Information-Driven Volatility

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Abstract: Modern asset pricing theory predicts an unambiguously positive relation between volatility and expected returns. Empirically, however, realized volatility often predicts expected returns with a negative sign, as exemplified by the volatility-managed portfolios of Moreira and Muir (2017). We show that information driven volatility induces a negatively correlation between past realized volatility and future volatility and future expected returns. We develop a simple asset pricing model based on this intuition and demonstrate that our model can account for several volatility-related asset pricing puzzles such as the return on volatility managed portfolios, the “variance risk premium” return predictability (Bollerslev, Tauchen, and Zhou (2009)), and the predictability of returns by implied volatility reduction on macroeconomic announcement days.

Keywords: volatility managed portfolios, variance risk premium, macroeconomic announcements, generalized risk sensitivity

JEL Code: D83, D84, G11, G12, G14

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†This is a preliminary draft. Please do not circulate.
1 Introduction

The relationship between volatility and expected returns, or the risk and return trade-off is a fundamental concept in finance. Most existing theories model time-varying volatility by assuming stochastic volatility of macroeconomic fundamentals, such as stochastic volatility in aggregate consumption. However, several empirical relationships between volatility and expected returns on financial markets prove to be puzzling from this traditional viewpoint of stochastic volatility. First, measured volatility, either implied volatility or realized volatility, has at best very weak predictive powers for future stock market returns, while standard theory implies that risk premium should be proportional to volatility. Second, as shown by Moreira and Muir (2017), volatility managed portfolios that take less risk when realized volatility is high and more risk when realized volatility is low produces large excess returns relative to the market. This is also contradictory to the textbook positive relationship between volatility and expected returns. Third, although neither implied nor realized volatility has strong predictive powers for returns, the difference between the two, often called the variance risk premium, has strong predictive powers for returns over the three-to-six-month horizon. Finally, the magnitude of drops in implied volatility on macroeconomic announcement days are positively correlated with contemporaneous returns on macroeconomic announcement days but negatively predicts returns in the future. The main purpose of this paper is to demonstrate that this mechanism can explain many puzzling facts about the volatility and expected return relationship in financial markets.

In this paper, we develop an asset pricing model where stochastic volatility originates not from changes in the volatility of macroeconomic fundamentals but variations of the informativeness of publicly available information events, such as macroeconomic announcements. To illustrate the difference between our theory and traditional models of stochastic volatility of macroeconomic fundamentals, we consider the following variance decomposition identity:

\[ \text{Var} \left[ X \right] = \text{Var} \left[ E \left( X | Y \right) \right] + E \left[ \text{Var} \left( X | Y \right) \right]. \quad (1) \]

We interpret \( X \) as macroeconomic fundamentals such as aggregate consumption and \( Y \) as public signals that are informative about \( X \), for example, macroeconomic announcements. Traditional models of stochastic volatility generate variations in volatility through changes in \( \text{Var} \left[ X \right] \). Our theory focuses on variations in the informativeness of macroeconomic news, that is, the \( E \left[ \text{Var} \left( X | Y \right) \right] \) term.

There are several motivations for our focus on the time-variation of the informativeness of macroeconomic news. First, the volatility puzzles we focus on mostly happen at monthly or higher frequencies. The volatility managed portfolios of Moreira and Muir (2017) re-balance every month. The variance risk premium predictability is about returns over a three-to-six-month horizon, and the announcement return predictability is at the daily frequency. Empirically, variations in the volatility of macroeconomic fundamentals operate at a much lower frequency. Evidence for the time variations in the volatility of aggregate consumption, investment, and output are often documented...
at the annual or lower frequencies. Therefore, variations in macroeconomic volatility are unlikely to be the only factor responsible for the volatility and return relationship at higher frequencies, such as the daily and monthly frequencies. In contrast, information arrives at the financial markets continuously and affects the volatility at much higher frequencies.

Second, traditional models with time-varying volatility in macroeconomic fundamentals and our model with time-varying information-driven volatility have drastically different implications on the relationship between realized volatility and expected returns. From a theoretical perspective, changes in $\text{Var}[X]$ affect the total quantity of risk. The standard textbook formula implies a positive relationship between volatility and expected return. Because volatility shocks are typically persistent, a high realization of $\text{Var}[X]$ will imply a high expected return going forward. However, this is precisely where this theory has difficulty in explaining the volatility-return relationship obtained at relatively high frequencies. By contrast, changes in $\text{Var}[E(X|Y)]$ affect the intertemporal distribution of risk and risk compensation. Holding the total amount of risk constant, a higher realization of $\text{Var}[E(X|Y)]$ is associated with a larger realization of the risk premium contemporaneously, but also implies a lower quantity of risk, $E[\text{Var}(X|Y)]$, and therefore risk compensation in the future, when combined with preferences that satisfy the property of generalized risk sensitivity of Ai and Bansal (2018).

To highlight the mechanism of information-driven volatility, we develop a parsimonious asset pricing model with homoscedasticity. In our model, the volatility of aggregate consumption is constant, and yet the financial market exhibits stochastic volatility because the informativeness of public signals is time varying. In this setup, high realized financial market volatility is due to a highly informative event and leads to lower uncertainty and a lower equity market risk compensation in the future. This mechanism can explain many of the documented puzzling facts about the short-run relationship between realized volatility and expected returns.

First, periods of high realized volatility is typically associated with lower expected returns in the future. This means that the volatility managed portfolio, a portfolio that takes more aggressive position on the market portfolio when realized volatility is low and invests more in the risk-free bond when market volatility is high will earn an extra return than the market due to the negative relationship between realized volatility and expected returns in the future.

Second, the difference between implied and realized variance reflects the informativeness of the upcoming informative event. It predicts returns because, holding realized variance constant, the release of a highly informative signal is associated with both a high implied variance before the event and the realization of larger risk compensation associated with the event. If the upcoming public signals are expected to be informative, the implied variance will rise, but realized variance stays the same. The empirically documented variance risk premium predictability can therefore be explained by the information channel without assuming high-frequency movements in macroeconomic volatility.

Third, the magnitude of reductions in implied variance across macroeconomic announcements is associated with higher realizations of the announcement premium and lower expected returns going
forward. When macroeconomic announcements are expected to be informative, the announcement premium is higher due to a higher uncertainty reduction captured by $\text{Var} \left[ E(X|Y) \right]$. Our calibrated model matches the above fact quite well.

It is important to emphasize that we do not intend to argue that time-varying macroeconomic volatility is absent or unimportant for understanding equity market risk compensations. Our purpose is to distinguish two notions of uncertainty: the variance of macroeconomic fundamentals and the variance of investors’ posterior beliefs. We argue that variations in the posterior beliefs affect the intertemporal distribution of risk and risk compensation and are probably more important in understanding short-horizon risk compensations in financial markets. Changes in the variance of macroeconomic fundamentals affect the total quantity of risk will undoubtedly have prominent impacts on risk and risk compensation, but the effects are likely to manifest only over longer horizons.


This paper is also closely related to the literature on generalized risk sensitivity and macroeconomic announcements. From the empirical perspective, pre-scheduled macroeconomic announcements are the most salient information events that are associated with significant realizations of market equity premiums and affect the volatility of stock market returns. The literature that documents a significant macroeconomic announcement premium, for example, Savor and Wilson (2013, 2014) and Lucca and Moench (2015) provides strong empirical support for the mechanism emphasized in this paper. From the theoretical point of view, Ai and Bansal (2018) demonstrates that the existence of announcement premium implies generalized risk sensitivity in preferences. In our setup, as we will demonstrate in Section 3 of the paper, generalized risk sensitivity is also necessary for information quality to affect the intertemporal distribution of risk compensation.

The empirical literature provides mixed evidence on the expected return-volatility relationship. French, Schwert, and Stambaugh (1987) and Campbell and Hentschel (1992) conclude that the data are consistent with a positive relation between conditional expected excess return and conditional variance, while Nelson (1991) and Glosten, Jagannathan, and Runkle (1993) find a negative relationship. Harvey (1989) provides evidence for the time variations in the relationship between expected excess returns and conditional variances, and Chan, Karolyi, and Stulz (1992) argue that there is no significant variance effect on expected returns. We demonstrate that variations in the volatility of macroeconomic fundamentals give rise to a positive relationship between realized
volatility and expected returns, but variations in information-driven volatility result in a negative relationship between past realized volatility and future expected returns. These offsetting effects can potentially explain the failure of the above empirical literature in finding a robustly positive relationship between variance and expected returns.

Several recent papers provide empirical evidence that is consistent with the information effect emphasized in this paper. Baker, Bloom, Davis, Kost, Sammon, and Viratyosin (2020) document that a higher clarity of news is associated with lower realized volatility in the future. Zhang and Zhao (2020) provide evidence that in periods where public information is imprecise, the realized macroeconomic announcement premium is low. Chaudhry (2021) shows that a higher implied volatility reduction on macroeconomic announcement days is associated with a lower expected return going forward.

