implies the set in Equation (34) is non-empty. Thus, for any $t < \tau^{l\prime}$, there exists an equilibrium in which the high type separates in period t.

As mentioned in the text, the optimal separating contract must be the one with the lowest cost, i.e.,

$$d_t^h = \Pi_t^l(1) - \Pi_t^l(0).$$

Next, we establish that the pooling equilibrium constructed in Proposition 2 is suboptimal compared to separating in some period τ_S. For $t < \tau^l$, any optimal pooling equilibrium must feature $d_t^P = 0$, otherwise, both types could improve by offering no payouts.[70] Thus, the equilibrium of Proposition 2 remains optimal.

We first show that the high type prefers to separate rather than continue pooling for any $t \geq \tau_S^h$.

Lemma 25 *For any $t < \tau^l$, type h prefers to separate with payouts in period t instead of pooling in period t and separating in period $t+1$ if and only if*

$$f_t(q_0) = c\left(\frac{(1-q_0)\,p_t^h}{q_0 p_t^h + (1-q_0)\,p_t^l} - 1\right) - \delta\lambda p_t^h\left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right) > 0.$$

Proof. Using Equation (15) and the fact that $\alpha_t^h = \bar{\alpha}_t^h$, if the high type separates in period t, her value is

$$\Pi_t^h = \lambda p_t^h\left(V - \frac{c}{\lambda p_t^h}\right) - k + \delta\left(1-\lambda p_t^h\right)\Pi_{t+1}^h(1) - \left(\Pi_t^l(1) - \Pi_t^l(0)\right),$$

while if she pools in period t and separates in period $t+1$, her value is

$$\Pi_t^{h\prime} = \lambda p_t^h\left(V - \frac{c}{\lambda p_t(q_0)}\right) - k + \delta\left(1-\lambda p_t^h\right)\left(\Pi_{t+1}^h(1) - \left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right)\right).$$

In the second case, type h offers the optimal pooling contract a_t^p before separation. By construction, no liquidation occurs before τ^l in the pooling equilibrium and therefore $q_t = q_0$. She prefers to separate earlier if and only if

$$\Pi_t^h - \Pi_t^{h\prime} = \lambda p_t^h\left(\frac{c}{\lambda p_t(q_0)} - \frac{c}{\lambda p_t^h}\right) - \left(\Pi_t^l(1) - \Pi_t^l(0)\right) + \delta\left(1-\lambda p_t^h\right)\left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right)$$

is positive. Using the fact that

$$\Pi_t^l(1) - \Pi_t^l(0) = \lambda p_t^l\left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t^h}\right) + \delta\left(1-\lambda p_t^l\right)\left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right),$$

[70] Recall that the equity share α_t^P provides all incentives to the investor while d_t^P simply serves as a transfer.

we have, after some algebra,

$$\frac{p_t^h}{(1-\delta)\left(p_t^h-p_t^l\right)}\left(\Pi_t^h-\Pi_t^{h\prime}\right)=c\left(\frac{(1-q_0)\,p_t^h}{q_0p_t^h+(1-q_0)\,p_t^l}-1\right)-\delta\lambda p_t^h\left(\Pi_{t+1}^l(1)-\Pi_{t+1}^l(0)\right)=f_t(q_0)\,. \tag{35}$$

Since $p_t^h>p_t^l$, $\Pi_t^h-\Pi_t^{h\prime}$ is positive if and only if $f_t(q_0)$ is positive. ■

The next lemma establishes the monotonicity of $f_t(\cdot)$ given the initial belief q_0.

Lemma 26 *Given q_0, f_t strictly increases in $1\le t<\tau^l$, i.e.,*

$$f_1(q_0)<f_2(q_0)<...<f_{\tau_l-1}(q_0).$$

Proof. We first show that p_t^l/p_t^h decreases in t. This is because

$$\frac{p_{t+1}^l}{p_{t+1}^h}=\frac{p_t^l}{p_t^h}\frac{1-\lambda p_t^h}{1-\lambda p_t^l}<\frac{p_t^l}{p_t^h}.$$

This implies

$$\frac{1-q_0}{q_0+(1-q_0)\frac{p_t^l}{p_t^h}}<\frac{1-q_0}{q_0+(1-q_0)\frac{p_{t+1}^l}{p_{t+1}^h}},$$

which is equivalent to

$$\frac{(1-q_0)\,p_t^h}{q_0p_t^h+(1-q_0)\,p_t^l}<\frac{(1-q_0)\,p_{t+1}^h}{q_0p_{t+1}^h+(1-q_0)\,p_{t+1}^l}.$$

For the second part, we first generate an upper bound of $\delta\left(\Pi_t^l(1)-\Pi_t^l(0)\right)$. To start, notice that for $t<\tau^l$,

$$\frac{c}{\lambda p_{t+1}^l}-\frac{c}{\lambda p_{t+1}^h}=\frac{c\left(1-\lambda p_t^h\right)}{(1-\lambda)\,p_t^h}-\frac{c\left(1-\lambda p_t^l\right)}{(1-\lambda)\,p_t^l}=\frac{1}{1-\lambda}\left(\frac{c}{\lambda p_t^l}-\frac{c}{\lambda p_t^h}\right),$$

and for $t\ge\tau^l$, we have

$$\lambda p_t^l\left(V-\frac{c}{\lambda p_t^l}\right)-k\le 0=\lambda p_t^l\left(\frac{c}{\lambda p_t^h}-\frac{c}{\lambda p_t^h}\right),$$

since type l liquidates when she reveals her type, so that

$$V-\frac{c}{\lambda p_t^h}-\frac{k}{\lambda p_t^l}\le\frac{c}{\lambda p_t^l}-\frac{c}{\lambda p_t^h}.$$

Therefore,[71]

$$\begin{aligned}
& \delta\left(\Pi_t^l(1)-\Pi_t^l(0)\right) \\
= \; & \delta\lambda p_t^l\left(\sum_{s=t}^{\tau^l-1}(\delta(1-\lambda))^{s-t}\left(\frac{c}{\lambda p_s^l}-\frac{c}{\lambda p_s^h}\right)\right. \\
& \left.+\sum_{s=\tau^l}^{\tau^{l\prime}-1}(\delta(1-\lambda))^{s-t}\left(V-\frac{c}{\lambda p_s^h}-\frac{k}{\lambda p_s^l}\right)\right) \\
\leq \; & \delta\lambda p_t^l\left(\frac{c}{\lambda p_t^l}-\frac{c}{\lambda p_t^h}\right)\sum_{s=t}^{\tau^{l\prime}-1}(\delta(1-\lambda))^{s-t} \\
\leq \; & \delta\lambda p_t^l\left(\frac{c}{\lambda p_t^l}-\frac{c}{\lambda p_t^h}\right).
\end{aligned}$$

Second, using a similar derivation, if $\tau^l<\tau^{l\prime}$ or if $t<\tau^l-1$, we have

$$\begin{aligned}
& \delta p_t^h\left(\Pi_t^l(1)-\Pi_t^l(0)\right)-\delta p_{t+1}^h\left(\Pi_{t+1}^l(1)-\Pi_{t+1}^l(0)\right) \\
= \; & \delta\lambda p_t^h p_t^l\left(\frac{c}{\lambda p_t^l}-\frac{c}{\lambda p_t^h}\right)-\left(1-\delta\frac{\left(1-\lambda p_t^h\right)\left(1-\lambda p_t^l\right)}{1-\lambda}\right)\delta p_{t+1}^h\left(\Pi_{t+1}^l(1)-\Pi_{t+1}^l(0)\right) \\
\geq \; & \delta\lambda p_t^h p_t^l\left(\frac{c}{\lambda p_t^l}-\frac{c}{\lambda p_t^h}\right)-\left(1-\delta\frac{\left(1-\lambda p_t^h\right)\left(1-\lambda p_t^l\right)}{1-\lambda}\right)\delta\lambda p_{t+1}^h p_{t+1}^l\left(\frac{c}{\lambda p_{t+1}^l}-\frac{c}{\lambda p_{t+1}^h}\right) \\
= \; & \delta\lambda p_t^h p_t^l\left(\frac{c}{\lambda p_t^l}-\frac{c}{\lambda p_t^h}\right)\frac{1-\lambda}{\left(1-\lambda p_t^h\right)\left(1-\lambda p_t^l\right)}\left(\frac{\left(1-\lambda p_t^h\right)\left(1-\lambda p_t^l\right)}{1-\lambda}-1\right) \\
> \; & 0.
\end{aligned}$$

The inequality comes from Assumption 3. If $\tau^l=\tau^{l\prime}$ and $t=\tau^l-1$, then $\Pi_{t+1}^l(1)-\Pi_{t+1}^l(0)=0$ and the expression is still positive. Together with the previous result, this generates the Lemma statement. ■

Next, we will characterize τ_S^h for two cases: $\tau^l=\tau^{l\prime}$ and $\tau^l<\tau^{l\prime}$.