Our paper is considerably related to the variance risk premium predictability literature. Bollerslev, Tauchen, and Zhou (2009) document the predictability of stock market returns by the difference between implied and realized variance, and develop a model of variance risk premium predictability based on stochastic volatility in the volatility of macroeconomic fundamentals. Drechsler and Yaron (2011) develop a model with stochastic volatility and stochastic jumps to quantitatively explain the variance risk premium predictability. Eraker and Wang (2015) estimate a non-linear diffusion model and study the variance risk premium predictability. Zhou (2018) provides a thorough review of this literature. The above literature has interpreted the difference between implied and realized variance as the difference between variance under the physical measure and that under the risk neutral measure and hence defined as the variance risk premium. We show that the difference between implied and realized variance can predict returns without assuming variance risk premium. In our model, the difference between the two reflects the informativeness of the upcoming announcement. It predicts returns because in our model, resolution of uncertainty is associated with realizations of risk premium.

The rest of the paper is organized as follows. In Section 2, we summarize the stylized facts between realized variance and expected returns. We present a simple two-period model to illustrate the impact of informativeness of macroeconomic news on the intertemporal distribution of risk and risk compensation in Section 3. Section 4 develops a dynamic model to account for the stylized facts, and Section 5 presents the quantitative results. Section 6 concludes.

2 Stylized Facts

In this section, we provide details on several stylized facts on volatility and stock market returns that motivate the development of our theory.

1. The volatility of macroeconomic fundamentals do not exhibit significant variations at the monthly or annual frequency.

In Figure 1, we plot the time series of monthly consumption growth and monthly stock market returns in the top panel. In the bottom panel, we plot the estimated conditional volatility
The top panel is the monthly consumption growth rates (solid line) and the S&P500 index returns (dash-dotted line) during the period of 1960.02-2019.12. The bottom panel is the estimated conditional volatility of the two series from a GARCH(1,1) model during the same sample period.

of the two time series from a GARCH (1,1) model. Compared to stock market returns, the variations of consumption growth are much smaller. The estimated conditional volatility of stock returns exhibits sharp variations over the monthly horizon, while that of aggregate consumption growth is virtually flat by comparison, as shown in the bottom panel.

2. Strategies that take more leverage when volatility is low and take less market risk exposure when volatility is high produces large alphas. This is contradictory to the positive relationship between volatility and expected returns predicted by standard theories.

In Figure 2, we follow Moreira and Muir (2017) and construct a volatility managed portfolio
Figure 2: The volatility managed portfolio

This figure plots the return on a volatility managed portfolio (solid line) and the return on the buy-and-hold market portfolio (dashed line) during the period of 1926.07-2015.12.

that is rebalanced every month according to past-month realized volatility. Consistent with their result, we show that a volatility managed market portfolio produces an average annual return of 9.54% per year from 1926 through 2015, while the average market return on a buy-and-hold strategy is 7.75% per year during the same period.

3. Even though realized market volatility or implied volatility do not seem to predict returns themselves, the difference between the two has been documented to have strong predictive powers for returns over three-to-six month horizons. This is the well-known variance risk premium predictability (Bollerslev, Tauchen, and Zhou (2009)). In Table 1, we report results of the following standard VRP predictability regression:

$$R_{t,t+\Delta} = \alpha + \beta [IV_t - RV_{t-21,t}] + \varepsilon_{t,t+\Delta},$$

where $R_{t,t+\Delta}$ is the cumulative market return from time $t$ to time $t + \Delta$, where $\Delta = 21, 42, 63, 84, 105, 126$ days. $IV_t$ is the forward-looking 30-day implied volatility (VIX index) at time $t$, and $RV_{t-21,t}$ is the past 30-day realized volatility. The regression coefficients are statistically significant and increasing up to six months.
Table 1: Return predictability by $IV - RV$

<table>
<thead>
<tr>
<th>Number of days</th>
<th>21</th>
<th>42</th>
<th>63</th>
<th>84</th>
<th>105</th>
<th>126</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IV_t - RV_t$</td>
<td>0.03</td>
<td>0.04</td>
<td>0.08</td>
<td>0.14</td>
<td>0.10</td>
<td>0.09</td>
</tr>
<tr>
<td>$(2.95)$</td>
<td>$(1.64)$</td>
<td>$(2.74)$</td>
<td>$(5.27)$</td>
<td>$(2.46)$</td>
<td>$(1.82)$</td>
<td></td>
</tr>
<tr>
<td>$R^2$ (%)</td>
<td>1.84</td>
<td>2.06</td>
<td>4.95</td>
<td>10.21</td>
<td>4.34</td>
<td>2.57</td>
</tr>
</tbody>
</table>

This table presents the results of the return predictability regression (2). Columns 2 to 6 represent returns on the left hand side of (2) with $\Delta = 1, 2, 3, 4, 5, 6$ months. Newey-West t-statistics are in parentheses.

4. Implied variance typically drops after pre-scheduled macroeconomic announcements, but occasionally increases after the announcements. The magnitude of implied variance reduction around the announcements is positively correlated with announcement returns and negatively correlated with returns in the future. In Table 2, we regress realized variance and realized returns on implied variance reduction on the previous FOMC announcement day:

$$RV_{t,t+\Delta} = \alpha_{RV} + \beta_{RV,1} [IV_t^- - IV_t^+] + \beta_{RV,2} RV_{t-1,t} + \varepsilon_{t,t+\Delta}; \quad (3)$$

$$R_{t,t+\Delta} = \alpha_R + \beta_{R,1} [IV_t^- - IV_t^+] + \beta_{R,2} RV_{t-1,t} + \varepsilon_{t,t+\Delta}.$$

In the above equations, $RV_{t,t+\Delta}$ is the realized variance from time $t$ to time $t + \Delta$, $R_{t,t+\Delta}$ is the cumulative return from $t$ to $t + \Delta$, for $\Delta = 1, 2, 3, 4, 5$ days. The term $IV_t^- - IV_t^+$ is the implied variance reduction on an FOMC announcement day, $t$. Note that the magnitude of implied variance reduction negatively predicts future realized returns and future realized variance.

Table 2: Return and variance predictability by $IV$ reduction

<table>
<thead>
<tr>
<th>Number of days</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$RV_{t,t+\Delta}$</td>
<td>-0.12</td>
<td>-0.08</td>
<td>-0.05</td>
<td>-0.05</td>
<td>-0.05</td>
</tr>
<tr>
<td>$(\times 3.74)$</td>
<td>$(\times 3.50)$</td>
<td>$(\times 2.46)$</td>
<td>$(\times 2.63)$</td>
<td>$(\times 2.76)$</td>
<td></td>
</tr>
<tr>
<td>$R^2$ (%)</td>
<td>0.66</td>
<td>0.59</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>$R_{t,t+\Delta}$</td>
<td>-0.03</td>
<td>-0.02</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
</tr>
<tr>
<td>$(\times 1.17)$</td>
<td>$(\times 1.78)$</td>
<td>$(\times 2.25)$</td>
<td>$(\times 2.10)$</td>
<td>$(\times 1.59)$</td>
<td></td>
</tr>
<tr>
<td>$R^2$ (%)</td>
<td>0.05</td>
<td>0.05</td>
<td>0.04</td>
<td>0.09</td>
<td>0.02</td>
</tr>
</tbody>
</table>

This table presents the results of the return predictability regression (3). Columns 2 to 6 represent the horizon of variance and returns on the left hand side of Equation (3) with $\Delta = 1, 2, 3, 4, 5$ days. Newey-West t-statistics are in parentheses.

### 3 A Three-Period Model

In this section, we present a three-period model to illustrate the mechanism through which information affects the intertemporal distribution of risk and risk compensation. Our example, as
illustrated in Figure 3, is adapted from the two-period model of Ai and Bansal (2018). For simplicity, we assume both $C_{-1}$ and $C_0$ are deterministic. Consumption in period 1, $C_1$, follows a lognormal distribution with $\ln C_1 \sim \mathcal{N}(\mu, \sigma^2)$. At time 0, there is a public announcement that provides a noisy signal about $\ln C_1$ of the form $s = \ln C_1 + \varepsilon$ with $\text{Var} [\varepsilon] = \sigma^2$.

Figure 3: A three-period model

This figure illustrates the timing of the two-period model. Consumptions in period $-1$ and that in period 0 are constant, whereas the consumption in period 1 follows a lognormal distribution.

In the lognormal setup, we can explicitly compute $\text{Var} [E [\ln C_1 | s]]$ and $E [\text{Var} [\ln C_1 | s]]$ in the variance decomposition formula of Equation (1). The expressions of the stochastic discount factors are also standard: $\Lambda_{-1,0} = \frac{\beta}{1-\beta} \left( \frac{C_0}{C_{-1}} \right)^{-1} \left( \frac{V_0}{m_{-1}} \right)^{1-\gamma}$ and $\Lambda_{0,1} = \frac{\beta}{1-\beta} \left( \frac{C_1}{C_0} \right)^{-1} \left( \frac{C_1}{m_0} \right)^{1-\gamma}$, where $\Lambda_{t,t+1}$ denotes the stochastic discount factor that prices date-$t+1$ consumption goods in terms of date-$t$ consumption goods. Here, we use $V_0 = C_0^{1-\beta} \left( E \left[ C_1^{1-\gamma} \right] \right)^{1-\gamma}$ for the date-0 utility of the agent. $m_0 = \left( E \left[ C_1^{1-\gamma} \right] \right)^{1-\gamma}$ is the date-0 certainty equivalent of future utility and $m_{-1} = \left( E \left[ V_0^{1-\gamma} \right] \right)^{1-\gamma}$ is the date-$-1$ certainty equivalent of future utility. The following proposition demonstrates how the precision of the signal, $\sigma^2$, affects the intertemporal distribution of risk and risk compensation.

**Proposition 1. (Intertemporal distribution of risk and risk compensation)**

\[
\frac{\partial}{\partial \sigma^2} \text{Var} [E [\ln C_1 | s]] > 0, \quad \frac{\partial}{\partial \sigma^2} E [\text{Var} [\ln C_1 | s]] < 0. \tag{4}
\]

Suppose $\gamma > 1$, both $\Lambda_{-1,0}$ and $\ln \Lambda_{0,1}$ are decreasing functions of $C_1$. In addition,

\[
\frac{\partial}{\partial \sigma^2} \text{Var} [\ln \Lambda_{-1,0}] > 0, \quad \frac{\partial}{\partial \sigma^2} \text{Var} [\ln \Lambda_{0,1}] < 0. \tag{5}
\]

Equation (4) is derived from the variance decomposition formula assuming the total amount of variance is fixed. The more precise is the signal released at the announcement, the larger fraction of risk is released at time 0, when the announcement is made. Therefore, the signal provided at time 0 only affects the intertemporal distribution of the risk.
The volatility of the stochastic discount factor is a measure of risk compensation. Equation (5) implies that under the assumption of $\gamma > 1$, the market price of risk is positive, in the sense that payoffs that are positively correlated with aggregate consumption $C_1$ require positive premiums. In addition, a higher precision of the signal corresponds to a higher risk compensation on date 0 but a lower risk compensation on date 1. Information quality affects the intertemporal distribution of risk compensation: when public announcements are informative, the announcement premium is higher, but the risk premium going forward will be lower.

The above channel depends crucially on the parameter restriction of $\gamma > 1$, which reflects the assumption of generalized risk sensitivity (Ai and Bansal (2018)). Note that both $C_{-1}$ and $C_0$ are deterministic. With expected utility, there will be no risk premium for returns received from period $-1$ to period 0, because $C_0$ is deterministic. We need preferences with generalized risk sensitivity so that the release of the macroeconomic announcement at date 0 is associated with the realization of risk premium. The recursive preference with a unit intertemporal elasticity of substitution (IES) is a special case where generalized risk sensitivity is equivalent to $\gamma > 1$. Therefore, to model the effect of information on the intertemporal distribution of risk compensation, generalized risk sensitivity is a necessary condition. In the rest of the paper, we develop a quantitative model and show that this channel can account for many of the stylized facts on volatility and expected returns we document in Section 2.

4 A Dynamic Model

In this section, we develop a simple dynamic model with time-varying information quality. Our purpose is to use a parsimonious model to demonstrate how stochastic shocks to the quality of information change the intertemporal distribution of risk and risk compensation and allow our model to account for the stylized facts we document in Section 2. We shut down all other mechanisms for time-varying risk premium by assuming homoscedasticity in all macroeconomic fundamentals and constant elasticity of substitution (CES) preferences. We do not intend to argue that these other mechanisms are not important in driving variations in risk premiums in the data. Instead, we abstract from other mechanisms of the time-varying risk premium for two reasons. First, it allows us to highlight the mechanism of information-driven volatility. Second, we believe, at the monthly or higher frequencies, time-varying information is much more likely to affect risk premium than time-varying risk aversion or time-varying volatility, both of which are likely to vary at only lower frequencies.

Preferences and endowment We consider an endowment economy where the representative agent has a CES recursive preference with a risk aversion $\gamma$ and an IES $\psi$. The aggregate endowment follows a diffusion process of the form:

$$\frac{dY_t}{Y_t} = \theta_t dt + \sigma_Y dB_{Y,t},$$  

(6)
where $B_{Y,t}$ is a standard Brownian motion and $\{\theta_t\}_{t \geq 0}$ is a two-state Markov process with the state space $\Theta = \{\theta_H, \theta_L\}$, where $\theta_H > \theta_L$. The transition matrix for $\theta_t$ over a small interval $\Delta$ is
\[
\begin{bmatrix}
  e^{-\lambda_H \Delta} & 1 - e^{-\lambda_H \Delta} \\
  1 - e^{-\lambda_L \Delta} & e^{-\lambda_L \Delta}
\end{bmatrix},
\]
where intensity $\lambda_H$ is the rate of transition from high to low state, and $\lambda_L$ is the rate of transition from low to high. We assume that the state variable $\theta_t$ is unobservable to investors. However, information about $\theta_t$ continuously arrives into the financial markets. Investors observe two sources of information about $\theta_t$. First, the aggregate consumption itself contains information about $\theta_t$. Second, pre-scheduled macroeconomic announcements are made at time $T, 2T, \ldots, nT$. We assume that announcements carry a noisy signal of $\theta_t$. The distribution of the signal is given as follows. At the announcement at time $nT$, if $\theta_{nT} = \theta_H$,
\[
s = \theta_H \quad \text{with} \quad \text{prob } \nu_n,
\]
and if $\theta = \theta_L$,
\[
s = \theta_H \quad \text{with} \quad \text{prob } 1 - \nu_n,
\]
Here $\nu_n \in [\frac{1}{2}, 1]$ is a parameter that measures the information quality. When $\nu_n = 1$, announcements carry perfectly accurate information, and $\nu_n = 0.5$ indicates that announcements are completely uninformative. For simplicity, we assume that $\nu_1, \nu_2, \ldots, \nu_n$ are i.i.d. over time.

**Asset prices** We define $\pi_t = P_t(\theta_t = \theta_H)$ as the probability of $\theta_t = \theta_H$ and $\hat{\theta}_t = E_t[\theta_t]$ to be the posterior mean of $\theta_t$. That is, $\hat{\theta}_t = \pi_t \theta_H + (1 - \pi_t) \theta_L$. The life-time utility of the representative agent can be written as a function of state variables of the form $V(\hat{\theta}_t, t, Y_t) = H(\hat{\theta}_t, t) Y_t$. In Appendix 7.2, we show that the function $H(\hat{\theta}_t, t)$ satisfies PDEs with appropriate boundary conditions. Given the value function, we can construct the pricing kernel $M(t)$. The law of motion of $M(t)$ in the interior of $(nT, (n + 1)T)$ can be written as
\[
\frac{dM(t)}{M(t)} = -r(\hat{\theta}, t) dt - \sigma_M(\hat{\theta}, t) d\hat{B}_{Y,t},
\]
where $r(\hat{\theta}, t)$ is the risk free rate:
\[
r(\theta, t) = \rho + \frac{1}{\psi} \theta - \frac{1}{2} \gamma \left( 1 + \frac{1}{\psi} \right) \sigma_Y^2 + \frac{1}{\psi} \frac{\gamma - 1}{\gamma - 1} \frac{H_T(\theta, t)}{H(\theta, t)} (\theta_H - \theta) (\theta - \theta_L)
\]
\[
+ \frac{\left( \frac{1}{\psi} - \gamma \right) \left( 1 - \frac{1}{\psi} \right)}{2 (1 - \gamma)^2} \left( \frac{H_T(\theta, t)}{H(\theta, t)} \right)^2 \frac{(\theta_H - \theta)^2 (\theta - \theta_L)^2}{\sigma_Y^2}.
\]
and \( \sigma_M(\hat{\theta},t) \) is the market price of risk:

\[
\sigma_M(\theta,t) = \gamma \sigma_Y - \frac{1}{\gamma} H_\theta(\theta,t) (\theta_H - \theta)(\theta - \theta_L) \frac{1}{\sigma_Y}.
\]  

The aggregate stock market is the claim to a dividend process of the form:

\[
\frac{dD_t}{D_t} = \left[ \xi \left( \hat{\theta}_t - \bar{\theta} \right) + \bar{\theta} \right] dt + \sigma_Y \hat{B}_{Y,t},
\]  

where \( \xi \) is the leverage parameter. The stock price is of the form \( p(\hat{\theta}_t,t) \) is the price-to-dividend ratio defined by

\[
p(\hat{\theta}_t,t) = E_t \left[ \int_0^\infty \frac{\pi_{t+s}}{\pi_t} \frac{D_{t+s}}{D_t} ds \right].
\]  

We provide the expression for the PDE together with the boundary conditions that determines the function \( p(\hat{\theta}_t,t) \) in Appendix 7.2. With the pricing kernel and the price-to-dividend ratio, the market risk premium is given by the following proposition:

**Proposition 2. (Equity premium)**

In the interior of \((nT, (n+1)T)\), the instantaneous risk premium of the asset is given by:

\[
E_t \left[ \frac{d}{dt} \frac{p(\hat{\theta}_t,t) D_t}{p(\hat{\theta}_t,t)} + D_t dt \right] - r(\hat{\theta}_t,t) dt = \sigma_M(\hat{\theta}_t,t) \left[ \frac{p(\theta_H - \hat{\theta}_t)(\hat{\theta}_t - \theta_L)}{p(\hat{\theta}_t,t) \sigma_Y} + \eta \sigma_Y \right].
\]

At an announcement time \( nT \), the announcement premium is given by:

\[
E_t \left[ \frac{p(\hat{\theta}_t^+,T^+)}{p(\hat{\theta}_t^-,T^-)} \right] - 1 = \frac{E_T \left[ H\left( \hat{\theta}_t^+,T^+ \right) \right]^{\frac{1}{1-\gamma}}}{E_T \left[ H\left( \hat{\theta}_t^+,T^+ \right) \right]^{\frac{1}{1-\gamma}}} E_T \left[ \frac{p(\hat{\theta}_t^+,T^+)}{p(\hat{\theta}_t^-,T^-)} \right] - 1.
\]

In our model, \( E_t \left[ H(\hat{\theta}_t^+,T^+) \right]^{\frac{1}{1-\gamma}} \) is the announcement stochastic discount factor. Note that the value function \( H(\hat{\theta},t) \) is increasing in \( \hat{\theta} \). Under the assumption \( \gamma > \frac{1}{\psi} \), the term \( H\left( \hat{\theta}_t^+,T^+ \right) \) is negatively correlated with \( p(\hat{\theta}_t^+,T^+) \). As a result, \( \text{Cov} \left[ H\left( \hat{\theta}_t^+,T^+ \right)^{\frac{1}{1-\gamma}}, p(\hat{\theta}_t^+,T^+) \right] < 0 \) and the announcement premium is positive.

**Comparative statics with respect to informativeness of announcements**
5 Quantitative Results

Parameter values In this section, we calibrate our model and evaluate its implications on the volatility-expected return relationship. We choose a discount rate $\rho = 2\%$, a risk aversion $\gamma = 10$, a IES $\psi = 2$ in line with the standard long-run risk literature. We set the volatility of consumption growth $\sigma_Y = 3\%$ to match the volatility of annual consumption growth in the U.S. in our sample period from 1990-2015. We set the value of the two Markov states $\theta_H = 2.5\%$, $\theta_L = -1.2\%$ and the transition probabilities $\lambda_H = 0.08$ and $\lambda_L = 0.26$ as in Ai and Kiku (2013), who estimate these parameters from aggregate consumption data. We choose a leverage parameter $\xi = 3$ as in Bansal and Yaron (2004).

Table 3: Calibrated Parameters

<table>
<thead>
<tr>
<th>Panel A. Preferences</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$ Time discount rate</td>
<td>2%</td>
</tr>
<tr>
<td>$\psi$ IES</td>
<td>2</td>
</tr>
<tr>
<td>$\gamma$ Relative risk aversion</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B. Consumption and dividend dynamics</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_Y$ Endowment growth volatility</td>
<td>3%</td>
</tr>
<tr>
<td>$\lambda_H$ Transition rate (high to low)</td>
<td>0.08</td>
</tr>
<tr>
<td>$\lambda_L$ Transition rate (low to high)</td>
<td>0.26</td>
</tr>
<tr>
<td>$\theta_H$ High endowment growth state</td>
<td>0.025</td>
</tr>
<tr>
<td>$\theta_L$ Low endowment growth state</td>
<td>-0.012</td>
</tr>
<tr>
<td>$\xi$ Leverage</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: Information</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_H$ High ann signal precision state</td>
<td>0.99</td>
</tr>
<tr>
<td>$\nu_L$ Low ann signal precision state</td>
<td>0.6</td>
</tr>
<tr>
<td>$\frac{1}{T}$ Frequency of announcements</td>
<td>8</td>
</tr>
</tbody>
</table>

This table displays the calibrated parameter in our model.

The parameters $\nu_H$ and $\nu_L$ govern the informativeness of the announcements. We set $\nu_H = 0.99$ and $\nu_L = 0.60$ so that our model matches the mean and standard deviation of implied variance reduction on announcement days. Finally, we choose $T = \frac{1}{8}$ so that there are eight announcements per year in our model, matching the frequency of FOMC announcements in the data. All calibrated parameters are listed in Table 3.

Basic statistics of announcement returns and volatility We list the asset pricing moments in the data and in our model in Table 4. Our model produces an average level of the risk-free rate of 1.28% per year, with a standard deviation of 0.48% per year, both moments are fairly close to their data counterparts. The average equity market premium in the model is 6.334% per year, and the standard deviation of market return is 16% per year. Our model produces a significant announcement premium. The average announce-day return is 63 bps and the average non-announcement day return is 1.5 bps. In the data, many other macroeconomic announcements also generate a significant return on announcement days. With only eight announcements per year, our model needs a slightly higher announcement-day return than the average FOMC announcement premium to generate a comparable level of the equity risk premium as in the data.
Table 4: Asset Pricing Moments

<table>
<thead>
<tr>
<th>Aggregate market returns</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[R_M] - r_f$</td>
<td>Equity premium</td>
<td>7.25%</td>
</tr>
<tr>
<td>$\text{Std}[R_M]$</td>
<td>Vol of market return</td>
<td>16.6%</td>
</tr>
<tr>
<td>$E[r_f]$</td>
<td>Average risk-free rate</td>
<td>1.32%</td>
</tr>
<tr>
<td>$\text{Std}[r_f]$</td>
<td>Vol of risk-free rate</td>
<td>0.67%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Announcement returns</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[R_A]$</td>
<td>A-day average return</td>
<td>40 bps</td>
</tr>
<tr>
<td>$E[R_N]$</td>
<td>NA-day average return</td>
<td>2 bps</td>
</tr>
<tr>
<td>$AC(R_A)$</td>
<td>AC(1) of A-day return</td>
<td>−0.01</td>
</tr>
<tr>
<td>$E[IV_1^- - IV_1^+]$</td>
<td>Av. IV reduction</td>
<td>2.5</td>
</tr>
<tr>
<td>$\text{Corr}(R_A, IV_1^- - IV_1^+)$</td>
<td>Corr between A-day return and IV reduction</td>
<td>0.73</td>
</tr>
<tr>
<td>$\text{Corr}(R_A, RV_{T+1}^{T+1})$</td>
<td>Corr between A-day return and RV</td>
<td>0.12</td>
</tr>
</tbody>
</table>

This table displays the asset pricing moments in the data and implied by the model.

Our model matches several features of the announcement-day returns in the data. First, the announcement returns are slightly negatively correlated in the data and in the model. In our model, when the previous announcement is more precise, the associated announcement premium is larger. However, this also means that the uncertainty going forward will be lower, and therefore, the premium for the next announcement will be smaller. A negative correlation between announcement returns is another indication of the information-driven volatility channel at work.

Second, implied variance drops sharply upon announcement. The average reduction of implied variance is 2.5 (monthly bps squared) in the data with a t-statistics of 4.1. The same moment in the model is 3.6 (monthly bps squared). In our model, the implied variance is a forward-looking measure of variance. Because announcements are typically associated with a significant response of the market valuation, the implied variance is high before announcements and low afterwards. The drop in implied variance reflects the informativeness of the announcement. To further illustrate this implied variance reduction associated with information revelation, in Table 5, we report the quantiles of the implied variance reduction in the data and that implied by our model.

Table 5: Implied Variance Reduction

<table>
<thead>
<tr>
<th>Q5</th>
<th>Q25</th>
<th>Q50</th>
<th>Q75</th>
<th>Q95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>-5.95</td>
<td>-0.11</td>
<td>1.28</td>
<td>4.34</td>
</tr>
<tr>
<td>Model</td>
<td>-9.65</td>
<td>-1.00</td>
<td>2.06</td>
<td>3.54</td>
</tr>
</tbody>
</table>

This table displays the quantile of implied variance drop on announcement days in the data and that in the model.

It is clear from the variance decomposition formula in Equation (1) that because $\text{Var} \left[ E(X|Y) \right] > 0$ implied variance will on average be lower after the announcements. If both $X$ and $Y$ are normally distributed, then $\text{Var} \left[ X|Y \right]$ will be a constant, and implied variance reduction must always
be positive. In general, the variance decomposition only requires that the average drop in implied variance to be positive. In the data, however, the 5th and the 25th percentiles of implied variance drops are both negative. Our model with a two-state Markov chain captures these features of the data as well. The average reduction of implied variance is unambiguously positive. However, there is a significant fraction of observations with increases in the implied variance after announcements.

Third, there is a strong positive correlation between implied variance reduction and announcement day returns both in the data and in our model. This positive correlation arises in our model for two reasons. First, there is an endogenous negative correlation between returns and variance in the model. Ceteris paribus, increases in uncertainty is associated with decreases in the price-to-dividend ratio. Second, and more importantly, because of the informativeness of announcements, $\nu_n$ is time varying. A high $\nu_n$ is associated with high implied variance reductions and high announcement premiums. The magnitude of the announcement premium is proportional to the amount of uncertainty reduction in our model. For the same reason, consistent with the data, the announcement-day return and announcement-day realized variance are positively correlated in our model.