Lemma 27 *If $\tau^l=\tau^{l\prime}$, there exists a threshold $\bar{q}$ such that the high type always prefers to pool if and only if $q_0\geq\bar{q}$.*

Proof. If $\tau^l=\tau^{l\prime}$, then

$$f_{\tau^l-1}(q_0)=c\left(\frac{(1-q_0)\,p_{\tau^l-1}^h}{q_0 p_{\tau^l-1}^h+(1-q_0)\,p_{\tau^l-1}^l}-1\right).$$

If $q_0>0.5$, the above expression is strictly negative. By continuity, there exists a $\bar{q}<0.5$ such that $f_{\tau^l-1}(q_0)\leq 0$ if and only if $q_0\geq\bar{q}$. Since $f_t(q_0)$ is strictly increasing in t, we have $f_t(q_0)<0$ for all $t<\tau^l-1$. Thus, for any equilibrium with separation in period $t\leq\tau^l-1$, we can increase the

[71] If $\tau^l>\tau^{l\prime}-1$, we use the convention $\sum_{s=\tau^l}^{\tau^{l\prime}-1}(...)=0$.

high type's payoff by separating in period $t+1$ instead. Applying the argument inductively implies that the high type prefers to never separate. ■

Corollary 28 *If $\tau^l = \tau^{l\prime}$ and $q_0 \geq \bar{q}$, the optimal contract is pooling.*

Lemma 29 *If $\tau^l = \tau^{l\prime}$ and $q_0 < \bar{q}$, we have $1 \leq \tau_S^h \leq \tau^l - 1$.*

Proof. Define $\tau_S = \min\{t | f_t(q_0) \geq 0\}$. Since $q_0 < \bar{q}$, this set is non-empty, and we have $\tau_S \leq \tau^l - 1$. ■

Now, consider the case $\tau^l < \tau^{l\prime}$. In Section B.2, we have shown that separating through α_t^h has the same payoff as the pooling equilibrium for the high type when $\tau^l \leq t < \tau^{l\prime}$. Since separating through payouts is cheaper, it is now preferred by the high type.

Lemma 30 *If $\tau^l < \tau^{l\prime}$, the high type strictly prefers separating in period τ^l and we have $\tau_S^h \leq \tau^l$.*

Proof. For all $t \geq \tau^l$, we can replicate the argument of Lemma 25. The difference is that separating in period $t+1$ has the following payoff structure, due to the change of α_t^p:

$$\lambda p_t^h \left(V - \frac{\lambda p_t^l V - k}{\lambda p_t^l} \right) - k + \delta \left(1 - \lambda p_t^h\right) \left(\Pi_{t+1}^h (1) - \left(\Pi_{t+1}^l (1) - \Pi_{t+1}^l (0)\right)\right).$$

Besides,

$$\Pi_t^l (1) - \Pi_t^l (0) = \lambda p_t^l \left(V - \frac{c}{\lambda p_t^h} \right) - k + \delta \left(1 - \lambda p_t^l\right) \left(\Pi_{t+1}^l (1) - \Pi_{t+1}^l (0)\right).$$

The difference in value for type h from separating in period t vs. period $t+1$ is

$$\Pi_t^{h\prime} - \Pi_t^h = \frac{\lambda \left(p_t^h - p_t^l\right)}{\lambda p_t^l} \left(\lambda p_t^l \left(V - \frac{c}{\lambda p_t^h} \right) - k \right) - \delta \lambda \left(p_t^h - p_t^l\right) \left(\Pi_{t+1}^l (1) - \Pi_{t+1}^l (0)\right).$$

Using the same argument as in Lemma 26, we can show

$$\Pi_{t+1}^l (1) - \Pi_{t+1}^l (0) < \frac{1}{\lambda} \left(\lambda p_{t+1}^l \left(V - \frac{c}{\lambda p_{t+1}^h} \right) - k \right) < \frac{1}{\lambda p_t^l} \left(\lambda p_t^l \left(V - \frac{c}{\lambda p_t^h} \right) - k \right).$$

Therefore, $\Pi_t^{h\prime} - \Pi_t^h$ is strictly positive for all $t \geq \tau^l$. ■

We have now established that there exists a period $\tau_S^h \leq \tau^{l'} - 1$ in which the high type prefers to separate. The low type, of course, prefers to never separate. Consider now the optimal contract, which maximizes the weighted average between the low and high type's value. For any weight γ, the optimal separation time satisfies $\tau_S \geq \tau_S^h$. For any earlier period, both the low and high type prefer to continue pooling and thus separating is Pareto dominated. If pooling is optimal, we can set $\tau_S = \bar{\tau}^l$. In that period, the low type liquidates with certainty in the pooling equilibrium, so pooling and separating are equivalent. Since we are optimizing over the finite set $\left\{\tau_S^h, \tau_S^h + 1, ..., \tau^{l\prime}\right\}$, an optimal separating time exists. Clearly, whenever γ is sufficiently small, pooling is optimal while whenever γ is sufficiently large, the optimal separating time satisfies $f_{\tau_S}(q_0) > 0$.

To conclude, we provide a sufficient condition for when $\tau^l = \tau^{l\prime}$.

Lemma 31 *We have $\tau^l = \tau^{l\prime}$ if and only if p_1^l/p_1^h exceeds a threshold.*

Proof. Given p_1^l, τ^l is pinned down as $\min\{t|\lambda p_t^l V - p_t^l c/p_t(q_0) - k \leq 0\}$. $\tau^l = \tau^{l\prime}$ implies

$$p_{\tau^l}^h \leq \frac{p_{\tau^l}^l c}{\lambda p_{\tau_l}^l V - k} = \tilde{p}_{\tau^l}^h.$$

Both $p_{\tau_l}^l$ and τ_l are determined given p_1^l. Therefore, we can determine an initial belief $\tilde{p}_1^h$ which is consistent with $\tilde{p}_{\tau^l}^h$ through backward induction using Bayes' rule:

$$p_t^h = \frac{p_{t+1}^h}{1 - \lambda + \lambda p_{t+1}^h}.$$

Now, we can pick $p_1^l/\tilde{p}_1^h$ sufficiently large (or, equivalently $\tilde{p}_1^h$ sufficiently small) to ensure the above inequality holds. ■

B.5 Proof of Proposition 5

Since the proof is similar to the proof of Proposition 4, we only provide a sketch.

Lemma 32 *With observable types, the high type pivots in period τ^h, which is the first period for which*

$$\Pi_1^h(1) - F \geq \Pi_t^h(1) \tag{36}$$

while the low type pivots in period τ^l,which is the first period for which

$$\Pi_1^l(0) - F \geq \Pi_t^l(0). \tag{37}$$

That τ^h and τ^l satisfy Equations (36) and (37) is straightforward, and we omit the proof.

Lemma 33 *In the optimal pooling equilibrium, the low type liquidates with positive probability l_t^l whenever $\underline{\tau}^l \leq t \leq \bar{\tau}^l$. The equity share is given by*

$$\alpha_t^P = \frac{c}{\lambda p_t(q_t) V}.$$

The period $\underline{\tau}^l$ is the first period in which

$$\Pi_t^l = \Pi_1^l(0) - F.$$

The proof is analogous to the proof of Proposition 2 and hence omitted. The only difference is that when the low type liquidates with positive probability, her value satisfies

$$\lambda p_t^l \left(1 - \alpha_t^P\right) V - k + \delta \left(1 - \lambda p_t^l\right) \left(\Pi_1^l(0) - F\right) = \Pi_1^l(0) - F$$

and the pooling share α_t^P is now given by

$$\alpha_t^P = \frac{\lambda p_t^l V - k}{\lambda p_t^l V} - \frac{\Pi_1^l(0) - F}{\lambda p_t^l V} \frac{1 - \delta\left(1 - \lambda p_t^l\right)}{1 - \delta}$$

for $\underline{\tau}^l \leq t \leq \bar{\tau}^l$.

The following Lemma allows us to restrict attention to contracts which separate before period τ^l.

Lemma 34 *We have $\tau^l \leq \underline{\tau}^l$. Separating via pivots is not incentive compatible in any period $t \geq \tau^l$.*

Proof. That $\tau^l \leq \underline{\tau}^l$ follows from the fact that $\Pi_t^l \geq \Pi_t^l(0)$ for all t. We have for $t \geq \tau^l$, $\Pi_1^l(1) - F \geq \Pi_1^l(0) - F \geq \Pi_t^l(0)$, which implies that separating in period $t \geq \tau^l$ is not incentive compatible. ■

We next show that the set $\left[\Pi_1^l(1) - \Pi_t^l(0), \Pi_1^h(1) - \Pi_t^h(0)\right]$ in Equation (16) is nonempty and characterize how this set changes over time.