To illustrate further the time-varying announcement premium and the time-varying implied variance reduction in our model, we construct an ex-ante measure of the expected implied variance reduction using the term structure of implied variance, and we run an announcement day return predictability regression:

$$R_A(T^+) = \alpha + \beta \times EVR_T - T + \epsilon_T.$$  

In the above regression, $R_A(T^+)$ is the announcement-day return, and $EVR_T$ is the expected variance reduction measure we construct using option prices before the announcement at time $T$. We provide the details of construction of $EVR_T$ in Appendix 7.4. The regression coefficient on $EVR_T$ is positive and significant with a t statistic of 3.07. The above regression has a $R^2$ of 10% in the data and 40% in our model.

**The risk-reallocation channel**  The key mechanism in our model is that information quality affects the intertemporal distribution of risk and risk compensation. As shown in Proposition 1, a more precise public signal predicts lower volatility and a lower expected return in the future. Pre-scheduled FOMC announcements affords a clearly identifiable information event that provides signals about macroeconomic fundamentals. As we demonstrate in Section 2, implied variance reduction on FOMC announcement days negatively predicts realized variance and expected returns going forward.

Our dynamic model matches this feature of the data very well. In our simulated model, we run the same return and variance predictability regression as in (3). We report our results in Table 6. In our model, consistent with empirical evidence, higher implied variance reductions on announcement days are associated with lower realized variance and lower expected returns going forward.
Table 6: Model Implied Return and Variance Predictability by IV Reduction

<table>
<thead>
<tr>
<th>Number of days</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$RV_{t,t+\Delta}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Data</td>
<td>-0.12</td>
<td>-0.08</td>
<td>-0.05</td>
<td>-0.05</td>
<td>-0.05</td>
</tr>
<tr>
<td>Model</td>
<td>-0.25</td>
<td>-0.24</td>
<td>-0.24</td>
<td>-0.23</td>
<td>-0.23</td>
</tr>
<tr>
<td>$R^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Data</td>
<td>0.66</td>
<td>0.59</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>Model</td>
<td>0.81</td>
<td>0.81</td>
<td>0.81</td>
<td>0.81</td>
<td>0.81</td>
</tr>
<tr>
<td>$R_{t,t+\Delta}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Data</td>
<td>-0.03</td>
<td>-0.02</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
</tr>
<tr>
<td>Model</td>
<td>-0.12</td>
<td>-0.10</td>
<td>-0.09</td>
<td>-0.08</td>
<td>-0.06</td>
</tr>
<tr>
<td>$R^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Data</td>
<td>0.05</td>
<td>0.05</td>
<td>0.04</td>
<td>0.09</td>
<td>0.02</td>
</tr>
<tr>
<td>Model</td>
<td>0.12</td>
<td>0.12</td>
<td>0.11</td>
<td>0.11</td>
<td>0.10</td>
</tr>
</tbody>
</table>

This table presents the results of the return predictability regression (3) in the data and those in our model. The columns labeled 1, 2, 3, 4, 5 represents the horizon of returns and variance on the left hand side of Equation (3) with $\Delta = 1, 2, 3, 4, 5$ days.

**Volatility managed portfolios** In our model of information-driven volatility, more precise FOMC announcements are associated with higher realized volatility on announcement days, but lower realized volatility and expected returns going forwards. In this section, we show that this feature of our model is consistent with the return on the volatility managed portfolios. Here, we replicate the volatility managed strategy constructed by Moreira and Muir (2017).

We simulate our model for 1000 months and we compute a cumulative return for each month denoted $f_{t+1}$, where time is measured in months. For each $t + 1$, we use the daily return of the previous month to construct the realized volatility:

$$RV_t^2(f) = \sum_{d=1/30}^{1} \left( f_{t+d} - \frac{1}{30} \sum_{d=1/22}^{1} f_{t+d} \right)^2 .$$

The buy-and-hold strategy is the market return constructed from the sequence of $\{f_{t+1}\}$. The volatility managed portfolio is constructed as

$$f_{t+1}^\sigma = \frac{c}{RV_t^2(f)} f_{t+1},$$

where the constant $c$ is chosen so that managed portfolio $\{f_{t+1}^\sigma\}$ and the market portfolio have the same unconditional standard deviation. In our simulated model, the average monthly return of the market portfolio is 0.606% and the average monthly return on the volatility managed portfolio is 0.465% per month. Both numbers match closely their empirical counterparts.

In addition, we run a CAPM regression of the volatility managed portfolio returns on the buy-and-hold market returns. We obtain a CAPM alpha of 3.76% and a beta of 0.70 at the annual level. Both numbers of close to their empirical counterparts in the data, 4.08% and 0.61, respectively.
Variance risk premium predictability  In this section, we report our model’s implications on return predictability regressions on the difference between implied and realized variance. The literature has interpreted the difference between implied and realized variance as so-called “variance risk premium (VRP)”, and labeled these predictability regressions as VRP predictability. Previous literature has developed models with time-varying volatility of macroeconomic fundamentals to account for the variance risk premium predictability, for example, Bollerslev, Tauchen, and Zhou (2009) and Drechsler and Yaron (2011).

As we emphasized earlier, the volatility of macroeconomic fundamentals is unlikely to change substantially over short horizons. Our model offers an alternative interpretation of the “VRP” predictability regressions. In our model, all macroeconomic shocks are homoskedastic, and therefore variance under the physical probability and that under the risk-neutral measure is the same. The difference between implied and realized variance in our model reflects the informativeness of the upcoming announcement. Because implied variance is a forward-looking measure of variance, it increases when the upcoming announcement is expected to be informative. The difference between implied and realized variance, therefore, reflects the informativeness of the upcoming announcement. It predicts returns because more informative announcements are associated with higher realizations of announcement premiums.

Table 7: Model implied return predictability by $IV - RV$:

<table>
<thead>
<tr>
<th>Number of days</th>
<th>21</th>
<th>42</th>
<th>63</th>
<th>84</th>
<th>105</th>
<th>126</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IV_i - RV_i$</td>
<td>Data</td>
<td>0.02</td>
<td>0.05</td>
<td>0.07</td>
<td>0.12</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>0.03</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.06</td>
</tr>
</tbody>
</table>

This table presents the results of the return predictability regressions defined in (2). Columns 3 to 8 represent returns on the left hand side of (2) with $\Delta = 1, 2, 3, 4, 5, 6$ months.

In Table 7, we report the results of the return predictability regression (2). As in the data, in our model, returns are predictable by the difference between $IV$ and $RV$. Our purpose is not to argue for the absence of the variance risk premium. The level of VIX implied variance in the data is clearly higher than the historical average of realized stock market volatility. Models of VRP predictability based only on time-varying volatility of volatility, however, have two difficulties. First, there is little evidence that macroeconomic volatility varies at high frequencies. Second, these models should also imply strong predictability of future returns by realized variances of past returns, which is not supported by data. Our purpose is to present an alternative mechanism for the return predictability regressions based on the information-driven volatility channel.

6 Conclusion

In this paper, we present a model of information-driven volatility. Traditional models of stochastic volatility typically imply a positive relationship between the realized variance of past returns and
the forward-looking future returns. However, empirical evidence often favors a negative relationship between the two, which is exemplified by the evidence on volatility managed portfolio returns. We develop a model of information-driven volatility. We show that when variations in stock market volatility are driven by information, high realized variances of past return typically predict lower future variances and lower future returns. We show that our model can account for several stylized facts on the variance-expected return relationships in the data.

This paper focuses on the impact of news on financial markets. We do not investigate the micro-foundation for the impact of for example, monetary policy news on the financial market and the macroeconomy. Some existing paper has made progress on this important question, for example Gu, Han, and Wright (2020). Integrating these micro-foundations into our setup to study the volatility and expect return dynamics associated with monetary policy announcements will be important direction for future research.
References


Gu, Chao, Han Han, and Randall Wright, 2020, The Effects of News When Liquidity Matters, International Economic Review.


Shaliastovich, Ivan, 2015, Learning, Confidence, and Option Prices, Journal of Econometrics 187, 18–42.


7 Appendix

7.1 The Two-Period Model

Below, we provide the proof for Proposition 1 of the paper.

Proof. Standard Bayesian updating implies that the posterior mean $E[\ln C_1|s] = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_\epsilon^2} \mu + \frac{\sigma_\epsilon^2}{\sigma_s^2 + \sigma_\epsilon^2} s$. In addition, $Var[\ln C_1|s] = \lambda \sigma^2$, and $Var[\ln C_1|s] = (1 - \lambda) \sigma^2$ where $\lambda = \frac{\sigma_\epsilon^2}{\sigma_s^2 + \sigma_\epsilon^2}$, which the variance decomposition formula in (1).