Lemma 35 *For δ sufficiently small, $\Pi_t^h(1) - \Pi_t^l(1)$ is decreasing in t and for all t, we have*

$$\Pi_1^l(1) - \Pi_t^l(0) \leq \Pi_1^h(1) - \Pi_t^h(0)$$

Proof. We have[72]

$$\begin{aligned} \Pi_t^h(1) - \Pi_t^l(1) &= \sum_{s=t}^{\tau_h} \delta^{(s-t)} \left[\prod_{t \leq u < s} \left(1 - \lambda p_u^h\right)\right] \left(\lambda p_s^h \left(1 - \bar{\alpha}_s^h\right) V - k\right) \\ &\quad - \sum_{s=t}^{\tau_l'} \delta^{(s-t)} \left[\prod_{t \leq u < s} \left(1 - \lambda p_u^l\right)\right] \left(\lambda p_s^l \left(1 - \bar{\alpha}_s^h\right) V - k\right). \end{aligned}$$

Since $\tau_l' \leq \tau_h$, we have for any $t \leq \tau_l'$

$$\Pi_t^h(1) - \Pi_t^l(1) = \lambda\left(p_t^h - p_t^l\right)\left(1 - \bar{\alpha}_t^h\right) V + \delta\left(\left(1 - \lambda p_t^h\right)\Pi_{t+1}^h(1) - \left(1 - \lambda p_{t+1}^l\right)\Pi_{t+1}^l\right).$$

As δ becomes small, this expression converges to

$$\Pi_t^h(1) - \Pi_t^l(1) = \lambda\left(p_t^h - p_t^l\right)\left(1 - \bar{\alpha}_t^h\right) V.$$

Then,

$$\Pi_t^h(1) - \Pi_t^l(1) - \left(\Pi_t^h(1) - \Pi_{t+1}^l(1)\right) = \lambda V\left(p_t^h - p_t^l - \left(p_{t+1}^h - p_{t+1}^l\right)\right) + c\left(\frac{p_t^l}{p_t^h} - \frac{p_{t+1}^l}{p_{t+1}^h}\right).$$

This expression is positive, because by Assumption 3, $p_t^h - p_t^l$ is decreasing in t and because p_t^l / p_t^h is also decreasing in t, which can be seen from Bayes' rule.

[72]Recall that τ_l' is defined as the time at which the low type liquidates when the investor's belief is $q_t = 1$.

To show that

$$\Pi_1^l(1) - \Pi_t^l(0) \le \Pi_1^h(1) - \Pi_t^h(0),$$

we can rearrange this expression as

$$\Pi_t^h(0) - \Pi_t^l(0) \le \Pi_1^h(1) - \Pi_1^l(1).$$

We have $\Pi_t^h(0) - \Pi_t^l(0) \le \Pi_t^h(1) - \Pi_t^l(1)$, which follows from a similar argument as in Lemma 21. Thus, a sufficient condition for the inequality to hold is that

$$\Pi_t^h(1) - \Pi_t^l(1) \le \Pi_1^h(1) - \Pi_1^l(1),$$

which in turn holds if $\Pi_t^h(1) - \Pi_t^l(1)$ is decreasing in t, which we just established. ■

Corollary 36 *For F sufficiently small, there exists a period t, such that*

$$F \in \left[\Pi_1^l(1) - \Pi_t^l(0), \Pi_1^h(1) - \Pi_t^h(0)\right].$$

This result implies that there are two periods $1 \le \underline{\tau}_{Piv} \le \bar{\tau}_{Piv}$, such that separating via a pivot is feasible whenever $\underline{\tau}_{Piv} \le t \le \bar{\tau}_{Piv}$. Intuitively, for $t < \underline{\tau}_{Piv}$, separation is not feasible, because it is too costly for the high type. As t increases beyond $\bar{\tau}_{Piv}$, we eventually have $F < \Pi_1^l(1) - \Pi_t^l(0)$, so that separation is not incentive-compatible, because the low type would pivot together with the high type.

Lemma 37 *For δ and F sufficiently small, there exists a period $\underline{\tau}_{Piv} \le \tau_S^h \le \bar{\tau}_{Piv}$, such that the high type prefers to separate by pivoting in period τ_S^h and prefers to continue pooling for all $t < \tau_S^h$.*

Proof. Lemma 34 implies that separation is not incentive compatible after period τ^l. Thus, we can wlog restrict attention to the case when $\bar{\tau}_{Piv} \le \tau^l \le \underline{\tau}^l$. This implies that at any time at which we consider separation, we have $q_t = q_0$.

If the high type separates in period t,her payoff is $\Pi_1^h(1) - F$, while if she pools in period t and separates in period $t+1$, her payoff is

$$\lambda p_t^h \left(1 - \alpha_t^P\right) V - k + \delta\left(1 - \lambda p_t^h\right)\left(\Pi_1^h(1) - F\right).$$

The high type prefers separating in period t rather than in period $t+1$ whenever

$$f_t := \left(1 - \delta\left(1 - \lambda p_t^h\right)\right)\left(\Pi_1^h(1) - F\right) - \left(\lambda p_t^h\left(1 - \alpha_t^P\right)V - k\right) \ge 0.$$

Since we have $t \le \underline{\tau}^l$, we can write

$$\begin{aligned}
\lambda p_t^h\left(1 - \alpha_t^P\right)V &= \lambda p_t^h V - c\frac{p_t^h}{p_t(q_0)} \\
&= \lambda p_t^h V - c\frac{1}{q_0 + (1 - q_0)\frac{p_t^l}{p_t^h}},
\end{aligned}$$

which is strictly decreasing in t, because both p_t^h and p_t^l/p_t^h are strictly decreasing. For δ sufficiently small, f_t is then strictly increasing in t and crosses zero at most once. Let τ_S^h denote the crossing time. If τ_S^h does not exist, then the high type prefers to never separate via a pivot.

We now show that τ_S^h exists. Pick a period t such that both Equation (IC_h^{Piv}) and Equation (IC_l^{Piv}) hold. By Corollary 36, such a period exists whenever F is sufficiently small. Using Equation (IC_l^{Piv}), we have

$$\begin{aligned} f_t \quad \geq \quad & \left(1-\delta\left(1-\lambda p_t^h\right)\right)\left(\Pi_1^h(1)-\left(\Pi_1^l(1)-\Pi_t^l(0)\right)\right) \\ & -\left(\lambda p_t^h\left(1-\alpha_t^P\right)V-k\right). \end{aligned}$$

As δ becomes small, the RHS approaches

$$\lambda\left(p_1^h-p_1^l\right)\left(1-\bar{\alpha}_1^h\right)V-\lambda\left(p_t^h-p_t^l\right)V+c\left(\frac{p_t^h}{p_t\left(q_0\right)}-1\right).$$

By Assumption 3, $p_t^h-p_t^l$ is decreasing. Thus, the expression is positive, which implies that $f_t \geq 0$ and that τ_S^h exists.

That τ_S^h is the optimal time to separate for the high type follows from a similar argument as in the proof of Proposition 4. Specifically, for $t<\tau_S^h$, the high type prefers to separate later, while for $t>\tau_S^h$, the prefers to separate earlier. Thus, separating in period τ_S^h maximizes her ex-ante value. ■

The following Lemma concludes our argument.

Lemma 38 *For γ sufficiently large, the optimal contract features separation via pivots.*

The proof follows the same lines as the proof of Proposition 4. The low type prefers to never separate via a pivot. Instead, she prefers to only pivot in period τ^l. The high type prefers separating via a pivot in period τ_S^h. For γ large, there exists a period τ_S such that $\underline{\tau}_{Piv} \leq \tau_S \leq \bar{\tau}_{Piv}$, so that pivoting in period τ_S is optimal.

B.6 Proof of Proposition 6

The proof of Proposition 6 is similar to the proof of Proposition 4. We hence only provide a sketch.