The expressions of the stochastic discount factor is also standard. The stochastic discount factor that prices date-0 consumption goods in terms of date-1 consumption goods is $\Lambda_{0,1} = \frac{\beta}{1 - \beta} \left( \frac{C_0}{C_1} \right)^{-1} \left( \frac{V_0}{m_1} \right)^{1-\gamma}$, and the stochastic discount factor that prices data-1 consumption goods in terms of date 0 consumption goods is given by: $\Lambda_{-1,0} = \frac{\beta}{1 - \beta} \left( \frac{C_0}{C_1} \right)^{-1} \left( \frac{C_1}{m_0} \right)^{1-\gamma}$, where $V_0 = C_0^{1 - \beta} \left( E \left[ C_1^{1 - \gamma} \right] \right)^{1/\gamma}$, $m_0 = \left( E \left[ C_1^{1 - \gamma} \right] \right)^{1/\gamma}$ is the date-0 utility of the agent, $m_0 = \left( E \left[ C_1^{1 - \gamma} \right] \right)^{1/\gamma}$ is the date-1 certainty equivalent of future at time 0 and $m_{-1} = \left( E \left[ V_0^{1 - \gamma} \right] \right)^{1/\gamma}$ is the date-1 certainty equivalent of future utility.

Therefore, $Var[\ln \Lambda_{0,1}] = \gamma^2 Var[\ln C_1|s] = \gamma^2 (1 - \lambda) \sigma^2$. Note also, $Var[\ln \Lambda_{-1,0}] = (1 - \gamma)^2 Var[\ln V_0]$, where

$$\ln V_0 = (1 - \beta) \ln C_0 + \frac{\beta}{1 - \gamma} \ln E \left[ C_1^{1 - \gamma} \right] = (1 - \beta) \ln C_0 + \beta \left\{ E[\ln C_1|s] + \frac{1}{2} (1 - \gamma) Var[\ln C_1|s] \right\}.$$ 

This implies that $Var[\ln \Lambda_{-1,0}] = (1 - \gamma)^2 \beta^2 Var[E[\ln C_1|s]] = (1 - \gamma)^2 \beta^2 \lambda \sigma^2$. Given the expression for $\lambda = \frac{\sigma_\epsilon^2}{\sigma_s^2 + \sigma_\epsilon^2}$, Proposition 1 can be easily proved. 

\hfill \square

7.2 The Infinite Horizon Model

The filtering equations Define $\pi_t = P_t (\theta_t = \theta_H)$, and $\hat{\theta}_t = E_t [\theta_t]$, that is, $\hat{\theta}_t = \pi_t \theta_H + (1 - \pi_t) \theta_L$, then

$$d\pi_t = [\lambda_L - (\lambda_H + \lambda_L) \pi_t] dt + \pi_t (1 - \pi_t) (\theta_H - \theta_L) \frac{1}{\sigma_y} d\tilde{B}_{Y,t}, \quad (13)$$

where $\tilde{B}_{Y,t}$ is the innovation process defined by:

$$d\tilde{B}_{Y,t} = \frac{1}{\sigma_y} \left[ \frac{dY_t}{Y_t} - \hat{\theta}_t dt \right]. \quad (14)$$

Note that the mapping between $\hat{\theta}$ and $\pi$ is one-to-one. So we can equivalently use $\hat{\theta}$ as the state
variable. By definition, \( \hat{\theta}_t = \pi_t \theta_H + (1 - \pi_t) \theta_L \). Using Ito’s lemma,

\[
d\hat{\theta}_t = (\lambda_H + \lambda_L) \left( \bar{\theta} - \hat{\theta}_t \right) dt + \left( \theta_H - \hat{\theta}_t \right) \left( \hat{\theta}_t - \theta_L \right) \frac{1}{\sigma_Y} dB_{Y,t},
\]

where \( \bar{\theta} \) is the steady-state mean of \( \theta \):

\[
\bar{\theta} = \frac{\lambda_L \theta_H + \lambda_H \theta_L}{\lambda_L + \lambda_H}.
\]

**Preference** We can write down the value function for recursive preference: \( V \left( \hat{\theta}, t, Y \right) = H \left( \hat{\theta}, t \right) Y \). The representative consumer’s preference is specified by a pair of aggregators \( (f, A) \) such that:

\[
dV_t = [-f(Y_t, V_t) - \frac{1}{2} A(V_t)||V_Y(t)||^2]dt + \sigma_Y(t)dB_t \tag{17}
\]

We adopt the convenient normalization \( A(V) = 0 \). Duffie and Epstein, and denote \( \tilde{f} \) the normalized aggregator. Under this normalization, \( \hat{f}(Y, V) \) is:

\[
\tilde{f}(Y, V) = \frac{\rho}{1 - 1/\psi} \frac{Y^{1 - 1/\psi} - ((1 - \gamma) V)^{1 - 1/\psi}}{((1 - \gamma) V)^{1 - 1/\psi - 1}} \tag{18}
\]

The HJB for recursive utility is

\[
f \left( Y_t, V \left( \hat{\theta}_t, t, Y \right) \right) + \mathcal{L} \left[ V \left( \hat{\theta}_t, t, Y \right) \right] = 0. \tag{19}
\]

Consider

\[
V \left( \hat{\theta}_t, t, Y \right) = \frac{1}{1 - \gamma} H \left( \hat{\theta}_t, t \right) Y_t^{1 - \gamma} \tag{20}
\]

where

\[
\frac{dY_t}{Y_t} = \hat{\theta}_t dt + \sigma_Y dB_{Y,t}, \tag{21}
\]

\[
d\hat{\theta}_t = \mu \left( \hat{\theta}_t \right) dt + \sigma \left( \hat{\theta}_t \right) \left( \frac{1}{\sigma_s} dB_{s,t} + \frac{1}{\sigma_Y} dB_{Y,t} \right), \tag{22}
\]

where \( \mu (\theta) = (\lambda_H + \lambda_L) (\bar{\theta} - \theta) \), \( \sigma (\theta) = (\theta_H - \theta) (\theta - \theta_L) \). Using generalized Ito’s formula, the HJB equation is written as

\[
0 = \frac{1}{1 - \gamma} H \left\{ H_t + H_\theta \left[ \mu_{\theta,t} + (1 - \gamma) \sigma_{\theta,t} \right] + \frac{1}{2} H_{\theta\theta} \sigma_{\theta,t}^2 \left( \frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2} \right) \right\} + \frac{\rho}{1 - 1/\psi} \left( H^{1 - 1/\psi - 1} - 1 \right) + \left( \hat{\theta}_t - \frac{1}{2} \gamma \sigma_Y^2 \right), \tag{23}
\]
At the boundary,
\[ H (\pi_T^-, T) = \mathbb{E} \left[ H (\pi_T^+, 0) \mid \pi_T^-, T \right] \] (24)

With this, we can write down the law of motion for the pricing kernel
\[
\frac{dM_t}{M_t} = \frac{df_V (Y, V)}{f_Y (Y, V)} + f_V (Y, V) \, dt
\] (25)

where \( f_Y (Y, V) = \rho H_t^{\frac{1}{1-\gamma}} Y_t^{-\gamma} \), and \( f_V (Y, V) = \rho \frac{1}{1-\gamma} H_t^{\frac{1-\gamma}{1-\nu}} - \rho \frac{1-\gamma}{\nu} \). Applying Ito’s lemma, we can derive the pricing kernel in Equation (26).

**Learning on the boundary** At the boundary, given the distribution of \( \theta, \pi \), we need to compute the distribution of \( \hat{\theta}_T^+, or \pi_T^- \). Applying Bayes’ rule,
\[
P^+ (\theta_i | s_j) = \frac{P (s_j | \theta_i) P^- (\theta_i)}{\sum_{\theta_i} P (s_j | \theta_i) P^- (\theta_i)}.
\]

That is, given that \( P^- (\theta_H) = \pi^- \), we have:
\[
P^+ (\theta_H | s_H) = \frac{\pi^- \nu}{\pi^- \nu + (1 - \pi^-) (1 - \nu)}; \quad P^+ (\theta_L | s_H) = \frac{(1 - \pi^-) (1 - \nu)}{\pi^- \nu + (1 - \pi^-) (1 - \nu)} = 1 - P^+ (\theta_H | s_H),
\] (26)

and
\[
P^+ (\theta_H | s_L) = \frac{\pi^- (1 - \nu)}{\pi^- (1 - \nu) + (1 - \pi^-) \nu}; \quad P^+ (\theta_L | s_L) = 1 - P^+ (\theta_H | s_L).
\] (27)

Now, given \( P^- (\theta_H) = \pi^- \), we need to compute the distribution of \( \pi^+ \). If we see \( s_H \), then,
\[
\pi^+ = \frac{\pi^- \nu + (1 - \pi^-) (1 - \nu)}{\pi^- \nu + (1 - \pi^-) (1 - \nu)}, \quad \text{and if we see} \ s_L, \ \pi^+ = \frac{\pi^- (1 - \nu)}{\pi^- (1 - \nu) + (1 - \pi^-) \nu}.
\]