Under symmetric information, each type liquidates at time τ^θ. We have $\tau^l \leq \tau^h$. Each type implements the prestige project at the time of liquidation, since by doing so she receives a higher outside option and does not need to suffer the decrease in project value. Thus, each type liquidates at the first time at which

$$\Pi_t^\theta \leq \pi.$$

A pooling equilibrium exists and has the same features as the equilibrium in Proposition 2. Specifically, the equity share is still given by

$$\alpha_t^P=\frac{c}{\lambda p_t\left(q_t\right)V},$$

and there exist two times $\underline{\tau}^l \leq \bar{\tau}^l$, such that in any period between $\underline{\tau}^l$ and $\bar{\tau}^l$, the low type randomizes between implementing the prestige project and liquidating or continuing. In any such period, the following indifference condition holds:

$$\lambda p_t^l \left(1-\alpha_t^P\right) V - k + \delta \left(1-\lambda p_t^l\right) \pi = \pi. \tag{38}$$

This equation is the analog of Equation (23) in the baseline model. It implies that the pooling equity share satisfies

$$\alpha_t^P = \frac{\lambda p_t^l V - k}{\lambda p_t^l V} - \pi \frac{1-\delta\left(1-\lambda p_t^l\right)}{\lambda p_t^l}. \tag{39}$$

We now consider a separating equilibrium in period t. Using the IC conditions for the low and high type, Equations (IC_l^{Pres}) and (IC_h^{Pres}), we can see that separating is incentive compatible whenever

$$\frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l} \le \lambda V_0 \le \frac{\Pi_t^h(1) - \Pi_t^h(0)}{p_t^h}.$$

A similar argument as in Lemma 21 implies that

$$\frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l} \le \frac{\Pi_t^h(1) - \Pi_t^h(0)}{p_t^h}.$$

Thus, for any period t, there exists a V_0, such that we can achieve separation in that period. It only remains to find conditions such that separation is optimal.

Following a similar argument as in Lemma 25, consider the high type's value from separating in period t versus separating in period $t+1$. If the high type separates in period $t < \bar{\tau}^l - 1$, her value is

$$\Pi_t^h(1) - \lambda p_t^h V_0,$$

while if she separates in period $t+1$, her value in period t is

$$\lambda p_t^h \left(1-\alpha_t^P\right) V - k + \delta\left(1-\lambda p_t^h\right)\left(\Pi_{t+1}^h(1) - (1-\delta)\lambda p_{t+1}^h V_0\right).$$

After some algebra, the difference between these two values is

$$f_t = c\left(\frac{p_t^h}{p_t(q_t)} - 1\right) - \lambda p_t^h V_0 \left(1-\delta(1-\lambda)\right).$$

Whenever this expression is positive, the high type prefers separating in period t rather than in period $t+1$.

For $t = \bar{\tau}^l - 1$, the high type knows that the low type will liquidate in period $\bar{\tau}^l$. Thus, separating in period $\bar{\tau}^l$ does not require costly signaling, and we have

$$f_{\bar{\tau}^l-1} = c\left(\frac{p_{\bar{\tau}^l-1}^h}{p_{\bar{\tau}^l-1}\left(q_{\bar{\tau}^l-1}\right)} - 1\right) - \lambda p_{\bar{\tau}^l-1}^h V_0$$

Next, we find a sufficient condition, such that $f_t > 0$. Consider the region $\underline{\tau}^l \le t \le \bar{\tau}^l$. On this region, the low type will liquidate if her type is discovered. Thus, the low type's IC condition becomes

$$\Pi_t^l(1) - \lambda p_t^l V_0 \le \pi,$$

since we have $\Pi_t^l(0) = \pi$. Let us pick V_0 such that the above inequality binds. Then, the high type prefers separating in period t over separating in the next period whenever

$$\hat{f}_t = c\left(\frac{p_t^h}{p_t(q_t)} - 1\right) - \frac{p_t^h}{p_t^l}\left(\Pi_t^l(1) - \pi\right)(1 - \delta(1-\lambda)) \geq 0$$

for $t < \bar{\tau}^l - 1$ or

$$\hat{f}_{\bar{\tau}^l-1} = c\left(\frac{p_{\bar{\tau}^l-1}^h}{p_{\bar{\tau}^l-1}(q_{\bar{\tau}^l-1})} - 1\right) - \frac{p_{\bar{\tau}^l-1}^h}{p_{\bar{\tau}^l-1}^l}\left(\Pi_{\bar{\tau}^l-1}^l(1) - \pi\right) \geq 0.$$

Here is the significance of $\hat{f}_t$. Whenever $\hat{f}_t$ is positive, there exists a V_0, such that f_t is positive. Thus, $\hat{f}_t > 0$ is a necessary and sufficient condition for the existence of a V_0 such that the high type prefers to separate in period t rather than in period $t+1$.

Pick period $t = \bar{\tau}^l - 1$. The low type liquidates in period $t + 1 = \bar{\tau}^l$, even if the investor's belief is $q_{t+1} = 1$. Thus, we have

$$\Pi_{\bar{\tau}^l-1}^l(1) = \lambda p_{\bar{\tau}^l-1}^l V - c\frac{p_{\bar{\tau}^l-1}^l}{p_{\bar{\tau}^l-1}^h} - k + \delta\left(1 - \lambda p_{\bar{\tau}^l-1}^l\right)\pi.$$

Plugging this expression into $\hat{f}_t$, we get,

$$\begin{aligned}
\hat{f}_{\bar{\tau}^l-1} &= -\frac{p_{\bar{\tau}^l-1}^h}{p_{\bar{\tau}^l-1}^l}\left(\lambda p_{\bar{\tau}^l-1}^l V - c\frac{p_{\bar{\tau}^l-1}^l}{p_{\bar{\tau}^l-1}(q_{\bar{\tau}^l-1})} - k + \delta\left(1 - \lambda p_{\bar{\tau}^l-1}^l\right)\pi - \pi\right) \\
&= -\frac{p_{\bar{\tau}^l-1}^h}{p_{\bar{\tau}^l-1}^l}\left(\Pi_{\bar{\tau}^l-1}^l - \pi\right) = 0,
\end{aligned}$$

where $\Pi_{\bar{\tau}^l-1}^l$ is the low type's value in period $\bar{\tau}^l - 1$ if there is pooling. Since $\bar{\tau}^l - 1 \geq \underline{\tau}^l$, we have $\Pi_{\bar{\tau}^l-1}^l = \pi$.

For any $\underline{\tau}^l \leq t < \bar{\tau}^l - 1$, we can write

$$\begin{aligned}
\hat{f}_t &= c\left(\frac{p_t^h}{p_t(q_t)} - 1\right) - \frac{p_t^h}{p_t^l}\left(\Pi_t^l(1) - \pi\right)(1 - \delta(1-\lambda)) \\
&= c\left(\frac{p_t^h}{p_t(q_t)} - 1\right) - \frac{p_t^h}{p_t^l}\left(\lambda\left(p_t^l V - c\frac{p_t^l}{p_t^h} - k\right) + \delta\left(1 - \lambda p_t^l\right)\Pi_{t+1}^l(1) - \pi\right) \\
&\quad +\delta(1-\lambda)\frac{p_t^h}{p_t^l}\left(\Pi_t^l(1) - \pi\right) \\
&= -\frac{p_t^h}{p_t^l}\left(\lambda p_t^l V - c\frac{p_t^l}{p_t(q_t)} - k + \delta\left(1 - \lambda p_t^l\right)\Pi_{t+1}^l(1) - \pi\right) \\
&\quad +\delta(1-\lambda)\frac{p_t^h}{p_t^l}\left(\Pi_t^l(1) - \pi\right) \\
&= \delta\frac{p_t^h}{p_t^l}\left((1-\lambda)\left(\Pi_t^l(1) - \pi\right) - \left(1 - \lambda p_t^l\right)\left(\Pi_{t+1}^l(1) - \pi\right)\right),
\end{aligned}$$

since Equation (38) implies that

$$\lambda p_t^l V - c\frac{p_t^l}{p_t(q_t)} - k - \pi = -\delta\left(1 - \lambda p_t^l\right)\pi.$$

Now, we pick $t = \bar{\tau}^l - 2$ and we pick $\pi \leq \Pi_{\bar{\tau}^l-1}^l(1)$, arbitrarily close to $\Pi_{\bar{\tau}^l-1}^l(1)$. Then, we have $\hat{f}_{\bar{\tau}^l-2} > 0$ and $\hat{f}_{\bar{\tau}^l-1} = 0$. Thus, there exists a V_0 such that the two IC conditions in Equation (IC_l^{Pres}) and Equation (IC_h^{Pres}) hold in period $\bar{\tau}^l - 2$ and such that $f_{\bar{\tau}^l-2} > 0$. This implies that the high type prefers separating in period $\bar{\tau}^l - 2$ over separating in any later period. This, in turn, implies that there exists a period $\tau_S^h \leq \bar{\tau}^l - 2$ at which the high type prefers to separate.

As before, the low type strictly prefers to pool until period $\bar{\tau}^l$. Thus, for γ sufficiently large, there exists an optimal separation period $\tau_S < \bar{\tau}^l - 1$ which maximizes the entrepreneur's ex-ante value.