So \( \pi^+ \) has only two possible realizations. The probability of seeing \( s_H \) is \( \pi^- \nu + (1 - \pi^-) (1 - \nu) \) and the probability of seeing \( s_L \) is \( \pi^- (1 - \nu) + (1 - \pi^-) \nu \). Therefore, if we use \( \pi \) as the state variable, the boundary condition for value function (see equation (24)) is
\[
H (\pi_T^-, T) = \mathbb{E} \left[ H (\pi_T^+, 0) \mid \pi_T^-, T \right] = \left[ \pi^- \nu + (1 - \pi^-) (1 - \nu) \right] H \left( \frac{\pi^- \nu}{\pi^- \nu + (1 - \pi^-) (1 - \nu)}, 0 \right)
+ \left[ \pi^- (1 - \nu) + (1 - \pi^-) \nu \right] H \left( \frac{\pi^- (1 - \nu)}{\pi^- (1 - \nu) + (1 - \pi^-) \nu}, 0 \right).
\] (28)

where \( h_{s_H} H (\pi_{s_H}^+, 0) = h_{s_L} H (\pi_{s_L}^+, 0) \)

and \( \pi_{s_H}^+ = \frac{\pi^- \nu}{\pi^- \nu + (1 - \pi^-) (1 - \nu)} \),
and \( \pi_{s_L}^+ = \frac{\pi^- (1 - \nu)}{\pi^- (1 - \nu) + (1 - \pi^-) \nu} \).

If we want to keep using \( \hat{\theta} \) as the state variable, note that \( \hat{\theta} = \pi \theta_H + (1 - \pi) \theta_L = \theta_L + \pi (\theta_H - \theta_L) \), that is, we can recover \( \pi \) from \( \hat{\theta} \): \( \pi = \frac{\hat{\theta} - \theta_L}{\theta_H - \theta_L} \). In addition, given \( \pi^+ \), we can compute
\[ \hat{\theta}^+ = \theta_L + \pi^+ (\theta_H - \theta_L). \] Therefore,

\[
h_{sH} \equiv \pi^- \nu + (1 - \pi^-) (1 - \nu) = \frac{\hat{\theta} - \hat{\theta}_L}{\hat{\theta}_H - \hat{\theta}_L} \nu + \frac{\theta_H - \hat{\theta}}{\theta_H - \theta_L} (1 - \nu), \tag{29}
\]

\[
h_{sL} \equiv \pi^- (1 - \nu) + (1 - \pi^-) \nu = \frac{\hat{\theta} - \hat{\theta}_L}{\theta_H - \hat{\theta}_L} (1 - \nu) + \frac{\theta_H - \hat{\theta}}{\theta_H - \theta_L} \nu. \tag{30}
\]

Also,

\[
\hat{\theta}_s^{+ \mid \pi_s^+ = \pi^+} = \theta_L + \frac{\pi^- \nu}{\pi^- \nu + (1 - \pi^-) (1 - \nu)} (\theta_H - \theta_L) \tag{31}
\]

\[
= \theta_L + \frac{(\hat{\theta} - \hat{\theta}_L) \nu (\theta_H - \theta_L)}{(\hat{\theta} - \theta_L) \nu + (\theta_H - \hat{\theta}) (1 - \nu)}, \tag{32}
\]

and

\[
\hat{\theta}_s^{+ \mid \pi_s^+ = \pi^+} = \theta_L + \frac{\pi^- (1 - \nu)}{\pi^- (1 - \nu) + (1 - \pi^-) \nu} (\theta_H - \theta_L) \tag{33}
\]

\[
= \theta_L + \frac{(\hat{\theta} - \theta_L) (1 - \nu) (\theta_H - \theta_L)}{(\hat{\theta} - \theta_L) (1 - \nu) + (\theta_H - \hat{\theta}) \nu}. \tag{34}
\]

We can rewrite that above as:

\[
H \left( \hat{\theta}^-, T \right) = h_{sH} H \left( \hat{\theta}_s^{+ \mid \pi_s^+ = \pi^+}, T \right) + h_{sL} H \left( \hat{\theta}_s^{+ \mid \pi_s^+ = \pi^+}, T \right) \tag{35}
\]

\[
= \left[ \frac{\hat{\theta} - \theta_L}{\hat{\theta}_H - \theta_L} \nu + \frac{\theta_H - \hat{\theta}}{\theta_H - \theta_L} (1 - \nu) \right] H \left( \theta_L + \frac{(\hat{\theta} - \hat{\theta}_L) \nu (\theta_H - \theta_L)}{(\hat{\theta} - \theta_L) \nu + (\theta_H - \hat{\theta}) (1 - \nu)}, 0 \right) \tag{36}
\]

\[
+ \left[ \frac{\hat{\theta} - \theta_L}{\theta_H - \hat{\theta}_L} (1 - \nu) + \frac{\theta_H - \hat{\theta}}{\theta_H - \theta_L} \nu \right] H \left( \theta_L + \frac{(\hat{\theta} - \theta_L) (1 - \nu) (\theta_H - \theta_L)}{(\hat{\theta} - \theta_L) (1 - \nu) + (\theta_H - \hat{\theta}) \nu}, 0 \right). \tag{37}
\]

**Asset Prices** Specify the dividend growth rate as follows

\[
\frac{dD_t}{D_t} = \left[ \xi \left( \hat{\theta}_t - \hat{\theta} \right) + \hat{\theta} \right] dt + \eta \sigma_Y d\tilde{B}_Y_t. \tag{38}
\]

Now the stock price can be solved as \( P \left( \hat{\theta}_t, t, D_t \right) = p \left( \hat{\theta}_t, t \right) D, \) where \( p \left( \hat{\theta}_t, t \right) \) is the price-to-dividend ratio, which is characterize by the form:

\[
M (t) D_t dt + \mathcal{L} \left[ M (t) p \left( \hat{\theta}_t, t \right) D_t \right] = 0. \tag{39}
\]
Therefore, the PDE for \( p \left( \hat{\theta}_t, t \right) \) is

\[
\varpi \left( \hat{\theta}_t, t \right) \ p_j = \ p_t + p_{\theta \theta} \left( \hat{\theta}_t, t \right) + \frac{1}{2} \ p_{\theta \theta \theta} \left( \hat{\theta}_t, t \right) \left( \frac{1}{\sigma^2_s} + \frac{1}{\sigma^2_Y} \right) + 1
\]

(40)

where

\[
\varpi \left( \hat{\theta}_t, t \right) = -\bar{\theta} (1 - \xi) + \rho - \frac{1}{2} \gamma \sigma^2_Y \left( \frac{1}{\psi} + 1 \right) + \gamma \eta \sigma^2_Y - \left( \xi - \frac{1}{\psi} \right) \hat{\theta}_t - \frac{1 - \gamma}{1 - \gamma} \sigma_{\theta, t} \left( \eta - 1 \right) \frac{H_{\theta}}{H} \\
+ \frac{\left( \frac{1}{\psi} - \gamma \right) \left( 1 - \frac{1}{\psi} \right)}{2 \left( 1 - \gamma \right)^2} \left( \frac{H_{\theta}}{H} \right)^2 \sigma^2_{\theta, t} \left( \frac{1}{\sigma^2_s} + \frac{1}{\sigma^2_Y} \right)
\]

(41)

\[
\varrho \left( \hat{\theta}_t, t \right) = \mu_{\theta, t} + (\eta - \gamma) \sigma_{\theta, t} + \frac{1}{\psi - \gamma} \frac{H_{\theta}}{H} \sigma^2_{\theta, t} \left( \frac{1}{\sigma^2_s} + \frac{1}{\sigma^2_Y} \right)
\]

(42)

At the boundary,

\[
p \left( \pi^{-}_T, T \right) = \mathbb{E}_T \left[ H \left( \pi^+_T, 0 \right) \right] \left( \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} \right) \frac{p \left( \pi^+_T, 0 \right) \right) \mathbb{E}_T \left[ H \left( \pi^+_T, 0 \right) \left| \pi^{-}_T \right. \right] \frac{\frac{1}{\psi} - \gamma}{1 - \gamma}
\]

(43)

\[
= \frac{h_{sH} \left( \pi^+_H, 0 \right) \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} \ p \left( \pi^+_H, 0 \right) + h_{sL} \left( \pi^+_L, 0 \right) \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} \ p \left( \pi^+_L, 0 \right)}{h_{sH} \left( \pi^+_H, 0 \right) + h_{sL} \left( \pi^+_L, 0 \right)} \left[ \frac{1}{\frac{\psi} - \gamma} \right]
\]

(44)

or in terms of \( \hat{\theta} \),

\[
p \left( \hat{\theta}_T, T \right) = \mathbb{E}_T \left[ H \left( \hat{\theta}^+_T, 0 \right) \right] \left( \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} \right) \frac{p \left( \hat{\theta}^+_T, 0 \right)}{\mathbb{E}_T \left[ H \left( \hat{\theta}^+_T, 0 \right) \right] \left( \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} \right)}
\]

(45)

\[
= \frac{h_{sH} \left( \hat{\theta}^+_H, 0 \right) \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} \ p \left( \hat{\theta}^+_H, 0 \right) + h_{sL} \left( \hat{\theta}^+_L, 0 \right) \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} \ p \left( \hat{\theta}^+_L, 0 \right)}{h_{sH} \left( \hat{\theta}^+_H, 0 \right) + h_{sL} \left( \hat{\theta}^+_L, 0 \right)} \left[ \frac{1}{\frac{\psi} - \gamma} \right]
\]

(46)

where \( h_{sH} = \frac{\hat{\theta}_H - \theta_H - \hat{\theta}_L (1 - \nu)}{\theta_H - \theta_L} \), \( h_{sL} = \frac{\hat{\theta}_H - \hat{\theta}_L (1 - \nu)}{\hat{\theta}_H - \hat{\theta}_L \nu} \). Also, \( \hat{\theta}^+_H = \theta_L + \frac{(\hat{\theta}^+_H - \theta_L) \nu (\hat{\theta}_H - \theta_L)}{(\hat{\theta} - \theta_L) \nu (\theta_H - \theta_L) (1 - \nu)} \)

and \( \hat{\theta}^+_L = \theta_L + \frac{(\hat{\theta}^+_L - \theta_L) (1 - \nu) (\theta_H - \theta_L)}{(\hat{\theta} - \theta_L) (1 - \nu) (\theta_H - \theta_L) \nu} \).

**Risk Premium** Conjecture the cumulated return of the following form:

\[
\frac{dR_t}{R_t} = \mu_{R, t} dt + \sigma_{R_Y, t} dB_{Y, t} + \sigma_{R_s, t} d\hat{B}_{s, t}
\]

(47)
The cumulative return can be computed as

\[
\frac{dR_t}{R_t} = \frac{1}{p(\hat{\theta}_t, t) D_t} \left[ D_t dt + d \left( p \left( \hat{\theta}_t, t \right) D_t \right) \right] = \frac{1}{p} dt + \frac{d(pD_t)}{pD_t}
\]  \hfill (48)

\[
d \left( p \left( \hat{\theta}_t, t \right) D_t \right) p(\hat{\theta}_t, t) D_t = \left\{ \frac{1}{p} \left[ p_t + p_\theta \mu_t, t + \frac{1}{2} p_\theta \sigma_{\theta_t}^2 \left( \frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2} \right) \right] + \xi \left( \hat{\theta}_t - \bar{\theta} \right) + \bar{\theta} + \frac{p_\theta}{p} \eta \sigma_{\theta_t} \right\} dt
\]

\[
+ \left( \frac{p_\theta \sigma_{\theta_t}}{p \sigma_Y} + \eta \sigma_Y \right) d\hat{B}_{Y,t} + \frac{p_\theta \sigma_{\theta_t}}{p \sigma_s} d\hat{B}_{s,t}
\]  \hfill (49)

Therefore

\[
\mu_{R,t} = \frac{1}{p} \left[ 1 + p_t + p_\theta \mu, t + \frac{1}{2} p_\theta \sigma_{\theta_t}^2 \left( \frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2} \right) \right] + \xi \left( \hat{\theta}_t - \bar{\theta} \right) + \bar{\theta} + \frac{p_\theta}{p} \eta \sigma_{\theta_t}
\]  \hfill (50)

\[
\sigma_{RY,t} = \frac{p_\theta \sigma_{\theta_t}}{p \sigma_Y} + \eta \sigma_Y
\]  \hfill (51)

\[
\sigma_{Rs,t} = \frac{p_\theta \sigma_{\theta_t}}{p \sigma_s}
\]  \hfill (52)

Together with the expression of pricing kernel, the risk premium is therefore

\[
\frac{dM_t}{M_t} = -r_t dt - \sigma_{MY,t} d\hat{B}_{Y,t} - \sigma_{M,s,t} d\hat{B}_{s,t}
\]  \hfill (53)

\[
\sigma_{MY,t} = \gamma \sigma_Y - \frac{1}{\gamma^2} H \sigma_{\theta_t} \sigma_Y
\]  \hfill (54)

\[
\sigma_{M,s,t} = -\frac{1}{\gamma^2} H \sigma_{\theta_t} \sigma_s
\]  \hfill (55)

\[
\mathbb{E}_t \left[ \frac{dR_t}{R_t} \right] - r_t = -\text{Cov}_t \left[ \frac{dM_t}{M_t}, \frac{dR_t}{R_t} \right]
\]  \hfill (56)

\[
\mu_{R,t} - r_t = \left( \gamma \sigma_Y - \frac{1}{\gamma^2} H \sigma_{\theta_t} \sigma_Y \right) \left( \frac{p_\theta \sigma_{\theta_t}}{p \sigma_Y} + \eta \sigma_Y \right) - \frac{1}{\gamma^2} H \sigma_{\theta_t} \sigma_s
\]  \hfill (57)

**Computing returns** On non-announcement days:

\[
R_{t,t+\Delta} = \frac{p \left( \theta_{t+\Delta}, t + \Delta \right) D_{t+\Delta} + \int_t^{t+\Delta} D_s ds}{p \left( \hat{\theta}_t, t \right) D_t}.
\]
Note that $D_{t+\Delta} = D_t e^{\int_t^{t+\Delta} (\xi \hat{\theta}_s + \bar{\theta}(1-\xi) - \frac{1}{2} \eta^2 \sigma_Y^2) ds + \int_t^{t+\Delta} \eta \sigma_Y d\hat{B}_Y, s}$. As an approximation,

$$\frac{D_{t+\Delta}}{D_t} = e^{\int_t^{t+\Delta} \frac{1}{2} [\hat{\theta}_t + \bar{\theta}(1-\xi) - \frac{1}{2} \eta^2 \sigma_Y^2] ds + \int_t^{t+\Delta} \eta \sigma_Y (\hat{B}_Y, s, t + \bar{\theta} - \hat{B}_Y)}$$

and

$$\frac{D_{t+\Delta} D_s ds}{D_t} = 2 \Delta \frac{D_{t+\Delta} + D_t}{D_t} = \frac{1}{2} \Delta \left( 1 + e^{\int_t^{t+\Delta} \frac{1}{2} [\hat{\theta}_t + \bar{\theta}(1-\xi) - \frac{1}{2} \eta^2 \sigma_Y^2] ds + \int_t^{t+\Delta} \eta \sigma_Y (\hat{B}_Y, s, t + \bar{\theta} - \hat{B}_Y)} \right)$$

On announcement day, assume that the announcement happens at the end of the day. First calculate non-announcement return $R_{T-\Delta, T}$ as above. Note calculate

$$R_A = \frac{p(\hat{\theta}_T^+, T^+)}{p(\hat{\theta}_T^-, T^-)}.$$

The total return equals $R_{T-\Delta, T} \times R_A$.

### 7.3 Forward Looking Measures of Variance

To compute implied variance, we need

$$Var_t \left[ \ln \left\{ p \left( \nu_t, \hat{\theta}_t, \tau \right) D_{\tau} \right\} - \ln \left\{ p \left( \nu_t, \hat{\theta}_t, t \right) D_t \right\} \right] = Var_t \left[ \ln p \left( \nu_t, \hat{\theta}_t, \tau \right) + \ln D_{\tau} \right]. \quad (58)$$

Note that $D_{\tau} = D_t e^{\int_t^{\tau} (\xi \hat{\theta}_s + \bar{\theta}(1-\xi) - \frac{1}{2} \eta^2 \sigma_Y^2) ds + \int_t^{\tau} \eta \sigma_Y d\hat{B}_Y, s}$. Therefore $\ln D_{\tau} = \ln D_t + \int_t^{\tau} (\xi \hat{\theta}_s + \bar{\theta}(1-\xi) - \frac{1}{2} \eta^2 \sigma_Y^2) ds + \int_t^{\tau} \eta \sigma_Y d\hat{B}_Y, s$. Therefore,

$$Var_t \left[ \ln p \left( \nu_t, \hat{\theta}_t, \tau \right) + \ln D_{\tau} \right]
= Var_t \left[ \ln p \left( \nu_t, \hat{\theta}_t, \tau \right) + \int_t^{\tau} (\xi \hat{\theta}_s + \bar{\theta}(1-\xi) - \frac{1}{2} \eta^2 \sigma_Y^2) ds + \int_t^{\tau} \eta \sigma_Y d\hat{B}_Y, s \right].
= Var_t \left[ \ln p \left( \nu_t, \hat{\theta}_t, \tau \right) \right] + 2 Cov_t \left[ \ln p \left( \nu_t, \hat{\theta}_t, \tau \right), \delta (\tau) - \delta (t) \right] + Var_t \left[ \delta (\tau) - \delta (t) \right], \quad (59)$$

where we denote $\delta (t) = \int_t^t (\xi \hat{\theta}_s + \bar{\theta}(1-\xi) - \frac{1}{2} \eta^2 \sigma_Y^2) ds + \int_t^t \eta \sigma_Y d\hat{B}_Y, s$.

### 7.4 Announcement Return Predictability